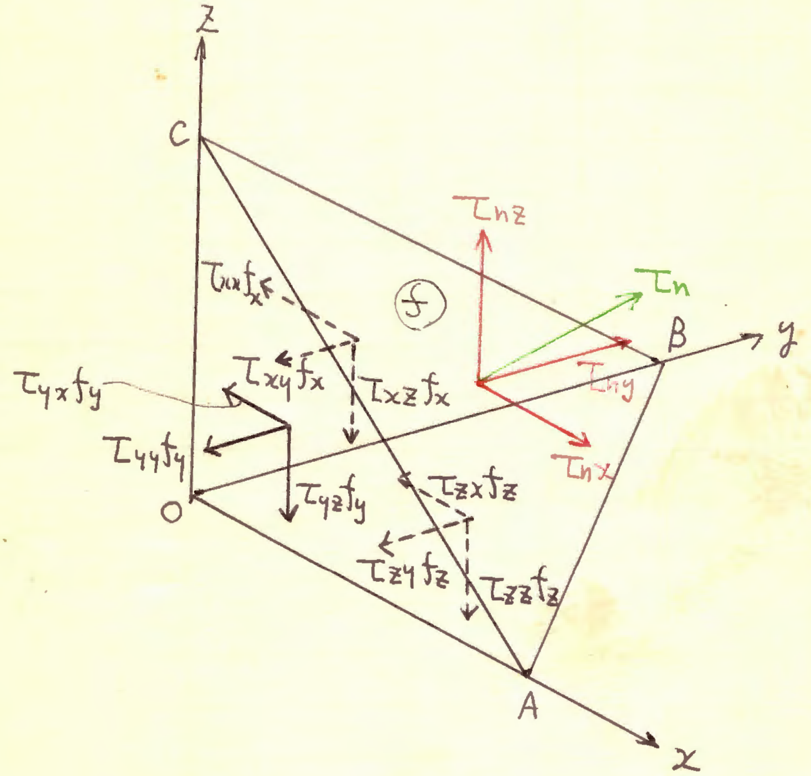
Chapter 2 Small Displacement Theory of Elasticity

§1. fundamental equations

(1) stress

☆ [物体内の一点は $\sigma = \{\sigma_x \ \ \sigma_y \ \ \sigma_z \ \ \tau_{yz} \ \ \tau_{xy} \ \ \tau_{zx}\}$ で表わされる。]



☆ body force can be neglected.

$$\tau_{nx} = \frac{f_x}{f}\tau_{xx} + \frac{f_y}{f}\tau_{yx} + \frac{f_z}{f}\tau_{zx}$$

$$= \tau_{xx}\cos(n,x) + \tau_{yx}\cos(n,y) + \tau_{zx}\cos(n,z)$$

$$= \sigma_x \cos(n,x) + \tau_{yx}\cos(n,y) + \tau_{zx}\cos(n,z)$$

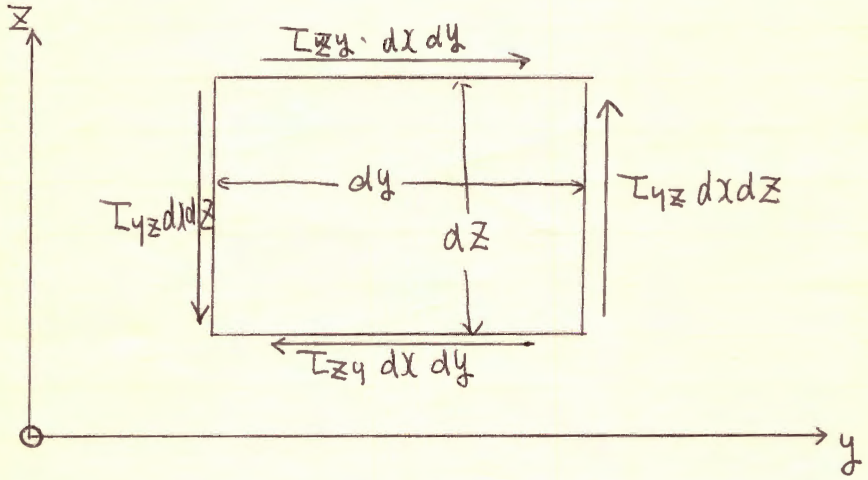
similarly

$$\tau_{ny} = \tau_{xy}\cos(n,x) + \sigma_y \cos(n,y) + \tau_{zy}\cos(n,z)$$

$$\tau_{nz} = \tau_{xz}\cos(n,x) + \tau_{yz}\cos(n,y) + \sigma_z \cos(n,z)$$

$$\therefore\ \tau_n = \begin{bmatrix} \tau_{nx} \\ \tau_{ny} \\ \tau_{nz} \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} \cos(n,x) \\ \cos(n,y) \\ \cos(n,z) \end{bmatrix}$$

この関係は，立体の大きさに無関係に成立する。



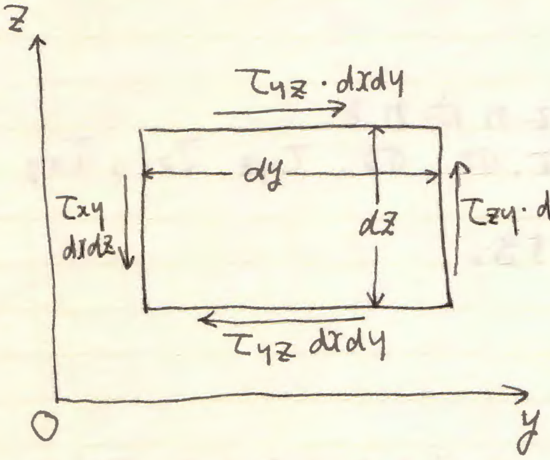
x 軸まわりのモーメントのつり合い。

$$(\tau_{zy}\,dx\,dy)\cdot dz - (\tau_{yz}\,dx\,dz)\,dy = 0$$

$$\therefore\ \tau_{zy} = \tau_{yz}$$

similarily $\left.\begin{aligned} \tau_{zx} &= \tau_{xz} \\ \tau_{xy} &= \tau_{yx} \end{aligned}\right\}$

この関係は，ABC面を平行移動して体積を無限に小さくしても変わらない。よって O点で n方向に垂直な平面での応力は τ_{nx} , τ_{ny} , τ_{nz} で表わされる。



次に，左図のような微小六面体を考える。そして，その中心を通る x 軸周りの偶力の釣合いを考えてみる。

各面での応力は，その合力が面の中心を通ると考えられるから，偶力をもつのは τ_{yz} , τ_{zy} のみである。

そして τ_{yz} からの偶力は $\tau_{yz} \cdot dx\,dy$ の力が長さ dy の腕で作用しているから $\tau_{yz}\,dx\,dy\,dz$. 同様にして τ_{zy} からは $-\tau_{zy}\,dx\,dy\,dz$ であるから，モーメントの釣合いを考えれば，

$$\tau_{yz} = \tau_{zy}$$

同様に $\tau_{zx} = \tau_{xz}$

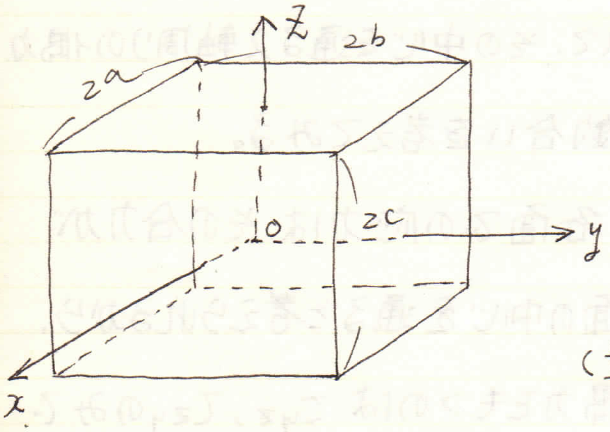
$$\tau_{xy} = \tau_{yx}$$

☆応力の釣り合いの式

物体内の応力の変化の条件.

直角座標系

x方向のつり合いを考える。



0での応力を
σ_x, σ_y, σ_z, τ_{yz}, τ_{zx}, τ_{xy}
とする。

($\pm a$, y, z)での応力を，Taylor

の展開公式より求める。(x方向のみ)

$$\sigma_x \pm a\frac{\partial\sigma_x}{\partial x} + y\frac{\partial\sigma_x}{\partial y} + z\frac{\partial\sigma_x}{\partial z} + \frac{1}{2!}(\pm\sigma_x^{x}a + \sigma_x^{y}y + \sigma_x^{z}z)^2 + \cdots$$

(x, $\pm b$, z)面.

$$\tau_{yx} + \tau_{yx}^{x} \pm \tau_{yx}^{y}b + \tau_{yx}^{z}z + \frac{1}{2!}(\tau_{yx}^{x}x \pm \tau_{yx}^{y}b + \tau_{yx}^{z}z)^2 + \cdots$$

(x, y, $\pm c$)面.

$$\tau_{zx} + \tau_{zx}^{x}x + \tau_{zx}^{y}y \pm \tau_{zx}^{z}c + \frac{1}{2!}(\tau_{zx}^{x}x + \tau_{zx}^{y}y \pm \tau_{zx}^{z}c)^2 + \cdots$$

体積力（重力, 慣性力など）　原点(0, 0, 0)で X, Y, Z

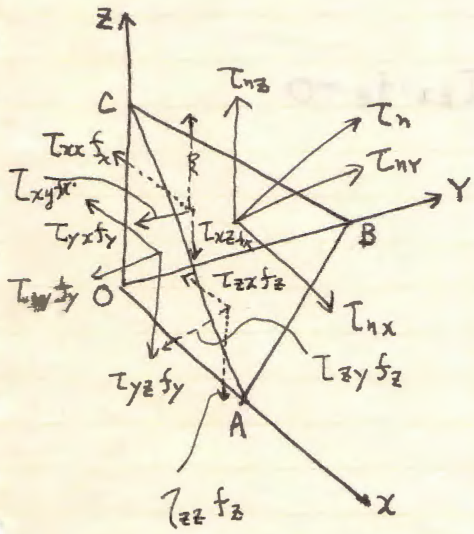
X方向　$X + X^{x}x + X^{y}y + X^{z}z + \frac{1}{2!}(X^{x}x + X^{y}y + X^{z}z)^2 + \cdots$

☆応力の成分

$$\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

$\tau_{yz}=\tau_{zy}$　　$\tau_{zx}=\tau_{xz}$　　$\tau_{xy}=\tau_{yx}$ の証明

$\{\sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{yz},\ \tau_{zx},\ \tau_{xy}\}$ で 1点$P^{(0)}$の応力を表わせることを証明する。



ABC面の面積をf
外への法線の方向をnとする。

面積　yz面　f_x
　　　zx面　f_y
　　　xy面　f_z

とする。

この四面体O－ABCが静止しているとすれば、これに作用している力はすべて釣り合っていなければならない。

今の場合，体積力は，四面体を微小にすれば，面力に比して1次ORDERの高い微小量となるので今の場合無視できる。

まずx方向のつりあいを考えると、ABC面での応力のx方向成分を

τ_{nx}とすると、各面でのx方向の力は、

BCO面では　$-\sigma_x \cdot f_x$

CAO面では　$-\tau_{yx} \cdot f_y$

ABO面では　$-\tau_{zx} \cdot f_z$

ABC面では　$\tau_{nx} \cdot f$

故に、$\tau_{nx} \cdot f - \sigma_x f_x - \tau_{yx} \cdot f_y - \tau_{zx} \cdot f_z = 0$

幾何学的関係から、

法線nの方向余弦

$$\frac{f_x}{f} = \cos(n, x)$$

$$\frac{f_y}{f} = \cos(n, y)$$

$$\frac{f_z}{f} = \cos(n, z)$$

であるから、

$$\tau_{nx} = \sigma_x \cos(n, x) + \tau_{yx} \cos(n, y) + \tau_{zx} \cos(n, z)$$

同様に

$$\tau_{ny} = \tau_{xy} \cos(n, x) + \sigma_y \cos(n, y) + \tau_{zy} \cos(n, z)$$

$$\tau_{nz} = \tau_{xz} \cos(n, x) + \tau_{yz} \cos(n, y) + \sigma_z \cos(n, z)$$

今，長さを一次のorderの微小量とすれば，体積力は三次のorderのものであり，面力は二次のorderである。

そこで，応力及び体積力の三次のorderのものを集めると，

$$\iiint\{(\sigma_x+\sigma_x^{\ x}a+\sigma_x^{\ y}y+\sigma_x^{\ z}z)-(\sigma_x-\sigma_x^{\ x}a+\sigma_x^{\ y}y+\sigma_x^{\ z}z)\}dydz$$

$$+\iiint\{(\tau_{yx}+\tau_{yx}^{\ x}x+\tau_{yx}^{\ y}b+\tau_{yx}^{\ z}z)-(\tau_{yx}+\tau_{yx}^{\ x}x-\tau_{yx}^{\ y}b+\tau_{yx}^{\ z}z)\}dzdx$$

$$+\iiint\{(\tau_{zx}+\tau_{zx}^{\ x}x+\tau_{zx}^{\ y}y+\tau_{zx}^{\ z}c)-(\tau_{zx}+\tau_{zx}^{\ x}x+\tau_{zx}^{\ y}y-\tau_{zx}^{\ z}c)\}dxdy$$

$$+\iiint X\,dxdydz \quad =0$$

$$左辺=\iint(2a\sigma_x^{\ x})dydz+\iint 2b\tau_{yx}^{\ y}dzdx+\iint 2c\tau_{zx}^{\ z}dxdy$$

$$+\iiint X\,dxdydz$$

$$=2a\sigma_x^{\ x}\cdot 2b\cdot 2c+2b\tau_{yx}^{\ y}\cdot 2c\cdot 2a+2c\tau_{zx}^{\ z}2a\cdot 2b$$

$$+8abcX$$

$$=8abc\sigma_x^{\ x}+8abc\tau_{yx}^{\ y}+8abc\tau_{zx}^{\ z}+8abc\bar{X}=0$$

$8abc$で割れば，

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\bar{X}=0$$

同様にして.

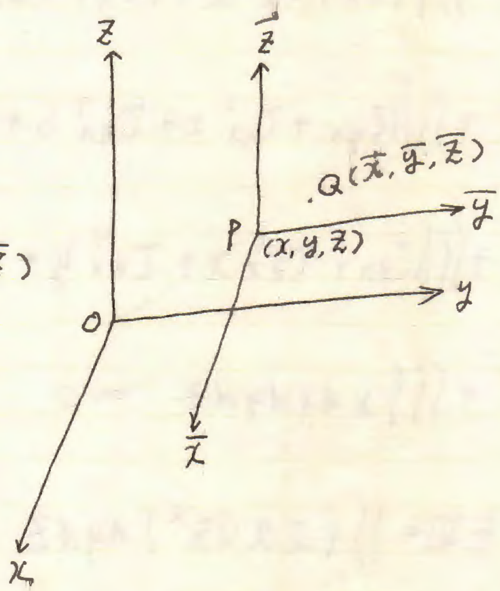
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \bar{Y} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \bar{Z} = 0$$

☆ 歪 (strain)

P点の変位 $u_p = u_p(x, y, z)$

Q点の変位 $u_Q = u_Q(x+\bar{x}, y+\bar{y}, z+\bar{z})$



P点の変位

$$\begin{cases} x\text{方向} & u_p(x, y, z) \\ y\text{方向} & v_p(x, y, z) \\ z\text{方向} & w_p(x, y, z) \end{cases}$$

Q点の変位

$$\begin{cases} x\text{方向} & u_Q(x+\bar{x}, y+\bar{y}, z+\bar{z}) \\ y\text{方向} & v_Q(x+\bar{x}, y+\bar{y}, z+\bar{z}) \\ z\text{方向} & w_Q(x+\bar{x}, y+\bar{y}, z+\bar{z}) \end{cases}$$

Taylor 展開.

$$\left.\begin{aligned} u_Q &= u_p + \left(\bar{x}\frac{\partial}{\partial x} + \bar{y}\frac{\partial}{\partial y} + \bar{z}\frac{\partial}{\partial z}\right)u + \frac{1}{2!}\left(\bar{x}\frac{\partial}{\partial x} + \bar{y}\frac{\partial}{\partial y} + \bar{z}\frac{\partial}{\partial z}\right)^2 u + \cdots \\ v_Q &= v_p + \left(\bar{x}\frac{\partial}{\partial x} + \bar{y}\frac{\partial}{\partial y} + \bar{z}\frac{\partial}{\partial z}\right)v + \frac{1}{2!}\left(\bar{x}\frac{\partial}{\partial x} + \bar{y}\frac{\partial}{\partial y} + \bar{z}\frac{\partial}{\partial z}\right)^2 v + \cdots \\ w_Q &= w_p + \left(\bar{x}\frac{\partial}{\partial x} + \bar{y}\frac{\partial}{\partial y} + \bar{z}\frac{\partial}{\partial z}\right)w + \frac{1}{2!}\left(\bar{x}\frac{\partial}{\partial x} + \bar{y}\frac{\partial}{\partial y} + \bar{z}\frac{\partial}{\partial z}\right)^2 w + \cdots \end{aligned}\right\} \quad -(1)$$

P点とQ点が極めて近傍に位置するとすれば、

$$u_Q - u_P = \left(\bar{x}\frac{\partial}{\partial x} + \bar{y}\frac{\partial}{\partial y} + \bar{z}\frac{\partial}{\partial z}\right)u$$

$$v_Q - v_P = \left(\bar{x}\frac{\partial}{\partial x} + \bar{y}\frac{\partial}{\partial y} + \bar{z}\frac{\partial}{\partial z}\right)v$$

$$w_Q - w_P = \left(\bar{x}\frac{\partial}{\partial x} + \bar{y}\frac{\partial}{\partial y} + \bar{z}\frac{\partial}{\partial z}\right)w$$

このとき

$\bar{x}_1 = \bar{x} + (u_Q - u_P)$ より、

$$\left.\begin{aligned}
\bar{x}_1 &= \left\{1 + \frac{\partial u}{\partial x}\right\}\bar{x} + \frac{\partial u}{\partial y}\bar{y} + \frac{\partial u}{\partial z}\bar{z} \\
\text{同様に}\quad \bar{y}_1 &= \frac{\partial v}{\partial x}\bar{x} + \left(1 + \frac{\partial v}{\partial y}\right)\bar{y} + \frac{\partial v}{\partial z}\bar{z} \\
\bar{z}_1 &= \frac{\partial w}{\partial x}\bar{x} + \frac{\partial w}{\partial y}\bar{y} + \left(1 + \frac{\partial w}{\partial z}\right)\bar{z}
\end{aligned}\right\} \quad \text{— (2)}$$

今　$P(x, y, z)$　　$Q(x+dx,\ y+dy,\ z+dz)$

が変形後

$P_1(x+u,\ y+v,\ z+w)$　　　$\left(\overline{PQ} = dl,\ \overline{P_1Q_1} = dl_1\right)$

$Q_1(x+dx+u+du,\ y+dy+v+dv,\ z+dz+w+dw)$

へ移ったとする。

(1)式において、$\left\{\begin{aligned} dx &\to \bar{x} \\ dy &\to \bar{y} \\ dz &\to \bar{z} \end{aligned}\right.$　$\left\{\begin{aligned} u_Q - u_P &= du \\ v_Q - v_P &= dv \\ w_Q - w_P &= dw \end{aligned}\right.$

とおきかえれば、高次項を無視して、

$$\left.\begin{aligned}
du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \\
dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \\
dw &= \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz
\end{aligned}\right\} \quad \text{— (3)}$$

となるから、

$$\overline{P_1 Q_1}^2 = dl_1^2 = (dx+du)^2 + (dy+dv)^2 + (dz+dw)^2$$

$$= \left\{ \left(1+\frac{\partial u}{\partial x}\right)dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz \right\}^2$$

$$+ \left\{ \frac{\partial v}{\partial x}dx + \left(1+\frac{\partial v}{\partial y}\right)dy + \frac{\partial v}{\partial z}dz \right\}^2$$

$$+ \left\{ \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \left(1+\frac{\partial w}{\partial z}\right)dz \right\}^2$$

$$= \left\{ \left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \right\} dx^2$$

$$+ \left\{ \left(\frac{\partial u}{\partial y}\right)^2 + \left(1+\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\} dy^2$$

$$+ \left\{ \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(1+\frac{\partial w}{\partial z}\right)^2 \right\} dz^2$$

$$+ 2\left\{ \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z} + \left(1+\frac{\partial v}{\partial y}\right)\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\left(1+\frac{\partial w}{\partial z}\right) \right\} dy\,dz$$

$$+ 2\left\{ \frac{\partial u}{\partial z}\left(1+\frac{\partial u}{\partial x}\right) + \frac{\partial v}{\partial z}\cdot\frac{\partial v}{\partial x} + \left(1+\frac{\partial w}{\partial z}\right)\frac{\partial w}{\partial x} \right\} dz\,dx$$

$$+ 2\left\{ \left(1+\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\left(1+\frac{\partial v}{\partial y}\right) + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \right\} dx\,dy$$

$$\therefore dl_1^2 = \left\{ 1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \right\} dx^2 \quad \Rightarrow (1+\gamma_{xx})dx^2$$

$$+ \left\{ 1 + 2\frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\} dy^2 \quad \Rightarrow (1+\gamma_{yy})^2 dy^2$$

$$+ \left\{ 1 + 2\frac{\partial w}{\partial z} + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right\} dz^2 \quad \Rightarrow (1+\gamma_{zz})^2 dz^2$$

$$+ 2\left\{ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial z} \right\} dy\,dz \quad \Rightarrow 2\gamma_{yz}\,dy\,dz$$

$$+ 2\left\{ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial z}\cdot\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial v}{\partial x} + \frac{\partial w}{\partial z}\cdot\frac{\partial w}{\partial x} \right\} dz\,dx \quad \Rightarrow 2\gamma_{zx}\,dz\,dx$$

$$+ 2\left\{ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y} \right\} dx\,dy \quad \Rightarrow 2\gamma_{xy}\,dx\,dy$$

$\therefore\ dl_1^2 = (1+\gamma_{xx})dx^2 + (1+\gamma_{yy})^{\cancel{2}}dy^2 + (1+\gamma_{zz})^{\cancel{2}}dz^2$

$\qquad + 2\gamma_{yz}\,dy\,dz + 2\gamma_{zx}\,dz\,dx + 2\gamma_{xy}\,dx\,dy \quad \cdots \quad (4)$

dy, $dz = 0$ とすると、 $dl_1^2 = (1+\gamma_{xx})dx^2$

$\lambda_x = \dfrac{dl_1 - dl}{dl} = \dfrac{\sqrt{1+\gamma_{xx}}\,dx - dx}{dx} = \sqrt{1+\gamma_{xx}} - 1 \cdots$ x方向~~[illegible]~~（歪）

同様に $\lambda_y = \sqrt{1+\gamma_{yy}} - 1$, $\lambda_z = \sqrt{1+\gamma_{zz}} - 1$

$P(x, y, z)$, $Q(x+dx, y, z)$, $R(x, y+dy, z)$.

変形後の PQの長さ $\sqrt{1+\gamma_{xx}}\,dx$

〃 PR 〃 $\sqrt{1+\gamma_{yy}}\,dy$.

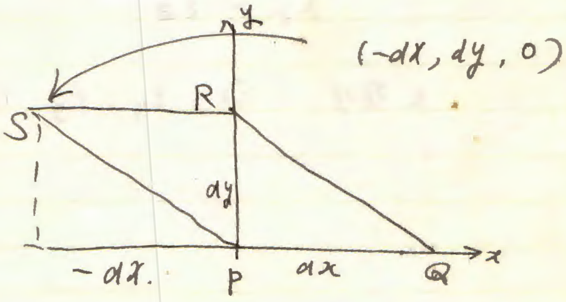
PQ と PR のなす角 $= \varphi_{xy}$ とすると、

$(1+\gamma_{xx})dx^2 + (1+\gamma_{yy})dy^2 - 2\sqrt{(1+\gamma_{xx})(1+\gamma_{yy})}\,dx\,dy\cos\varphi_{xy}$

$RQ = SP$

$SP = (1+\gamma_{xx})dx^2 + (1+\gamma_{yy})dy^2$

$\quad - 2\gamma_{xy}\,dx\,dy$.



$\therefore \cos\varphi_{xy} = \dfrac{\gamma_{xy}}{\sqrt{(1+\gamma_{xx})(1+\gamma_{yy})}}$

同様に

$\cos\varphi_{yz} = \dfrac{\gamma_{yz}}{\sqrt{(1+\gamma_{yy})(1+\gamma_{zz})}}$, $\cos_{zx} = \dfrac{\gamma_{zx}}{\sqrt{(1+\gamma_{zz})(1+\gamma_{xx})}}$

故に，γ_{xx}，γ_{yy}，γ_{zz} は 直歪を与える量

γ_{xy}，γ_{yz}，γ_{zx} は せんだん歪を与える量

である。

高次項を省略すれば，

$$\gamma_{xx} = 2\frac{\partial u}{\partial x} \qquad \gamma_{yy} = 2\frac{\partial v}{\partial y} \qquad \gamma_{zz} = 2\frac{\partial w}{\partial z} \qquad \text{—— (5)}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{zy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \cdots (6)$$

今，(5)式において，

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \text{——} \quad (7)$$

とすれば，

$$\lambda_x = \sqrt{1+\gamma_{xx}} - 1 = \sqrt{1+2\varepsilon_x} - 1 \fallingdotseq \varepsilon_x$$

$$\lambda_y \fallingdotseq \varepsilon_y$$
$$\lambda_z \fallingdotseq \varepsilon_z$$

となり，ε_x，ε_y，ε_y は そのまま，x, y, z方向歪となる。

☆ 応力・ひずみ関係 (stress-strain relations)

$$\varepsilon_{x_1} = \frac{1}{E}\sigma_x$$

$$\varepsilon_{x_2} = \frac{1}{E}\cdot(-\nu\sigma_y) \qquad \varepsilon_{x_2} = \frac{1}{E}(-\nu\sigma_z)$$

$$\therefore\ \varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] = \varepsilon_{x_1} + \varepsilon_{x_2} + \varepsilon_{x_3}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

$$G = \frac{E}{1(1+\nu)}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \frac{G}{1-2\nu} \begin{bmatrix} 2(1-\nu) & 2\nu & 2\nu & & & \\ 2\nu & 2(1-\nu) & 2\nu & & & \\ 2\nu & 2\nu & 2(1-\nu) & & & \\ & & & 1-2\nu & & \\ & & & & 1-2\nu & \\ & & & & & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & & & \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & & & \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & & & \\ & & & \frac{1}{G} & & \\ & & & & \frac{1}{G} & \\ & & & & & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix}$$

#

適合条件式

ひずみと変位の関係式 (6),(7) より, u, v, w の第二偏微分係数を求める。

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial \varepsilon_x}{\partial x}$$

$$\frac{\partial u}{\partial y} = \gamma_{xy} - \frac{\partial v}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \varepsilon_y}{\partial x}$$

$$\frac{\partial u}{\partial z} = \gamma_{zx} - \frac{\partial w}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial z^2} = \frac{\partial \gamma_{zx}}{\partial z} - \frac{\partial^2 w}{\partial x \cdot \partial z} = \frac{\partial \gamma_{zx}}{\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)$$

$$= \frac{\partial \gamma_{zx}}{\partial z} - \frac{\partial \varepsilon_z}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \varepsilon_x}{\partial y}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial \varepsilon_y}{\partial y}$$

$$\frac{\partial^2 v}{\partial z^2} = \frac{\partial \gamma_{yz}}{\partial z} - \frac{\partial^2 w}{\partial z \partial y} = \frac{\partial \gamma_{yz}}{\partial z} - \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial z}\right) = \frac{\partial \gamma_{yz}}{\partial z} - \frac{\partial \varepsilon_z}{\partial y}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial \gamma_{zx}}{\partial x} - \frac{\partial^2 u}{\partial z \partial x} = \frac{\partial \gamma_{zx}}{\partial x} - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial \gamma_{zx}}{\partial x} - \frac{\partial \varepsilon_x}{\partial z}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial \gamma_{yz}}{\partial y} - \frac{\partial^2 v}{\partial y \partial z} = \frac{\partial \gamma_{yz}}{\partial y} - \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial \gamma_{yz}}{\partial y} - \frac{\partial \varepsilon_y}{\partial z}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{\partial \varepsilon_z}{\partial z}$$

$$\frac{\partial u}{\partial z} = \gamma_{zx} - \frac{\partial w}{\partial x}$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial}{\partial x}\left(\gamma_{yz} - \frac{\partial v}{\partial z}\right)$$

$$= \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial^2 v}{\partial x \partial z} = \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial}{\partial z}\left(\gamma_{xy} - \frac{\partial u}{\partial y}\right) = \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial^2 u}{\partial z \partial y}$$

$$\therefore \quad \frac{\partial^2 u}{\partial y \partial z} = \frac{1}{2}\left[-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right]$$

$$\frac{\partial u}{\partial x} = \varepsilon_x \quad \therefore \quad \frac{\partial^2 u}{\partial z \partial x} = \frac{\partial \varepsilon_x}{\partial z}$$

$$\frac{\partial u}{\partial x} = \varepsilon_x \quad \therefore \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial \varepsilon_x}{\partial y}$$

$$\frac{\partial^2 v}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial z}\left(\gamma_{xy} - \frac{\partial u}{\partial y}\right) = \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial^2 u}{\partial z \partial y} = \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial}{\partial y}\left(\gamma_{zx} - \frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial^2 w}{\partial y \partial x} = \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial}{\partial x}\left(\gamma_{yz} - \frac{\partial v}{\partial z}\right)$$

$$= \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial^2 v}{\partial x \partial z}$$

$$\therefore \quad \frac{\partial^2 v}{\partial z \partial x} = \frac{1}{2}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right]$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial \varepsilon_y}{\partial x}$$

$$\frac{\partial^2 v}{\partial y \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial \varepsilon_y}{\partial z}$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(\gamma_{yz} - \frac{\partial v}{\partial z}\right) = \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial^2 v}{\partial x \partial z}$$

$$= \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial}{\partial z}\left(\gamma_{xy} - \frac{\partial u}{\partial y}\right) = \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial^2 u}{\partial z \partial y}$$

$$= \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial}{\partial y}\left(\gamma_{zx} - \frac{\partial w}{\partial x}\right)$$

$$= \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial^2 w}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 w}{\partial x \partial y} = \frac{1}{2}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right]$$

$$\frac{\partial^2 w}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial z}\right) = \frac{\partial \varepsilon_z}{\partial y}$$

$$\frac{\partial^2 w}{\partial z \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right) = \frac{\partial \varepsilon_z}{\partial x}$$

すなわち、

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial \varepsilon_x}{\partial x},\quad \frac{\partial^2 u}{\partial y^2}=\frac{\partial \gamma_{xy}}{\partial y}-\frac{\partial \varepsilon_y}{\partial x},\quad \frac{\partial^2 u}{\partial z^2}=\frac{\partial \gamma_{zx}}{\partial z}-\frac{\partial \varepsilon_z}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial \gamma_{xy}}{\partial x}-\frac{\partial \varepsilon_x}{\partial y},\quad \frac{\partial^2 v}{\partial y^2}=\frac{\partial \varepsilon_y}{\partial y},\quad \frac{\partial^2 v}{\partial z^2}=\frac{\partial \gamma_{yz}}{\partial z}-\frac{\partial \varepsilon_z}{\partial y}$$

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial \gamma_{zx}}{\partial z}-\frac{\partial \varepsilon_x}{\cancel{\partial y}\,\partial z},\quad \frac{\partial^2 w}{\partial y^2}=\frac{\partial \gamma_{yz}}{\partial y}-\frac{\partial \varepsilon_y}{\partial z},\quad \frac{\partial^2 w}{\partial z^2}=\frac{\partial \varepsilon_z}{\partial z}$$

$$\frac{\partial^2 u}{\partial y\partial z}=\frac{1}{2}\left[-\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{zx}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right],\quad \frac{\partial^2 u}{\partial z\partial x}=\frac{\partial \varepsilon_x}{\partial z},\quad \frac{\partial^2 u}{\partial x\partial y}=\frac{\partial \varepsilon_x}{\partial y}$$

$$\frac{\partial^2 v}{\partial z\partial x}=\frac{1}{2}\left[\frac{\partial \gamma_{yz}}{\partial x}-\frac{\partial \gamma_{zx}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right],\quad \frac{\partial^2 v}{\partial x\partial y}=\frac{\partial \varepsilon_y}{\partial x},\quad \frac{\partial^2 v}{\partial y\partial z}=\frac{\partial \varepsilon_y}{\partial z}$$

$$\frac{\partial^2 w}{\partial x\partial y}=\frac{1}{2}\left[\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{zx}}{\partial y}-\frac{\partial \gamma_{xy}}{\partial z}\right],\quad \frac{\partial^2 w}{\partial y\partial z}=\frac{\partial \varepsilon_z}{\partial y},\quad \frac{\partial^2 w}{\partial z\partial x}=\frac{\partial \varepsilon_z}{\partial x}$$

ここで、

$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x\partial y}\right)$ より

$$\frac{\partial^2 \gamma_{xy}}{\partial x\partial y}-\frac{\partial^2 \varepsilon_x}{\partial x^2}=\frac{\partial^2 \varepsilon_x}{\partial y^2} \qquad \text{—(1)}$$

$\frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial z\partial y}\right)$ より

$$\frac{\partial^2 \gamma_{xy}}{\partial z\partial y}-\frac{\partial^2 \varepsilon_y}{\partial z\partial x}=\cancel{\frac{1}{2}\Big[-\frac{\partial^2 \gamma_{yz}}{\ }}\ \frac{1}{2}\frac{\partial}{\partial y}\left[-\frac{\gamma_{yz}}{\partial x}+\frac{\partial \gamma_{zx}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right] \qquad \text{—(2)}$$

$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial z^2}\right)=\frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial x\partial z}\right)$ より

$$\frac{\partial^2 \gamma_{zx}}{\partial x\partial z}-\frac{\partial^2 \varepsilon_z}{\partial x^2}=\frac{\partial^2 \varepsilon_x}{\partial z^2} \qquad \text{—(3)}$$

$\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial y \partial z}\right)$ より

$$\frac{\partial^2 \gamma_{zx}}{\partial y \partial z} - \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{1}{2} \cdot \frac{\partial}{\partial z}\left[-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right] \quad \text{—④}$$

同様にして,

$$\cancel{\frac{\partial^2 \varepsilon_y}{\partial x \partial y}} =$$

$$\frac{\partial^2 \gamma_{yz}}{\partial x \partial z} - \frac{\partial^2 \varepsilon_{xz}}{\partial x \partial y} = \frac{1}{2} \cdot \frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right] \quad \text{—⑤}$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} - \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \varepsilon_y}{\partial z^2} \quad \text{—⑥} \checkmark$$

$$\frac{\partial^2 \gamma_{xy}}{\partial y \partial x} - \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \varepsilon_y}{\partial x^2} \quad \text{—⑦} \checkmark$$

$$\frac{\partial^2 \gamma_{xy}}{\partial z \partial x} - \frac{\partial^2 \varepsilon_x}{\partial z \partial y} = \frac{1}{2} \cdot \frac{\partial}{\partial x}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right] \quad \text{—⑧}^{\circ}$$

$$\frac{\partial^2 \gamma_{zx}}{\partial y \partial z} - \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right] \quad \text{—⑨}^{\circ}$$

$$\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} - \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \varepsilon_z}{\partial x^2} \quad \text{—⑩} \checkmark$$

$$\frac{\partial \gamma_{yz}}{\partial z \partial y} - \frac{\partial^2 \varepsilon_y}{\partial z^2} = \frac{\partial^2 \varepsilon_z}{\partial y^2} \quad \text{—⑪} \checkmark$$

$$\frac{\partial \gamma_{yz}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_y}{\partial x \cdot \partial z} = \frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right] \quad \text{—⑫} \checkmark$$

⑥, ⑪ は同様に、 $$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}$$

③, ⑩ は同様に、 $$\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2}$$

①, ⑦ は同様に、 $$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

⑧, ⑨ は同様に、 $$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left[-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right]$$

②, ⑫ は同様に, $$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right]$$

④, ⑤ は同様に, $$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right]$$

よって,
$$R_x = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} - \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$
$$R_y = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} - \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$
$$R_z = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$
$$U_x = -\frac{\partial^2 \varepsilon_x}{\partial y \partial z} + \frac{1}{2}\frac{\partial}{\partial x}\left[-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right]$$
$$U_y = -\frac{\partial^2 \varepsilon_y}{\partial z \partial x} + \frac{1}{2}\frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right]$$
$$U_z = -\frac{\partial^2 \varepsilon_z}{\partial x \partial y} + \frac{1}{2}\frac{\partial}{\partial z}\left[\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right]$$

とおけば、適合条件は.

$$\begin{bmatrix} R_x & U_z & U_y \\ U_z & R_y & U_x \\ U_y & U_x & R_z \end{bmatrix} = [0]$$

変位法，

$$\sigma_x = 2G\left[\frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$$

$$\sigma_y = 2G\left[\frac{\partial v}{\partial y} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$$

$$\sigma_z = 2G\left[\frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$$

$$\tau_{yz} = G\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

(☆)

3
適合，
応力は[illegible]
→ つり合い．

$$\frac{\partial \sigma_x}{\partial x} = 2G\left[\frac{\partial^2 u}{\partial x^2} + \frac{\nu}{1-2\nu}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}\right)\right]$$

$$\frac{\partial \sigma_y}{\partial y} = 2G\left[\frac{\partial^2 v}{\partial y^2} + \frac{\nu}{1-2\nu}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z \partial y}\right)\right]$$

$$\frac{\partial \sigma_z}{\partial z} = 2G\left[\frac{\partial^2 w}{\partial z^2} + \frac{\nu}{1-2\nu}\left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial z^2}\right)\right]$$

$$\frac{\partial \tau_{yz}}{\partial y} = G\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 v}{\partial z \partial y}\right) \qquad \frac{\partial \tau_{yz}}{\partial z} = G\left(\frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial \tau_{zx}}{\partial z} = G\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z}\right) \qquad \frac{\partial \tau_{zx}}{\partial x} = G\left(\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial^2 w}{\partial x^2}\right)$$

$$\frac{\partial \tau_{xy}}{\partial x} = G\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial x}\right) \qquad \frac{\partial \tau_{xy}}{\partial y} = G\left(\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}\right)$$

x方向のつり合い

$$\cancel{2}G\left[\cancel{2}\frac{\partial^2 u}{\partial x^2}+\frac{2\nu}{1-2\nu}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y\partial x}+\frac{\partial^2 w}{\partial z\partial x}\right)\right]$$

$$+G\left(\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}\right)+G\left(\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial x\partial z}\right)+\overline{X}$$

$$=G\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\left(\frac{2\nu}{1-2\nu}+1\right)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right)+\overline{X}=0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\Delta u.$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=\frac{\partial e}{\partial x}.$$

$$\left(e=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

とすれば

$$\Delta u+\frac{1}{1-2\nu}\frac{\partial e}{\partial x}+\frac{\overline{X}}{G}=0$$

・同様に

$$\Delta v+\frac{1}{1-2\nu}\frac{\partial e}{\partial y}+\frac{\overline{Y}}{G}=0$$

$$\Delta w+\frac{1}{1-2\nu}\frac{\partial e}{\partial z}+\frac{\overline{Z}}{G}=0$$

(2,18)

(2,13), (2,14) に (＊) を代入すれば

$$G\left[\left(2\frac{\partial u}{\partial x}+\frac{2\nu}{1-2\nu}e\right)l+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)m+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)n\right]=\overline{X_\nu}$$

同様に $G\left[\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)l+\cancel{\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)m}+\left(2\frac{\partial v}{\partial y}+\frac{2\nu}{1-2\nu}e\right)m+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)n\right]=\overline{Y_\nu}$

$$G\left[\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)l+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)m+\left(2\frac{\partial w}{\partial z}+\frac{2\nu}{1-2\nu}e\right]=\overline{Z_\nu}$$

(2,21)

応力法.

$$\frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{1}{E}\left[\frac{\partial^2 \sigma_z}{\partial y^2} - \nu\left(\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial y^2}\right)\right]$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} = \frac{1}{E}\left[\frac{\partial^2 \sigma_y}{\partial z^2} - \nu\left(\frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial^2 \sigma_x}{\partial z^2}\right)\right]$$

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{2(1+\nu)}{E}\frac{\partial^2 \tau_{yz}}{\partial y \partial z}$$

$$\therefore R_x = \frac{1}{E}\left[\frac{\partial^2 \sigma_z}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial z^2} - \nu\left(\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial^2 \sigma_x}{\partial z^2}\right) - 2(1+\nu)\frac{\partial^2 \tau_{yz}}{\partial y \partial z}\right] = 0$$

$$\therefore \frac{\partial^2 \sigma_z}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial z^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}\right) = 2\frac{\partial^2 \tau_{yz}}{\partial y \partial z}$$

$$R_y = \frac{1}{E}\left[\frac{\partial^2 \sigma_x}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial x^2} - \nu\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial x^2}\right) - 2(1+\nu)\frac{\partial^2 \tau_{zx}}{\partial z \partial x}\right] = 0$$

$$\therefore \cancel{\frac{\partial^2 \sigma_z}{\partial x}}\ \frac{\partial^2 \sigma_x}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial x^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2 S}{\partial z^2} + \frac{\partial^2 S}{\partial x^2}\right) = 2\frac{\partial^2 \tau_{zx}}{\partial z \partial x}$$

$$R_z = \frac{1}{E}\left[\frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} - \nu\left(\frac{\partial^2 \sigma_z}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) - 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y}\right] = 0$$

$$\therefore \frac{\partial^2 \sigma_y}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial y^2} - \frac{\nu}{1+\nu}\left(\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2}\right) = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

U_x
U_y
U_z
についても同様

すなわち，適合条件を応力成分で表わせば，

$$(イ)\ \frac{\partial^2\sigma_z}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial z^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 S}{\partial y^2}+\frac{\partial^2 S}{\partial z^2}\right)=2\frac{\partial^2\tau_{yz}}{\partial y\partial z}\quad\leftarrow R_x=0$$

$$(ロ)\ \frac{\partial^2\sigma_x}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial x^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 S}{\partial z^2}+\frac{\partial^2 S}{\partial x^2}\right)=2\frac{\partial^2\tau_{zx}}{\partial z\partial x}\quad\leftarrow R_y=0$$

$$(ハ)\ \frac{\partial^2\sigma_y}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial y^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}\right)=2\frac{\partial^2\tau_{xy}}{\partial x\partial y}\quad\leftarrow R_z=0$$

$$(ニ)\ \frac{\partial^2\sigma_x}{\partial y\partial z}-\frac{\nu}{1+\nu}\frac{\partial^2 S}{\partial y\partial z}=\frac{\partial}{\partial x}\left[-\frac{\partial\tau_{yz}}{\partial x}+\frac{\partial\tau_{zx}}{\partial y}+\frac{\partial\tau_{xy}}{\partial z}\right]\leftarrow U_x=0$$

$$(ホ)\ \frac{\partial^2\sigma_y}{\partial z\partial x}-\frac{\nu}{1+\nu}\frac{\partial^2 S}{\partial z\partial x}=\frac{\partial}{\partial y}\left[\frac{\partial\tau_{yz}}{\partial x}-\frac{\partial\tau_{zx}}{\partial y}+\frac{\partial\tau_{xy}}{\partial z}\right]\leftarrow U_y=0$$

$$(ヘ)\ \frac{\partial^2\sigma_z}{\partial x\partial y}-\frac{\nu}{1+\nu}\frac{\partial^2 S}{\partial x\partial y}=\frac{\partial}{\partial z}\left[\frac{\partial\tau_{yz}}{\partial x}+\frac{\partial\tau_{zx}}{\partial y}-\frac{\partial\tau_{xy}}{\partial z}\right]\leftarrow U_z=0$$

ただし（ $S=\sigma_x+\sigma_y+\sigma_z$ ）

釣合い方程式において，簡単のために $\bar{X}=\bar{Y}=\bar{Z}=0$ とすると，

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=0\quad\text{——}\ (ト)$$

$$\frac{\partial\tau_{xy}}{\partial x}+\cancel{\tau_{yz}}\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0\quad\text{——}\ (チ)$$

$$\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}=0\quad\text{——}\ (リ)$$

(ト), (チ) より, ~~$\partial\tau_{xy}$~~

$$\frac{\partial\tau_{xy}}{\partial y}=-\frac{\partial\sigma_x}{\partial x}-\frac{\partial\tau_{zx}}{\partial z}$$

$$\frac{\partial\tau_{xy}}{\partial x}=-\frac{\partial\sigma_y}{\partial y}-\frac{\partial\tau_{yz}}{\partial z}$$

第1式をx, 第2式をyで微分して辺々加えると,

$$2\frac{\partial\tau_{xy}}{\partial x\partial y}=-\frac{\partial}{\partial z}\left(\frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}\right)-\frac{\partial^2\sigma_x}{\partial x^2}-\frac{\partial^2\sigma_y}{\partial y^2}$$

(リ)により右辺第1項を置換すると,

$$2\frac{\partial\tau_{xy}}{\partial x\partial y}=\frac{\partial^2\sigma_z}{\partial z^2}-\frac{\partial^2\sigma_x}{\partial x^2}-\frac{\partial^2\sigma_y}{\partial y^2}$$

これを(ハ)式に代入すると,

$$\frac{\partial^2\sigma_y}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial y^2}-\frac{\partial^2\sigma_z}{\partial z^2}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\nu}{1+\nu}\left(\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}\right)$$

$$=\Delta S-\frac{\partial^2 S}{\partial z^2}-\Delta\sigma_z-\frac{\nu}{1+\nu}\left(\Delta S-\frac{\partial^2 S}{\partial z^2}\right)$$

$$=\frac{1}{1+\nu}\left(\Delta S-\frac{\partial^2 S}{\partial z^2}\right)-\Delta\sigma_z=0 \qquad (A)$$

同様にして,
(リ), (ロ) から次式を得る。

$$\frac{1}{1+\nu}\left(\Delta S-\frac{\partial^2 S}{\partial x^2}\right)-\Delta\sigma_x=0 \qquad (B)$$

$$\frac{1}{1+\nu}\left(\Delta S-\frac{\partial^2 S}{\partial y^2}\right)-\Delta\sigma_y=0 \qquad (C)$$

(A), (B), (C) を加えて

$$\frac{3}{1+\nu}\Delta S-\frac{1}{1+\nu}\Delta S-\Delta S=\frac{1-\nu}{1+\nu}\Delta S=0 \qquad \therefore\ \Delta S=\cancel{\frac{1+\nu}{1-\nu}}=0$$

(A), (B), (C) に代入して,

$$\left.\begin{aligned}&\Delta\sigma_x+\frac{1}{1+\nu}\frac{\partial^2 S}{\partial x^2}=0\\&\Delta\sigma_y+\frac{1}{1+\nu}\frac{\partial^2 S}{\partial y^2}=0\\&\Delta\sigma_z+\frac{1}{1+\nu}\frac{\partial^2 S}{\partial z^2}=0\end{aligned}\right\}\ (☆)$$

(ト)をy，(チ)をxで 微分して辺々加える。

$$\frac{\partial^2 \tau_{xy}}{\partial y^2} + \frac{\partial^2 \tau_{xy}}{\partial x^2} = -\frac{\partial^2 \sigma_x}{\partial x \partial y} - \frac{\partial^2 \tau_{zx}}{\partial z \partial y} - \frac{\partial^2 \sigma_y}{\partial x \partial y} - \frac{\partial^2 \tau_{yz}}{\partial z \partial x}$$

$$= -\frac{\partial^2 \sigma_x}{\partial x \partial y} - \frac{\partial^2 \sigma_y}{\partial x \partial y} - \underline{\frac{\partial}{\partial z}\left(\frac{\partial \tau_{zx}}{\partial y} + \frac{\partial \tau_{yz}}{\partial x}\right)}$$

これと(ハ)式をあわせると，

$$\frac{\partial^2 \sigma_z}{\partial x \partial y} - \frac{\nu}{1+\nu}\frac{\partial^2 S}{\partial x \partial y} + \frac{\partial^2 \tau_{\cancel{yz}\,xy}}{\partial z^2 \cancel{\partial x}} = -\frac{\partial^2 \sigma_x}{\partial x \partial y} - \frac{\partial^2 \sigma_y}{\partial x \partial y} - \frac{\partial^2 \tau_{xy}}{\partial y^2} - \frac{\partial^2 \tau_{xy}}{\partial x^2}$$

$$\Delta \tau_{xy} - \frac{\nu}{1+\nu}\frac{\partial^2 S}{\partial x \partial y} + \frac{\partial^2 S}{\partial x \partial y} = 0$$

$$\therefore \quad \Delta \tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2 S}{\partial x \partial y} = 0$$

同様に

$$\Delta \tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2 S}{\partial y \partial z} = 0$$

$$\Delta \tau_{zx} + \frac{1}{1+\nu}\frac{\partial^2 S}{\partial z \partial x} = 0$$

(☆)

2-4 二次元弾性問題。

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial \varepsilon_x}{\partial x} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\gamma_{xy} - \frac{\partial v}{\partial x}\right) = \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$= \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \varepsilon_y}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(\gamma_{xy} - \frac{\partial u}{\partial y}\right) = \frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \varepsilon_x}{\partial y}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial \varepsilon_y}{\partial y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial \varepsilon_x}{\partial y}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial \varepsilon_y}{\partial x}$$

$$\cancel{\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y}\right) \text{より}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x \partial y}\right) \text{より} \quad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \varepsilon_x}{\partial y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x \partial y}\right) \text{より} \quad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} - \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

$$\therefore \quad \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0 \qquad (2.35)$$

Airy の応力関数 $F(x,y)$

$$\sigma_x = \frac{\partial^2 F}{\partial x^2} \qquad \sigma_y = \frac{\partial^2 F}{\partial y^2} \qquad \tau_{xy} = \frac{\partial^2 F}{\partial x \partial y} \qquad (2.34)$$

$$E\left(\varepsilon_x + \varepsilon_y - \frac{G}{E}\gamma_{xy}\right) = \sigma_x - \nu\sigma_y + \sigma_y - \nu\sigma_x - \tau_{xy}$$

より

又

$$X_\nu = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)\cdot\frac{dy}{dS} + \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\cdot\frac{dx}{dS} = \frac{d}{dS}\left(\frac{\partial F}{\partial y}\right)$$

$$Y_\nu = -\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\frac{dy}{dS} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)\cdot\frac{dx}{dS} = -\frac{d}{dS}\left(\frac{\partial F}{\partial x}\right) \qquad (2.37)$$

$$\therefore \quad X_\nu = \frac{d}{dS}\left(\frac{\partial F}{\partial y}\right) = \overline{X_\nu}$$

$$Y_\nu = -\frac{d}{dS}\left(\frac{\partial F}{\partial x}\right) = \overline{Y_\nu} \qquad (2.38)$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad — \quad (1)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \qquad — \quad (2)$$

2-5 ひずみエネルギ関数

図2-1より，応力のなす仕事を求める。

P°S°T°面とその対面．

$$
\begin{aligned}
&-\sigma_x dydz\,du + \sigma_x dydz\left\{du + \frac{\partial(du)}{\partial x}dx\right\} \\
&-\tau_{xz}\, dydz\,dw + \tau_{xz}\, dydz\left\{dw + \frac{\partial(dw)}{\partial x}dx\right\} \\
&-\tau_{xy}\, dydz\,dv + \tau_{xy}\, dydz\left\{dv + \frac{\partial(dv)}{\partial x}dx\right\} \\
&=\left\{\sigma_x \frac{\partial(du)}{\partial x} + \tau_{xz}\frac{\partial(dw)}{\partial x} + \tau_{xy}\frac{\partial(dv)}{\partial x}\right\}dxdydz
\end{aligned}
$$

P°T°R°面とその対面．

$$
\left\{\sigma_y \frac{\partial(dv)}{\partial y} + \tau_{yz}\frac{\partial(dw)}{\partial y} + \tau_{yx}\frac{\partial(\partial u)}{\partial y}\right\}dxdydz
$$

R°S°P°面とその対面

$$
\left\{\sigma_z \frac{\partial(dw)}{\partial z} + \tau_{zx}\frac{\partial(du)}{\partial z} + \tau_{zy}\frac{\partial(dv)}{\partial z}\right\}dxdydz
$$

合計すると．

$$
\begin{aligned}
&\left[\sigma_x \frac{\partial(du)}{\partial x} + \sigma_y \frac{\partial(dv)}{\partial y} + \sigma_z\frac{\partial(dw)}{\partial z} + \tau_{yz}\left\{\frac{\partial(dw)}{\partial y} + \frac{\partial(dv)}{\partial z}\right\}\right. \\
&\qquad\left. + \tau_{zx}\left\{\frac{\partial(du)}{\partial z} + \frac{\partial(dw)}{\partial x}\right\} + \tau_{xy}\left\{\frac{\partial(dv)}{\partial x} + \frac{\partial(du)}{\partial y}\right\}\right]dxdydz \\
&=\left[\sigma_x d\left(\frac{\partial u}{\partial x}\right) + \sigma_y d\left(\frac{\partial v}{\partial y}\right) + \sigma_z d\left(\frac{\partial w}{\partial z}\right) + \tau_{yz} d\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \tau_{zx} d\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right. \\
&\qquad\left. + \tau_{xy} d\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right]dxdydz \\
&=\left(\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z + \tau_{yz} d\gamma_{yz} + \tau_{zx} d\gamma_{zx} + \tau_{xy} d\gamma_{xy}\right)dxdydz
\end{aligned}
\tag{2.40}
$$

(2.11)式より，

$$A(\varepsilon_x, \varepsilon_y, \cdots, \gamma_{xy}) = \frac{1}{2}\{\varepsilon^*\}^T[A]\{\varepsilon\} = \frac{1}{2}\{\sigma\}^T\{\varepsilon\}$$

$$= \frac{1}{2}\left[\sigma_x \varepsilon_x + \cdots + \gamma_{xy}\tau_{xy}\right]$$

$\because$

$$\sigma_x \varepsilon_x = E\left[\frac{\varepsilon_x}{1+\nu} + \frac{\nu}{(1-2\nu)(1+\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]\varepsilon_x$$

$$= E\left[\frac{\varepsilon_x^2}{1+\nu} + \frac{\nu}{(1-2\nu)(1+\nu)}\varepsilon_x(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y \varepsilon_y = E\left[\frac{\varepsilon_y^2}{1+\nu} + \frac{\nu}{(1-2\nu)(1+\nu)}\varepsilon_y(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_z \varepsilon_z = E\left[\frac{\varepsilon_z^2}{1+\nu} + \frac{\nu}{(1-2\nu)(1+\nu)}\varepsilon_z(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$

$$\tau_{yz}\cdot\gamma_{yz} = G\gamma_{yz}^2, \quad \tau_{zx}\cdot\gamma_{zx} = G\gamma_{zx}^2, \quad \tau_{xy}\cdot\gamma_{xy} = G\gamma_{xy}^2$$

$$\therefore A = \frac{E}{2(1+\nu)}(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + \frac{\nu E}{2(1-2\nu)(1+\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2$$

$$+ \frac{G}{2}(\gamma_{yz}^2 + \gamma_{zx}^2 + \gamma_{xy}^2)$$

$$= G(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + \frac{\nu E}{2(1-2\nu)(1+\nu)}(\varepsilon_x + \varepsilon_y + \varepsilon_z)^2$$

$$+ \frac{G}{2}(\gamma_{yz}^2 + \gamma_{zx}^2 + \gamma_{xy}^2) \qquad (2.44)$$

P28 問題, 二次元弾性.

$$\sigma_x = \frac{E}{(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y) \quad \sigma_y = \frac{E}{(1-\nu^2)}(\varepsilon_y + \nu\varepsilon_x), \quad \tau_{xy} = G\gamma_{xy} \qquad (2.28)$$

$$A = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})$$

$$\sigma_x\varepsilon_x = \frac{E}{1-\nu^2}(\varepsilon_x^2 + \nu\varepsilon_x\varepsilon_y)$$

$$\sigma_y\varepsilon_y = \frac{E}{1-\nu^2}(\varepsilon_y^2 + \nu\varepsilon_x\varepsilon_y)$$

$$\tau_{xy}\gamma_{xy} = G\gamma_{xy}^2$$

$$\therefore \quad A = \frac{E}{2(1-\nu^2)}(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y) + G\gamma_{xy}^2$$

$$\cancel{= \frac{E}{2(1-\nu^2)}(\varepsilon_x + \varepsilon_y)^2 + G\gamma_{xy}^2 -}$$

$$= \frac{E}{2(1-\nu^2)}(\varepsilon_x^2 + \varepsilon_y^2 + 2\varepsilon_x\varepsilon_y + 2\nu\varepsilon_x\varepsilon_y - 2\varepsilon_x\varepsilon_y) + \frac{G}{2}\gamma_{xy}^2$$

$$= \frac{E}{2(1-\nu^2)}(\varepsilon_x + \varepsilon_y)^2 - \frac{1-\nu}{1-\nu^2}E\varepsilon_x\varepsilon_y + \frac{G}{2}\gamma_{xy}^2$$

$$\frac{1-\nu}{1-\nu^2}E\varepsilon_x\varepsilon_y = \frac{E}{1+\nu}\varepsilon_x\varepsilon_y = 2G\varepsilon_x\varepsilon_y$$

$$\therefore \quad A = \frac{E}{2(1-\nu^2)}(\varepsilon_x + \varepsilon_y)^2 + \frac{G}{2}(\gamma_{xy}^2 - 4\varepsilon_x\varepsilon_y) \qquad (2.46)$$

2.6 コンプリメンタリエネルギ関数。

(2,12) 式より.

$$\sigma_x \varepsilon_x = \frac{1}{E}\left[\sigma_x^2 - \nu\sigma_x(\sigma_y + \sigma_z)\right]$$

$$\sigma_y \varepsilon_y = \frac{1}{E}\left[\sigma_y^2 - \nu\sigma_y(\sigma_x + \sigma_z)\right]$$

$$\sigma_z \varepsilon_z = \frac{1}{E}\left[\sigma_z^2 - \nu\sigma_z(\sigma_x + \sigma_y)\right]$$

$$\tau_{yz}\gamma_{yz} = \frac{1}{G}\tau_{yz}^2, \quad \tau_{zx}\gamma_{zx} = \frac{1}{G}\tau_{zx}^2, \quad \tau_{xy}\gamma_{xy} = \frac{1}{G}\tau_{xy}^2$$

$$\therefore B = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \cdots + \tau_{xy}\gamma_{xy})$$

$$= \frac{1}{2}\{\sigma\}^T\{\varepsilon\}$$

$$= \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_y\sigma_z + \sigma_z\sigma_x + \sigma_x\sigma_y)\right] + \frac{1}{2G}(\tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2)$$

$$= \frac{1}{2E}\left[(\sigma_x + \sigma_y + \sigma_z)^2 - 2(1+\nu)(\sigma_y\sigma_z + \sigma_z\sigma_x + \sigma_x\sigma_y) + \frac{E}{G}(\tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2)\right]$$

$$\therefore B = \frac{1}{2E}\left[(\sigma_x + \sigma_y + \sigma_z)^2 + 2(1+\nu)(\tau_{yz}^2 + \tau_{zx}^2 + \tau_{xy}^2 - \sigma_y\sigma_z - \sigma_z\sigma_x - \sigma_x\sigma_y)\right]$$

(2.52)

P30

問題1。

$$\varepsilon_x = \frac{\delta}{l}, \quad \sigma_x = \frac{P}{A_0}$$

$$A = \frac{1}{2}E\varepsilon_x^2 \qquad B = \frac{1}{2E}\sigma_x^2$$

$$A = \frac{1}{2}E\varepsilon_x^2 = \frac{E\delta^2}{2l^2}$$

ここで $\iiint A\,dx\,dy\,dz = \iiint \frac{E\delta^2}{2l^2}\,dx\,dy\,dz$

$$= \frac{E\delta^2}{2l^2}\iiint dx\,dy\,dz = \frac{E\delta^2}{2l^2}\cdot A_0 l = \frac{EA_0}{2l}\delta^2 = U$$

Q.E.D.

また

$$B = \frac{2}{E}\sigma_x^2 = \frac{2P^2}{EA_0^2}$$

$$\therefore \iiint B\,dx\,dy\,dz = \iiint \frac{\cancel{2}P^2}{EA_0^2}\,dx\,dy\,dz = \frac{\cancel{2}P^2}{2EA_0^2}\cdot A_0 l = \frac{\cancel{2}l}{2EA_0}P^2$$

$$= V$$

Q.E.D.

問題2.

(2.29)式

$$E\varepsilon_x = \sigma_x - \nu\sigma_y \qquad G\gamma_{xy} = \tau_{xy}$$
$$E\varepsilon_y = \sigma_y - \nu\sigma_x$$

$$B = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})$$

$$\sigma_x\varepsilon_x = \frac{1}{E}(\sigma_x^2 - \nu\sigma_x\sigma_y)$$

$$\sigma_y\varepsilon_y = \frac{1}{E}(\sigma_y^2 - \nu\sigma_x\sigma_y)$$

$$\tau_{xy}\gamma_{xy} = \frac{1}{G}\tau_{xy}^2$$

$$\therefore B = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y - 2(1+\nu)\sigma_x\sigma_y + \frac{E}{G}\tau_{xy}^2\right)$$
$$= \frac{1}{2E}\left[(\sigma_x + \sigma_y)^2 + 2(1+\nu)(\tau_{xy}^2 - \sigma_x\sigma_y)\right]$$

(2.54)

仮想仕事の原理.

$$-\iiint_V \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \bar{X} \right) \delta u \right.$$

$$+ \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \bar{Y} \right) \delta v$$

$$\left. + \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \bar{Z} \right) \delta w \right] dx\,dy\,dz$$

$$+ \iint_{S_\sigma} \left[(X_\nu - \bar{X}_\nu)\delta u + (Y_\nu - \bar{Y}_\nu)\delta v + (Z_\nu - \bar{Z}_\nu)\delta w \right] dS = 0$$

幾何学的境界条件. $\delta u = 0$, $\delta v = 0$, $\delta w = 0$; S_u上.

$$\iiint_V \frac{\partial \sigma_x}{\partial x} \delta u \, dx\,dy\,dz = \iint_S \sigma_x \cancel{\frac{\partial \delta u}{\partial x}} \delta u \cdot dy \cdot dz - \iiint_V \sigma_x \cdot \frac{\partial \delta u}{\partial x} dx\,dy\,dz$$

$$= \iint_S \sigma_x l \, \delta u \, dS - \iiint_V \sigma_x \frac{\partial \delta u}{\partial x} dx\,dy\,dz$$

$$= \iint_{S_\sigma} \sigma_x l \, \delta u \, dS - \iiint_V \sigma_x \cdot \delta \varepsilon_x \, dx\,dy\,dz$$

同様に

$$\iiint_V \frac{\partial \tau_{xy}}{\partial y} \delta u \, dx\,dy\,dz = \iint_{S_\sigma} \tau_{xy} m \, \delta u \, dS - \iiint_V \tau_{xy} \delta\left(\frac{\partial u}{\partial y}\right) dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{zx}}{\partial z} \delta u \, dx\,dy\,dz = \iint_{S_\sigma} \tau_{zx} \cancel{m}\, n \cdot \delta u \, dS - \iiint_V \tau_{zx} \cdot \delta\left(\frac{\partial u}{\partial z}\right) dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{xy}}{\partial x} \delta v \, dx\,dy\,dz = \iint_{S_\sigma} \tau_{xy} \cdot l \cdot \delta v \, dS - \iiint_V \tau_{xy} \cdot \delta\left(\frac{\partial v}{\partial x}\right) dx\,dy\,dz$$

$$\iiint_V \frac{\partial \sigma_y}{\partial y} \delta v \, dx\,dy\,dz = \iint_{S_\sigma} \sigma_y \cdot m \cdot \delta v \cdot dS - \iiint_V \sigma_y \cdot \delta \varepsilon_y \cdot dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{yz}}{\partial z} \delta v \, dx\,dy\,dz = \iint_{S_\sigma} \tau_{yz} \cdot n \cdot \delta v \cdot dS - \iiint_V \tau_{yz} \cdot \delta\left(\frac{\partial v}{\partial z}\right) dx\,dy\,dz$$

$$\iiint \frac{\partial \tau_{zx}}{\partial x} \delta w \, dx\,dy\,dz = \iint_{S_\sigma} \tau_{zx} \cdot l \cdot \delta w \cdot dS - \iiint_V \tau_{zx} \cdot \delta\left(\frac{\partial w}{\partial x}\right) dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{yz}}{\partial y}\delta w\, dx\,dy\,dz = \iint_{S_\sigma} \tau_{yz}\cdot m\cdot \delta w\cdot dS - \iiint_V \tau_{yz}\cdot \delta\left(\frac{\partial w}{\partial y}\right) dx\,dy\,dz$$

$$\iiint_V \frac{\partial \sigma_z}{\partial z}\delta w\, dx\,dy\,dz = \iint_{S_\sigma} \sigma_z\cdot n\cdot \delta w\cdot dS - \iiint_V \sigma_z\cdot \delta\varepsilon_z\cdot dx\,dy\,dz$$

オ一項の総和 〃

$$\iiint_V \Big[\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + \underbrace{\delta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)}^{\delta\gamma_{yz}}\cdot\tau_{yz} + \tau_{zx}\cdot\underbrace{\delta\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)}^{\delta\gamma_{zx}}$$

$$+ \tau_{xy}\cdot\underbrace{\delta\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)}_{\delta\gamma_{xy}}\Big]\, dx\,dy\,dz$$

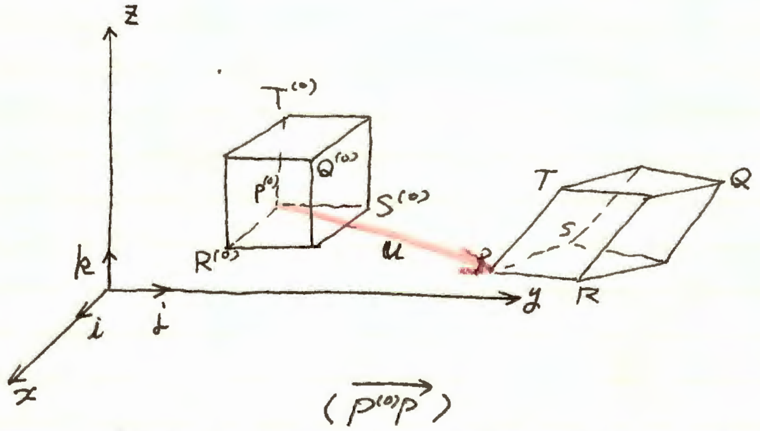
$$-\iint_{S_\sigma}\Big[\underbrace{(\sigma_x l + \tau_{xy} m + \tau_{zx} n)}_{X_\nu}\delta u + \underbrace{(l\tau_{xy} + m\sigma_y + n\tau_{yz})}_{Y_\nu}\delta v$$

$$+ \underbrace{(l\tau_{zx} + m\tau_{yz} + n\sigma_z)}_{Z_\nu}\delta w\Big]\, dS$$

$$\therefore \iiint_V (\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + \tau_{yz}\cdot\delta\gamma_{yz} + \tau_{zx}\cdot\delta\gamma_{zx} + \tau_{xy}\delta\gamma_{xy})\,dx\,dy\,dz$$

$$-\iiint_V (\bar{X}\delta u + \bar{Y}\delta v + \bar{Z}\delta w)\, dV$$

$$-\iint_{S_\sigma} (\bar{X}_\nu\delta u + \bar{Y}_\nu\delta v + \bar{Z}_\nu\delta w)\, dS = 0$$



$(\overrightarrow{P^{(0)}P})$

変位ベクトル　$\mathbb{u} = u\mathbb{i} + v\mathbb{j} + w\mathbb{k}$ ---- (1)

$\overrightarrow{P^{(0)}R^{(0)}} = \overline{PR}\cdot\mathbb{i} = dx\cdot\mathbb{i}$ ----- (2)

$\mathbb{P}^{(0)}$ (位置ベクトル) $= x\mathbb{i} + y\mathbb{j} + z\mathbb{k}$ --- (3)

$\mathbb{P} = (x+u)\mathbb{i} + (y+v)\mathbb{j} + (z+w)\mathbb{k}$ ---- (4)

$\mathbb{R}^{(0)} = \mathbb{P}^{(0)} + \overrightarrow{P^{(0)}R^{(0)}} = (x+dx)\mathbb{i} + y\mathbb{j} + z\mathbb{k}$ ···(5)

今

? $$\overrightarrow{R^{(0)}R} = \Big\{1 + \underbrace{dx\frac{\partial}{\partial x} + \frac{1}{2!}\Big(dx\frac{\partial}{\partial x}\Big)^2 + \cdots}_{du}\Big\}u\mathbb{i}$$
$$+ \Big\{1 + dx\frac{\partial}{\partial x} + \frac{1}{2!}\Big(dx\frac{\partial}{\partial x}\Big)^2 + \cdots\Big\}v\mathbb{j}$$
$$+ \Big\{1 + dx\frac{\partial}{\partial x} + \frac{1}{2!}\Big(dx\frac{\partial}{\partial x}\Big)^2 + \cdots\Big\}w\mathbb{k}$$

☆テーラー展開

より.

$$\mathbb{R} = x + u + du$$

$$\mathbb{R} = (x + dx + u + du)\mathbb{i} + (y + v + dv)\mathbb{j} + (z + w + dw)\mathbb{k}.$$

$$\overrightarrow{PR} = \overrightarrow{P^\circ R^\circ} + \overrightarrow{R^\circ R} - \overrightarrow{P^\circ P}$$

$$= \left\{ dx + u + dx\frac{\partial u}{\partial x} - u \right\} \boldsymbol{i}$$

$$+ \left\{ v + dx\frac{\partial v}{\partial x} - v \right\} \boldsymbol{j}$$

$$+ \left\{ w + dx\frac{\partial w}{\partial x} - w \right\} \boldsymbol{k}$$

$$= \left(dx + dx\frac{\partial u}{\partial x} \right) \boldsymbol{i} + dx\frac{\partial v}{\partial x} \boldsymbol{j} + dx\frac{\partial w}{\partial x} \boldsymbol{k} \qquad — (☆)$$

$$\therefore \overline{PR} = \sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}\ dx$$

また $\overline{P^\circ R^\circ} = dx$

$$\therefore \varepsilon_x = \frac{\sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}\ dx - dx}{dx}$$

$f(1+\gamma_{xx})$

$= f(1) + f'(1)\cdot\gamma_{xx}$

$= 1 + \frac{1}{2}\gamma_{xx} + -\frac{1}{4}\gamma_{xx}^2$

$$= \sqrt{\left(1+\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1$$

$f(x) = x^{\frac{1}{2}}$

$f' = \frac{1}{2}x^{-\frac{1}{2}}$

$f'' = -\frac{1}{4}x^{-\frac{3}{2}}$

$$= \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1$$

ここで $\gamma_{xx} = 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2$ とおくと

二項定理

$$\varepsilon_x = \sqrt{1 + \gamma_{xx}} - 1 \qquad (1+\gamma_{xx})^{\frac{1}{2}}$$

$$\fallingdotseq 1 + \frac{1}{2}\gamma_{xx} - 1 = \frac{1}{2}\gamma_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \quad //$$

$+ \quad \gamma_{xx}^2 \qquad \frac{1}{2} \sim 2$

$$1 + \frac{1}{2}\gamma_{xx} - \frac{1}{4}\gamma_{xx}^2 + \frac{1}{8}\gamma_{xx}^3 - \frac{1}{16}\gamma_{xx}^4 + \frac{1}{32}\gamma_{xx}^5$$

同様にして

$$\varepsilon_y = \frac{\overline{PS} - \overline{P^\circ S^\circ}}{\overline{P^\circ S^\circ}} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial z}\right)^2 = \frac{1}{2}\gamma_{yy}$$

$$\varepsilon_z = \frac{\overline{PT} - \overline{P^\circ T^\circ}}{\overline{P^\circ T^\circ}} = \frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 = \frac{1}{2}\gamma_{zz}$$

(☆)と同様にして.

$$\overrightarrow{PS} = dy\cdot\frac{\partial u}{\partial y}\,i + \left(dy + dy\frac{\partial v}{\partial y}\right)j + dy\frac{\partial w}{\partial y}\,k$$

$$\overrightarrow{PT} = dz\frac{\partial u}{\partial z}\,i + dz\frac{\partial v}{\partial z}\,j + \left(dz + dz\frac{\partial w}{\partial z}\right)k$$

内積 $$\overrightarrow{PS}\cdot\overrightarrow{PT} = \left(\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial z}\right)dy\,dz$$

$$= \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial z}\right)dy\,dz$$

また $|\overrightarrow{PS}| = \sqrt{1+\gamma_{yy}}\,dy$ ま

$|\overrightarrow{PT}| = \sqrt{1+\gamma_{zz}}\,dz$

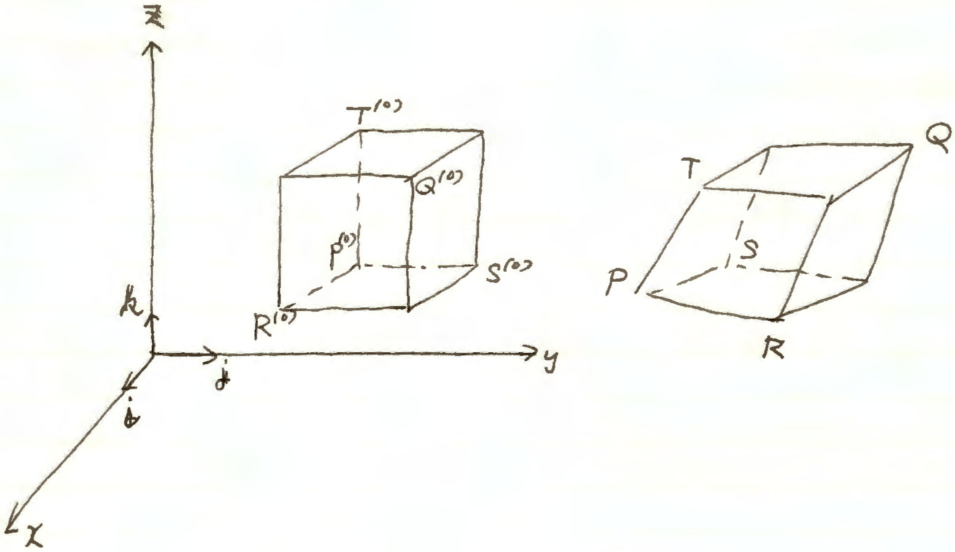
$$\therefore \cos\angle SPT = \frac{\gamma_{yz}}{\sqrt{1+\gamma_{yy}}\sqrt{1+\gamma_{zz}}}.$$

ただし $$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\cdot\frac{\partial w}{\partial z}$$

$\cos\angle SPT \fallingdotseq \frac{\pi}{2} - \angle SPT \Rightarrow$?

$1 \gg \gamma_{yy},\ 1 \gg \gamma_{zz}$

$\cos\angle SPT = \gamma_{yz} = \frac{\pi}{2} - \angle SPT.$



$$\mathbb{P}^{(0)} = x\mathbb{i} + y\mathbb{j} + z\mathbb{k}$$

$$\mathbb{U}_P = u\mathbb{i} + v\mathbb{j} + w\mathbb{k}.$$

$$\mathbb{R}^{(0)} = \mathbb{P}^{(0)} + \overrightarrow{P^{(0)}R^{(0)}} = (x+dx)\mathbb{i} + y\mathbb{j} + z\mathbb{k}.$$

$$\mathbb{P} = \mathbb{P}^{(0)} + \mathbb{U}_P = (x+u)\mathbb{i} + (y+v)\mathbb{j} + (z+w)\mathbb{k}.$$

$$\mathbb{R} = \mathbb{R}^{(0)} + \mathbb{U}_R = (x+dx+u+du)\mathbb{i} + (y+v+dv)\mathbb{j} + (z+w+dw)\mathbb{k}.$$

ただし,

$$\left[\begin{array}{l} du = dx\dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(dx\dfrac{\partial u}{\partial x}\right)^2 + \cdots\cdots \\ dv = dx\dfrac{\partial v}{\partial x} + \dfrac{1}{2}\left(dx\dfrac{\partial v}{\partial x}\right)^2 + \cdots\cdots \\ dw = dx\dfrac{\partial w}{\partial x} + \dfrac{1}{2}\left(dx\dfrac{\partial w}{\partial x}\right)^2 + \cdots\cdots \end{array}\right.$$

$\therefore \overrightarrow{PR} = \mathbb{R} - \mathbb{P}$

$$= (dx + du)\,\mathbb{i} + dv\,\mathbb{j} + dw\,\mathbb{k}.$$

$$\therefore |\overrightarrow{PR}| = \sqrt{(dx+du)^2 + dv^2 + dw^2}$$

$$= \sqrt{dx^2 + 2du\,dx + du^2 + dv^2 + dw^2}$$

$$\lim_{dx\to 0} \frac{|PR| - |P^\circ R^\circ|}{|P^\circ R^\circ|} = \frac{\sqrt{dx^2 + 2du\,dx + du^2 + dv^2 + dw^2} - dx}{dx}$$

$$\lim_{dx\to 0} = \sqrt{1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1$$

$$\left[\begin{array}{l} \dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\cdot dx \left(\dfrac{\partial u}{\partial x}\right)^2 + \dfrac{1}{6}\, dx^2 \left(\dfrac{\partial u}{\partial x}\right)^3 + \cdots\cdots \\[2ex] \dfrac{dv}{dx} = \dfrac{\partial v}{\partial x} + \dfrac{1}{2}\cdot dx \cdot \left(\dfrac{\partial v}{\partial x}\right)^2 + \dfrac{1}{6}\cdot dx^2 \left(\dfrac{\partial v}{\partial x}\right)^3 + \cdots \\[2ex] \dfrac{dw}{dx} = \dfrac{\partial w}{\partial x} + \dfrac{1}{2}\cdot dx \cdot \left(\dfrac{\partial w}{\partial x}\right)^2 + \dfrac{1}{6}\cdot dx^2 \left(\dfrac{\partial w}{\partial x}\right)^3 + \cdots\cdots \end{array}\right.$$

$$\therefore \lim_{dx\to 0} \frac{|PR| - |P^\circ R^\circ|}{|P^\circ R^\circ|} = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1$$

$$\varepsilon_x = \sqrt{1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1$$

$$\varepsilon_x = \sqrt{1 + \gamma_{xx}} - 1$$

$$= (1 + \gamma_{xx})^{\frac{1}{2}} - 1$$

$$= \left(1 + \frac{1}{2}\gamma_{xx} - \frac{1}{8}\gamma_{xx}^2 + \frac{1}{16}\gamma_{xx}^3 - \frac{5}{128}\gamma_{xx}^4 + \cdots\right) - 1$$

$$= \frac{1}{2}\gamma_{xx} - \frac{1}{8}\gamma_{xx}^2 + \frac{1}{16}\gamma_{xx}^3 - \frac{5}{128}\gamma_{xx}^4 + \cdots\cdots$$

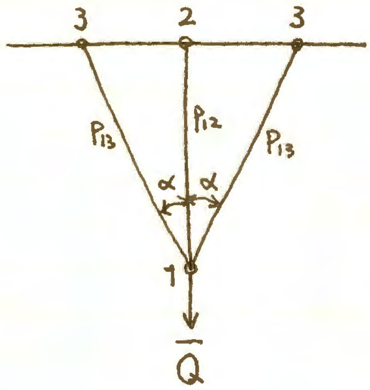
$$= \frac{1}{2}\gamma_{xx} + \frac{1}{2!}\cdot f''(\theta x)\cdot x^n$$

$$= \frac{1}{2}\gamma_{xx} + \frac{1}{2!}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot(1 + \theta\gamma_{xx})^{-\frac{3}{2}}\cdot\gamma_{xx}^2$$

$$= \frac{1}{2}\gamma_{xx} - \frac{1}{8}(1 + \theta\gamma_{xx})^{-\frac{3}{2}}\cdot\gamma_{xx}^2$$

$$= \frac{1}{2}\gamma_{xx} - \frac{1}{8}\cdot\frac{\gamma_{xx}^2}{\sqrt[3]{(1 + \theta\gamma_{xx})^2}} \qquad (0 < \theta < 1)$$

鋼構造演習.



(i) equation of equilibrium (釣合い方程式)

$$P_{12} + 2P_{13}\cos\alpha = \overline{Q} \tag{1.4}$$

(ii) equation of compatibility (変形の適合条件式)

$$\left.\begin{aligned} \delta_{12} &= u_1 \\ \delta_{13} &= u_1 \cos\alpha \end{aligned}\right\} \tag{1.5}$$

u_1を消去すれば.

$$\delta_{13} - \delta_{12}\cos\alpha = 0 \tag{1.6}$$

となる。

(iii) 弾性条件.

$$P_{12} = \frac{A_0 E}{l_{12}}\delta_{12}, \quad P_{13} = \frac{A_0 E}{l_{13}}\delta_{13} \tag{1.7}$$

— 変位法 (displacement method) —

未知節点変位 u_1

(1,5), (1,7) から δ_{12}, δ_{13} を消去すれば、

$$P_{12} = \frac{A_0 E}{l} u_1 \quad , \quad P_{13} = \frac{A_0 E}{l} \cos^2\alpha \cdot u_1 \qquad (1.8)$$

が得られ、これを (1.4) に代入して、

$$\frac{A_0 E}{l} u_1 + 2\frac{A_0 E}{l} \cos^3\alpha \, u_1 = \overline{Q} \qquad (1.9)$$

$$\therefore \ u_1 = \frac{\overline{Q} \cdot l}{A_0 E (1 + 2\cos^3\alpha)} \qquad (1.10)$$

— 応力法 (force method) —

未知部材力 P_{12}, P_{13}

(1.7) より

$$\delta_{12} = \frac{l}{A_0 E} P_{12} \quad , \quad \delta_{13} = \frac{l}{A_0 E \cos\alpha} P_{13} \qquad (1.11)$$

となり、これを (1.6) に代入すれば

$$\frac{P_{13} l}{A_0 E \cos\alpha} - \frac{P_{12} l \cos\alpha}{A_0 E} = 0$$

$$\therefore \ \frac{P_{13} l}{A_0 E \cos^2\alpha} - \frac{P_{12} l}{A_0 E} = 0 \qquad (1.12)$$

(1,4), (1,12) より P_{12}, P_{13} をもとめる。

$$P_{13} - P_{12}\cos^2\alpha = 0 \qquad (1.12)'$$

$$2P_{13}\cos^3\alpha + P_{12}\cos^2\alpha = \bar{Q}\cos^2\alpha \qquad (1.4)'$$

$(1.12)' + (1.4)'$ $\quad (2\cos^3\alpha + 1)P_{13} = \bar{Q}\cos^2\alpha$

$$\left.\begin{aligned} \therefore\ P_{13} &= \frac{\bar{Q}\cos^2\alpha}{2\cos^3\alpha+1} \\ P_{12} &= \frac{P_{13}}{\cos^2\alpha} = \frac{\bar{Q}}{2\cos^3\alpha+1} \end{aligned}\right\} \qquad (1.13)$$

変位法 ⟶ 釣合い方程式を未知節点変位で表わす。

応力法 ⟶ 適合条件式を未知部材力で表わし，釣合い方程式と併せて解く。

ひずみエネルギ と コンプリメンタリエネルギ.

軸力 P と 伸び δ の関係.

$$P = P(\delta) \qquad \text{軸力を伸びの関数とする.} \qquad (1.14)$$

or

$$\delta = \delta(P) \qquad \text{伸びを軸力の関数とする} \qquad (1.15)$$

〜定義〜

ひずみエネルギ $U = \int_0^{\delta} P(\delta)\, d\delta$ (1.16)

コンプリメンタリエネルギ $V = \int_0^{P} \delta(P)\, dP$ (1.17)

$$\frac{\partial U}{\partial \delta} = P \qquad (1.18)$$

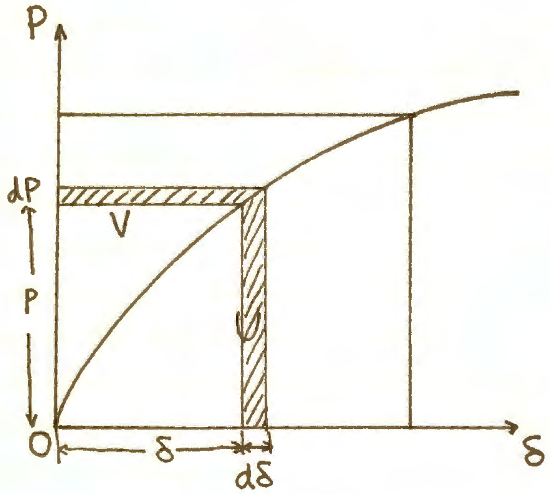
$$\frac{\partial V}{\partial P} = \delta \qquad (1.19)$$

弾性条件より.

$$P = \frac{A_0 E}{l}\delta \quad \text{or} \quad \delta = \frac{l}{A_0 E}P \qquad (1.20)\ (1.21)$$

$$\therefore \quad U = \int_0^{\delta} \frac{A_0 E}{l}\delta\, d\delta = \frac{A_0 E}{2l}\delta^2 \qquad (1.22)$$

$$V = \int_0^P \frac{l}{A_0 E} P\,dP = \frac{l}{2A_0 E} P^2 \qquad (1.23)$$



<Potential>.　位置エネルギ：物体間に作用する遠達力によって生ずる。

↓

相対的位置.

一物体に着目して、これが他の物体の作用のもとに動くと考えられる場合には，後者を座標系の基準として，質点の座標が決まり，質点の位置エネルギーは座標の関数 $U = U(x, y, z)$ として表わされる。これを，その質点に対するPotentialという。

質点が力の場の中で動いたとき，場の力 $\mathbb{F}$ がなした仕事は，位置エネルギの減少に等しい。

dS の変位の結果，ポテンシャルが $U + dU \rightarrow U$ に変ったとすると，

$$-dU = \mathbb{F} \cdot dS$$

$$\therefore \quad F_s = -\frac{\partial U}{\partial s}$$

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$$

x方向分力

1-4 変分原理の応用.

1-4:1 principle of minimum potential energy

節点変位によってひずみエネルギを表わす。　適合条件 弾性条件

$$U_{12} = \frac{A_0 E}{2l}\delta_{12}^2 = \frac{A_0 E}{2l}u_1^2 \qquad (1.25)$$

$$U_{13} = \frac{A_0 E}{2l/\cos\alpha}\cdot\delta_{13}^2 = \frac{A_0 E}{2l}\cos^3\alpha\cdot u_1^2 \qquad (1.26)$$

外力 $\bar{Q}$ のポテンシャルは，質点のなされた仕事に等しい.

$$-\bar{Q}\cdot u_1 \qquad (1.28)$$

Total Potential Energy Π.

$$\Pi = U_{12} + 2U_{13} - \bar{Q}\cdot u_1$$

$$= \frac{A_0 E}{2l}u_1^2 + 2\times\frac{A_0 E}{2l}\cos^3\alpha\cdot u_1^2 - \bar{Q}\cdot u_1 \qquad (1.29)$$

"Πを最小にするu_1が正解を与える" — 停留条件.

$$\frac{\partial\Pi}{\partial u_1} = \frac{A_0 E}{l}u_1 + 2\frac{A_0 E}{l}\cos^3\alpha\, u_1 - \bar{Q} = 0 \qquad (1.30)$$

(1.30)は，式(1.4)すなわち釣り合い条件式を，適合条件式(1.5)弾性条件式(1.7)で表わした形になっている。すなわち停留条件は，釣合い方程式と等価である。

テーラーの定理.

$$\Delta z = f(x+dx, y+dy) - f(x, y)$$

$$= \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right) f(x, y)$$

$$+ \frac{1}{2!}\left(dx\cdot\frac{\partial}{\partial x} + dy\cdot\frac{\partial}{\partial y}\right)^2 f(x, y)$$

$$+ \cdots\cdots$$

$$+ \frac{1}{r!}\left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^r f(x, y)$$

$$+ \cdots\cdots$$

$$+ \frac{1}{(n-1)!}\left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^{(n-1)} f(x, y)$$

$$+ \frac{1}{n!}\left(dx\cdot\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^n f(x+\theta dx, y+\theta dy)$$

$$(0 < \theta < 1)$$

マクローリン

$$f(x) = f(0) + f'(0)x + \cdots + \frac{1}{n!} f^{(n)}(0)\, x^n + \cdots\cdots$$

問題.

$$\Pi_I = \frac{1}{2}\frac{A_0E}{l_{12}}\delta_{12}^2 + 2\times\frac{1}{2}\frac{A_0E}{l_{13}}\delta_{13}^2 - \bar{Q}u_1$$
$$-\alpha(\delta_{12}-u_1) - 2\beta(\delta_{13}-u_1\cos\alpha) \qquad (1.31)$$

停留条件.

$$\frac{\partial\Pi_I}{\partial\delta_{12}} = \frac{A_0E}{l_{12}}\delta_{12} - \alpha = 0 \qquad (1.32a)$$

$$\frac{\partial\Pi_I}{\partial\delta_{13}} = \frac{A_0E}{l_{13}}\delta_{13} - \beta = 0 \qquad (1.32b)$$

$$\frac{\partial\Pi_I}{\partial u_1} = \alpha + 2\beta\cos\alpha - \bar{Q} = 0 \qquad (1.32c)$$

$$\frac{\partial\Pi_I}{\partial\alpha} = \delta_{12} - u_1 = 0 \qquad (1.32d)$$

$$\frac{\partial\Pi_I}{\partial\beta} = \delta_{13} - u_1\cos\alpha = 0 \qquad (1.32e)$$

$$\alpha = \frac{A_0E}{l_{12}}\delta_{12} = P_{12}$$

$$\beta = \frac{A_0E}{l_{13}}\delta_{13} = P_{13}$$

$$\therefore\ \Pi_I = \Pi_{II} = \frac{1}{2}\frac{A_0E}{l_{12}}\delta_{12}^2 + 2\cdot\frac{1}{2}\frac{A_0E}{l_{13}}\delta_{13}^2 - \bar{Q}u_1$$
$$- P_{12}(\delta_{12}-u_1) - 2P_{13}(\delta_{13}-u_1\cos\alpha)$$

$$\delta_{12} = \frac{l_{12}}{A_0 E} P_{12} \quad , \quad \delta_{13} = \frac{l_{13}}{A_0 E} P_{13}$$

とすれば

$$\Pi_{II} = \frac{1}{2} \cdot \frac{A_0 E}{l_{12}} \cdot \left(\frac{l_{12}}{A_0 E} P_{12} \right)^2 + 2 \times \frac{1}{2} \frac{A_0 E}{l_{13}} \left(\frac{l_{13}}{A_0 E} P_{13} \right)^2 - \bar{Q} u_1$$

$$- P_{12} \left(\frac{l_{12}}{A_0 E} P_{12} - u_1 \right) - 2 P_{13} \left(\frac{l_{13}}{A_0 E} P_{13} - u_1 \cos\alpha \right)$$

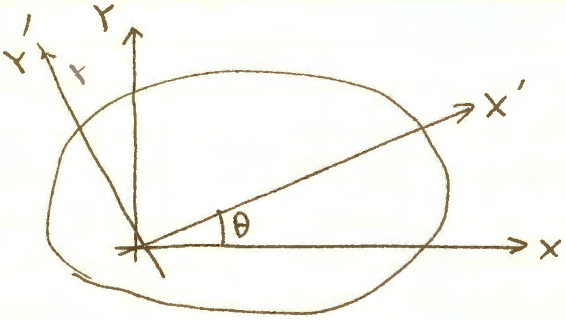
$$= \frac{1}{2} \frac{l_{12}}{A_0 E} P_{12}^2 - \frac{l_{12}}{A_0 E} P_{12}^2$$

$$+ \frac{l_{13}}{A_0 E} P_{13}^2 - 2 \frac{l_{13}}{A_0 E} P_{13}^2$$

$$- \bar{Q} u_1 + P_{12} u_1 + 2 P_{13} u_1 \cos\alpha$$

$$\therefore \Pi_{II} = -\frac{1}{2} \cdot \frac{l_{12}}{A_0 E} P_{12}^2 - 2 \times \frac{1}{2} \cdot \frac{l_{13}}{A_0 E} P_{13}^2$$

$$+ u_1 \left(P_{12} + 2 P_{13} \cos\alpha - \bar{Q} \right)$$



$$x' = x\cos\theta + y\sin\theta .$$
$$y' = y\cos\theta - x\sin\theta .$$

$$I_{x'} = \int_A (y\cos\theta - x\sin\theta)^2 dA$$

$$I_{y'} = \int_A (x\cos\theta + y\sin\theta)^2 dA$$

$$I_{x'y'} = \int_A (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta) dA .$$

$$\int_A y^2 dA = I_x , \quad \int_A x^2 dA = I_y \quad \int_A xy\, dA = I_{xy}$$

とすると.

主軸まわりの断面二次モーメント

$$\begin{cases} I_{x'} = \cos^2\theta\, I_x + \sin^2\theta\, I_y - 2\cos\theta\sin\theta\, I_{xy} \\ \quad = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y)\cos 2\theta - I_{xy}\sin 2\theta \\ I_{y'} = \cos^2\theta\, I_y + \sin^2\theta\, I_x + 2\cos\theta\sin\theta\, I_{xy} \\ \quad = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta + I_{xy}\sin 2\theta \end{cases}$$

$$I_{x'y'} = (I_x - I_y)\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta)$$
$$= \frac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy}\cos 2\theta \quad \text{—— ①}$$

ここで $x'-y'$ を主軸とすると $I_{x'y'} = 0$

①に代入して $\tan 2\theta = -\dfrac{2I_{xy}}{I_x - I_y}$ //

主軸

$$M_1 = M_x \cos\alpha + M_y \sin\alpha \quad \text{——} ①$$
$$M_2 = -M_x \sin\alpha + M_y \cos\alpha. \quad \text{——} ②$$

$$I_1 = I_x \cos^2\alpha + I_y \sin^2\alpha - 2I_{xy} \sin\alpha \cos\alpha \quad \text{——} ③$$
$$I_2 = I_x \sin^2\alpha + I_y \cos^2\alpha + 2I_{xy} \sin\alpha \cos\alpha \quad \text{——} ④$$

$$x' = x\cos\alpha + y\sin\alpha \quad \text{——} ⑤$$
$$y' = -x\sin\alpha + y\cos\alpha \quad \text{——} ⑥$$

$$\sigma_b = \frac{M_1 y'}{I_1} + \frac{M_2 x'}{I_2}$$

$$\frac{M_1 y'}{I_1} = \frac{(M_x\cos\alpha + M_y\sin\alpha)(-x\sin\alpha + y\cos\alpha)}{I_x\cos^2\alpha + I_y\sin^2\alpha - 2I_{xy}\sin\alpha\cos\alpha}$$
$$= \frac{-M_x x\sin\alpha\cos\alpha - M_y x\sin^2\alpha + M_x y\cos^2\alpha + M_y y\sin\alpha\cos\alpha}{I_x\cos^2\alpha + I_y\sin^2\alpha - 2I_{xy}\sin\alpha\cos\alpha}$$

$$\frac{M_2 x'}{I_2} = \frac{M_x x\sin^2\alpha - M_y x\sin\alpha\cos\alpha - M_x y\sin\alpha\cos\alpha + M_y y\cos^2\alpha}{I_x \sin^2\alpha + I_y\cos^2\alpha + 2I_{xy}\sin\alpha\cos\alpha}$$

$$\int y\,dA = \int_{-\frac{h}{2}}^{\frac{h}{2}} y \cdot b\,dy = b\cdot\frac{1}{2}\left(\cancel{\frac{h^2}{4}} - \right|$$

(1) *principal of virtual work.* — 釣合い方程式 / S_σ上の力学的境界条件) と等価

$$\iiint_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} + \tau_{xy}\delta\gamma_{xy})\,dV$$

$$- \iiint_V (\bar{X}\delta u + \bar{Y}\delta v + \bar{Z}\delta w)\,dV - \iint_{S_\sigma} (\bar{X}_\nu \delta u + \bar{Y}_\nu \delta v + \bar{Z}_\nu \delta w)\,dS = 0$$

付帯条件.

(1) 幾何学的境界条件.

$$\delta u = 0,\quad \delta v = 0,\quad \delta w = 0 \qquad ;\ S_u\text{上}.$$

(2) 適合条件.

$$\delta\varepsilon_x = \delta\left(\frac{\partial u}{\partial x}\right) = \frac{\partial \delta u}{\partial x}$$

$$\delta\varepsilon_y = \delta\left(\frac{\partial v}{\partial y}\right) = \frac{\partial \delta v}{\partial y}$$

$$\delta\varepsilon_z = \delta\left(\frac{\partial w}{\partial z}\right) = \frac{\partial \delta w}{\partial z}$$

$$\delta\gamma_{yz} = \delta\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) = \frac{\partial \delta w}{\partial y} + \frac{\partial \delta v}{\partial z}$$

$$\delta\gamma_{zx} = \delta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{\partial \delta u}{\partial z} + \frac{\partial \delta w}{\partial x}$$

$$\delta\gamma_{xy} = \delta\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{\partial \delta v}{\partial x} + \frac{\partial \delta u}{\partial y}$$

釣合い条件.

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \bar{X} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\tau_{yz}}{\partial z} + \bar{Y} = 0 \qquad (1)$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \bar{Z} = 0$$

力学的境界条件.

$$X_\nu - \overline{X_\nu} = 0 \ , \ Y_\nu - \overline{Y_\nu} = 0 \ , \ Z_\nu - \overline{Z_\nu} = 0 \qquad ; S_\sigma 上. \quad (2)$$

(1), (2) より.

$$-\iiint_V \Big[\Big(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \bar{X} \Big) \delta u + \Big(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \bar{Y} \Big) \delta v$$
$$+ \Big(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \bar{Z} \Big) \delta w \Big] dx\,dy\,dz$$
$$+ \iint_{S_\sigma} \left[(X_\nu - \overline{X_\nu}) \delta u + (Y_\nu - \overline{Y_\nu}) \delta v + (Z_\nu - \overline{Z_\nu}) \delta w \right] dS = 0 \qquad (3)$$

ただし S_u 上で $\delta u = 0$, $\delta v = 0$, $\delta w = 0$.

(Gauss の発散定理 参照)

$$dy\,dz = \pm l\,dS \qquad dz\,dx = \pm m\,dS. \qquad dx\,dy = \pm n\,dS$$

$$(l = \cos(x,\nu), \quad m = \cos(y,\nu), \quad n = \cos(z,\nu))$$

部分積分.

$$\iiint_V \frac{\partial \sigma_x}{\partial x}\delta u\,dx\,dy\,dz = \iint_{S_\sigma} \sigma_x l\,\delta u\,dS - \iiint_V \sigma_x \frac{\partial \delta u}{\partial x}\,dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{xy}}{\partial y}\delta u\,dx\,dy\,dz = \iint_{S_\sigma} \tau_{xy} m\,\delta u\,dS - \iiint_V \tau_{xy} \frac{\partial \delta u}{\partial y}\,dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{zx}}{\partial z}\delta u\,dx\,dy\,dz = \iint_{S_\sigma} \tau_{zx} n\,\delta u\,dS - \iiint_V \tau_{zx} \frac{\partial \delta u}{\partial z}\,dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{xy}}{\partial x}\delta v\,dx\,dy\,dz = \iint_{S_\sigma} \tau_{xy} l\,\delta v\,dS - \iiint_V \tau_{xy} \frac{\partial \delta v}{\partial x}\,dx\,dy\,dz$$

$$\iiint_V \frac{\partial \sigma_y}{\partial y}\delta v\,dx\,dy\,dz = \iint_{S_\sigma} \sigma_y m\,\delta v\,dS - \iiint_V \sigma_y \frac{\partial \delta v}{\partial y}\,dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{yz}}{\partial z}\delta v\,dx\,dy\,dz = \iint_{S_\sigma} \tau_{yz} n\,\delta v\,dS - \iiint_V \tau_{yz} \frac{\partial \delta v}{\partial z}\,dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{zx}}{\partial x}\delta w\,dx\,dy\,dz = \iint_{S_\sigma} \tau_{zx} l\,\delta w\,dS - \iiint_V \tau_{zx} \frac{\partial \delta w}{\partial x}\,dx\,dy\,dz$$

$$\iiint_V \frac{\partial \tau_{yz}}{\partial y}\delta w\,dx\,dy\,dz = \iint_{S_\sigma} \tau_{yz} m\,\delta w\,dS - \iiint_V \tau_{yz} \frac{\partial \delta w}{\partial y}\,dx\,dy\,dz$$

$$\iiint_V \frac{\partial \sigma_z}{\partial z}\delta w\,dx\,dy\,dz = \iint_{S_\sigma} \sigma_z n\,\delta w\,dS - \iiint_V \sigma_z \frac{\partial \delta w}{\partial z}\,dx\,dy\,dz$$

+)

×(−1)

$$\iiint_V (\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} + \tau_{xy}\delta\gamma_{xy})\,dV$$

$$-\iint_{S_\sigma}\Big[(\sigma_x l + \tau_{xy} m + \tau_{zx} n)\delta u + (\tau_{xy} l + \sigma_y m + \tau_{yz} n)\delta v$$

$$+ (\tau_{zx} l + \tau_{yz} m + \sigma_z n)\delta w\Big]\,dS$$

よって 式(3)は.

$$\iiint_V (\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \cdots\cdots + \tau_{xy} \delta \gamma_{xy}) dV$$

$$- \iiint_V (\bar{X} \delta u + \bar{Y} \delta v + \bar{Z} \delta w) dV$$

$$+ \iint_{S_\sigma} \left[(X_\nu - \sigma_x l - \tau_{xy} m - \tau_{zx} n) \delta u + (Y_\nu - \tau_{xy} l - \sigma_y m - \tau_{yz} n) \delta v \right.$$

$$\left. + (Z_\nu - \tau_{zx} l - \tau_{yz} m - \sigma_z n) \delta w \right] dS$$

$$- \iint_{S_\sigma} (\bar{X}_\nu \delta u + \bar{Y}_\nu \delta v + \bar{Z}_\nu \delta w) dS = 0$$

$$\therefore \iiint_V (\sigma_x \delta \varepsilon_x + \cdots\cdots + \tau_{xy} \delta \gamma_{xy}) dV$$

$$- \iiint_V (\bar{X} \delta u + \bar{Y} \delta v + \bar{Z} \delta w) dV$$

$$- \iint_{S_\sigma} (\bar{X}_\nu \delta u + \bar{Y}_\nu \delta v + \bar{Z}_\nu \delta w) dS = 0$$

仮想変位 δu, δv, δw の間になす応力と体積力のなす仮想仕事

(1) $P^{(0)}S^{(0)}T^{(0)}$ 及び対面

$$-\sigma_x \delta u\, dy\, dz - \tau_{xy}\delta v\, dy\, dz - \tau_{xz}\delta w\, dy\, dz$$

$$+\left(\sigma_x+\frac{\partial \sigma_x}{\partial x}dx\right)\left(\delta u+\frac{\partial \delta u}{\partial x}dx\right)dy\, dz+\left(\tau_{xy}+\frac{\partial \tau_{xy}}{\partial x}dx\right)\left(\delta v+\frac{\partial \delta v}{\partial x}dx\right)dy\, dz$$

$$+\left(\tau_{xz}+\frac{\partial \tau_{xz}}{\partial x}dx\right)\left(\delta w+\frac{\partial \delta w}{\partial x}dx\right)dy\, dz$$

$$=\left(\sigma_x\frac{\partial \delta u}{\partial x}+\tau_{xy}\frac{\partial \delta v}{\partial x}+\tau_{xz}\frac{\partial \delta w}{\partial x}\right)dx\, dy\, dz$$

$$+\left(\frac{\partial \sigma_x}{\partial x}\delta u+\frac{\partial \tau_{xy}}{\partial x}\delta v+\frac{\partial \tau_{xz}}{\partial x}\delta w\right)dx\, dy\, dz$$

$$+\left(\frac{\partial \sigma_x}{\partial x}\cdot\frac{\partial \delta u}{\partial x}+\frac{\partial \tau_{xy}}{\partial x}\cdot\frac{\partial \delta v}{\partial x}+\frac{\partial \tau_{xz}}{\partial x}\cdot\frac{\partial \delta w}{\partial x}\right)dx^2 dy\, dz$$

(2) $P^{(0)}R^{(0)}T^{(0)}$ 及び対面.

$$-\tau_{yx}\cdot\delta u\, dx\, dz-\sigma_y\delta v\, dx\, dz-\tau_{yz}\delta w\, dx\, dz$$

$$+\left(\tau_{yx}+\frac{\partial \tau_{yx}}{\partial y}dy\right)\left(\delta u+\frac{\partial \delta u}{\partial y}dy\right)dx\, dz+\left(\sigma_y+\frac{\partial \sigma_y}{\partial y}dy\right)\left(\delta v+\frac{\partial \delta v}{\partial y}dy\right)dx\, dz$$

$$+\left(\tau_{yz}+\frac{\partial \tau_{yz}}{\partial y}dy\right)\left(\delta w+\frac{\partial \delta w}{\partial y}dy\right)dx\, dz$$

$$=\left(\tau_{xy}\frac{\partial \delta u}{\partial y}+\sigma_y\frac{\partial \delta v}{\partial y}+\tau_{yz}\frac{\partial \delta w}{\partial y}\right)dx\, dy\, dz$$

$$+\left(\frac{\partial \tau_{xy}}{\partial y}\delta u+\frac{\partial \sigma_y}{\partial y}\delta v+\frac{\partial \tau_{yz}}{\partial y}\delta w\right)dx\, dy\, dz$$

$$+\left(\frac{\partial \tau_{xy}}{\partial y}\cdot\frac{\partial \delta u}{\partial y}+\frac{\partial \sigma_y}{\partial y}\frac{\partial \delta v}{\partial y}+\frac{\partial \tau_{yz}}{\partial y}\cdot\frac{\partial \delta w}{\partial y}\right)dx\, dy^2 dz$$

(3) $P^{(0)}R^{(0)}S^{(0)}$ Bび 対面.

$$-\tau_{zx}\,\delta u\,dx\,dy - \tau_{zy}\,\delta v\,dx\,dy - \sigma_z\,\delta w\,dx\,dy$$

$$+\left(\tau_{zx}+\frac{\partial \tau_{zx}}{\partial z}dz\right)\left(\delta u+\frac{\partial \delta u}{\partial z}dz\right)dx\,dy+\left(\tau_{yz}+\frac{\partial \tau_{yz}}{\partial z}dz\right)dx\,dy\left(\delta v+\frac{\partial \delta v}{\partial z}dz\right)$$

$$+\left(\sigma_z+\frac{\partial \sigma_z}{\partial z}dz\right)\left(\delta w+\frac{\partial \delta w}{\partial z}dz\right)dx\,dy$$

$$=\left(\tau_{zx}\frac{\partial \delta u}{\partial z}+\tau_{yz}\frac{\partial \delta v}{\partial z}+\sigma_z\frac{\partial \delta w}{\partial z}\right)dx\,dy\,dz$$

$$+\left(\frac{\partial \tau_{zx}}{\partial z}\delta u+\frac{\partial \tau_{yz}}{\partial z}\delta v+\frac{\partial \sigma_z}{\partial z}\delta w\right)dx\,dy\,dz$$

$$+\left(\frac{\partial \tau_{zx}}{\partial z}\cdot\frac{\partial \delta u}{\partial z}+\frac{\partial \tau_{yz}}{\partial z}\cdot\frac{\partial \delta v}{\partial z}+\frac{\partial \sigma_z}{\partial z}\cdot\frac{\partial \delta w}{\partial z}\right)dx\,dy\,dz^2$$

~~総和~~. 体積力による仕事.

$$(\bar{X}\,\delta u+\bar{Y}\,\delta v+\bar{Z}\,\delta w)\,dx\,dy\,dz$$

総和.

$$\Bigg[\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+\bar{X}\right)\delta u+\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+\bar{Y}\right)\delta v$$

$$+\left(\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+\bar{Z}\right)\delta w$$

$$+\sigma_x\frac{\partial \delta u}{\partial x}+\sigma_y\frac{\partial \delta v}{\partial y}+\sigma_z\frac{\partial \delta w}{\partial z}+\tau_{yz}\left(\frac{\partial \delta w}{\partial y}+\frac{\partial \delta v}{\partial z}\right)+\tau_{zx}\left(\frac{\partial \delta u}{\partial z}+\frac{\partial \delta w}{\partial x}\right)$$

$$+\tau_{xy}\left(\frac{\partial \delta v}{\partial x}+\frac{\partial \delta u}{\partial y}\right)\Bigg]dx\,dy\,dz$$

$$=(\sigma_x\,\delta\varepsilon_x+\sigma_y\,\delta\varepsilon_y+\sigma_z\,\delta\varepsilon_z+\tau_{yz}\,\delta\gamma_{yz}+\tau_{zx}\,\delta\gamma_{zx}+\tau_{xy}\,\delta\gamma_{xy})\,dx\,dy\,dz$$

体積力のなす仮想仕事.

$$\iiint_V (\bar{X}\delta u + \bar{Y}\delta v + \bar{Z}\delta w)\,dV$$

応力のなす仮想仕事は、相となる微小六面体間で相殺され、$\delta u, \delta v, \delta w$ は S_u 上で 0 であるから、

$$\iint_{S_\sigma} (\bar{X}_\nu \delta u + \bar{Y}_\nu \delta v + \bar{Z}_\nu \delta w)\,dS$$

となる。

$$\therefore \iiint_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} + \tau_{xy}\delta\gamma_{xy})\,dV$$

$$= \iiint_V (\bar{X}\delta u + \bar{Y}\delta v + \bar{Z}\delta w)\,dV + \iint_{S_\sigma} (\bar{X}_\nu \delta u + \bar{Y}_\nu \delta v + \bar{Z}_\nu \delta w)\,dS$$

(2) *principle of complementary virtual work.*

— 適合条件 + S_u上の幾何学的境界条件と等価.

$$\iiint_V (\varepsilon_x \delta\sigma_x + \varepsilon_y \delta\sigma_y + \varepsilon_z \delta\sigma_z + \gamma_{yz}\delta\tau_{yz} + \gamma_{zx}\delta\tau_{zx} + \gamma_{xy}\delta\tau_{xy})dV$$

$$-\iint_{S_u} (\bar{u}\,\delta X_\nu + \bar{v}\,\delta Y_\nu + \bar{w}\,\delta Z_\nu)\,dS = 0$$

付帯条件.

(1) 釣合い条件.

$$\frac{\partial \delta\sigma_x}{\partial x} + \frac{\partial \delta\tau_{xy}}{\partial y} + \frac{\partial \delta\tau_{zx}}{\partial z} = 0$$

$$\frac{\partial \delta\tau_{xy}}{\partial x} + \frac{\partial \delta\sigma_y}{\partial y} + \frac{\partial \delta\tau_{yz}}{\partial z} = 0$$

$$\frac{\partial \delta\tau_{zx}}{\partial x} + \frac{\partial \delta\tau_{yz}}{\partial y} + \frac{\partial \delta\sigma_z}{\partial z} = 0$$

(2) S_σ上の力学的境界条件.

$$\delta X_\nu = 0, \quad \delta Y_\nu = 0, \quad \delta Z_\nu = 0 \quad ; \; S_\sigma\text{上}.$$

(i) 変位の適合条件.

$$\varepsilon_x - \frac{\partial u}{\partial x} = 0$$

$$\varepsilon_y - \frac{\partial v}{\partial y} = 0$$

$$\varepsilon_z - \frac{\partial w}{\partial z} = 0$$

$$\gamma_{yz} - \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0$$

$$\gamma_{zx} - \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

$$\gamma_{xy} - \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad ; \quad V\text{内}$$

(ii) S_u上の幾何学的境界条件.

$$u - \bar{u} = 0 \quad , \quad v - \bar{v} = 0 \quad , \quad w - \bar{w} = 0 \qquad ; \quad S_u\text{上}$$

微小な応力変分 $\delta\sigma_{ij}$

(i), (ii) より.

$$\iiint_V \Big[\left(\varepsilon_x - \frac{\partial u}{\partial x}\right)\delta\sigma_x + \left(\varepsilon_y - \frac{\partial v}{\partial y}\right)\delta\sigma_y + \left(\varepsilon_z - \frac{\partial w}{\partial z}\right)\delta\sigma_z$$

$$+ \left(\gamma_{yz} - \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\delta\tau_{yz} + \left(\gamma_{zx} - \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\delta\tau_{zx}$$

$$+ \left(\gamma_{xy} - \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\delta\tau_{xy} \Big] dx\,dy\,dz$$

$$+ \iint_{S_u} \left[(u - \bar{u})\delta X_\nu + (v - \bar{v})\delta Y_\nu + (w - \bar{w})\delta Z_\nu \right] dS = 0 \qquad \text{(iii)}$$

部分積分.

$$\iiint_V \frac{\partial u}{\partial x}\delta\sigma_x \, dx\,dy\,dz = \iint_S u\,\delta\sigma_x\, l\, dS - \iiint_V u\frac{\partial \delta\sigma_x}{\partial x} dx\,dy\,dz$$

$$\iiint_V \frac{\partial v}{\partial y}\delta\sigma_y \, dx\,dy\,dz = \iint_S v\,\delta\sigma_y\, m\, dS - \iiint_V v\frac{\partial \delta\sigma_y}{\partial y} dx\,dy\,dz$$

$$\iiint_V \frac{\partial w}{\partial z}\delta\sigma_z \, dx\,dy\,dz = \iint_S w\,\delta\sigma_z\, n\, dS - \iiint_V w\frac{\partial \delta\sigma_z}{\partial z} dx\,dy\,dz$$

$$\iiint_V \frac{\partial w}{\partial y}\delta\tau_{yz} \, dx\,dy\,dz = \iint_S w\,\delta\tau_{yz}\, m\, dS - \iiint_V w\frac{\partial \delta\tau_{yz}}{\partial y} dx\,dy\,dz$$

$$\iiint_V \frac{\partial v}{\partial z}\delta\tau_{yz} \, dx\,dy\,dz = \iint_S v\,\delta\tau_{yz}\, n\, dS - \iiint_V v\frac{\partial \delta\tau_{yz}}{\partial z} dx\,dy\,dz$$

$$\iiint_V \frac{\partial u}{\partial z}\delta\tau_{zx} \, dx\,dy\,dz = \iint_S u\,\delta\tau_{zx}\, n\, dS - \iiint_V u\frac{\partial \delta\tau_{zx}}{\partial z} dx\,dy\,dz$$

$$\iiint_V \frac{\partial w}{\partial x}\delta\tau_{zx} \, dx\,dy\,dz = \iint_S w\,\delta\tau_{zx}\, l\, dS - \iiint_V w\frac{\partial \delta\tau_{zx}}{\partial x} dx\,dy\,dz$$

$$\iiint_V \frac{\partial v}{\partial x}\delta\tau_{xy} \, dx\,dy\,dz = \iint_S v\,\delta\tau_{xy}\, l\, dS - \iiint_V v\frac{\partial \delta\tau_{xy}}{\partial x} dx\,dy\,dz$$

$$\iiint_V \frac{\partial u}{\partial y}\delta\tau_{xy} \, dx\,dy\,dz = \iint_S u\,\delta\tau_{xy}\, m\, dS - \iiint_V u\frac{\partial \delta\tau_{xy}}{\partial y} dx\,dy\,dz$$

+)

$\times(-1)$

(iii) は、次のようになる。

$$\iiint_V (\varepsilon_x\delta\sigma_x + \varepsilon_y\delta\sigma_y + \varepsilon_z\delta\sigma_z + \gamma_{yz}\delta\tau_{yz} + \gamma_{zx}\delta\tau_{zx} + \gamma_{xy}\delta\tau_{xy})\, dV$$

$$+\iiint_V \Big[\Big(\frac{\partial\delta\sigma_x}{\partial x} + \frac{\partial\delta\tau_{xy}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}\Big)u + \Big(\frac{\partial\delta\tau_{xy}}{\partial x} + \frac{\partial\delta\sigma_y}{\partial y} + \frac{\partial\delta\tau_{yz}}{\partial z}\Big)v$$

$$+\Big(\frac{\partial\delta\tau_{zx}}{\partial x} + \frac{\partial\delta_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z}\Big)w\Big]\, dV$$

$$-\iint_{S_u\,S}\Big[\underbrace{(\delta\sigma_x l + \delta\tau_{xy} m + \delta\tau_{zx} n)}_{\delta X_\nu}u + \underbrace{(\delta\tau_{xy} + \delta\sigma_y + \delta\tau_{yz})}_{\delta Y_\nu}v$$

why

S_σ上で零

$$+\underbrace{(\delta\tau_{zx} l + \delta\tau_{yz} m + \delta\sigma_z\cdot n)}_{\delta Z_\nu}w\Big]\, dS + \iint_{S_u}\big[(u-\bar{u})\delta X_\nu + (v-\bar{v})\delta Y_\nu$$

$$+(w-\bar{w})\delta Z_\nu\big]\, dS$$

(3) *principle of minimum potential energy.*

ひずみエネルギ関数.

$$A(\varepsilon_x, \varepsilon_y, \cdots, \gamma_{xy}) \longrightarrow A(u, v, w)$$

〈適合条件により〉

potential energy.

$$\Pi_p = \iiint_V [A(u,v,w) - (\bar{X}u + \bar{Y}v + \bar{Z}w)]\,dx\,dy\,dz$$

$$- \iint_{S_\sigma} (\bar{X}_\nu u + \bar{Y}_\nu v + \bar{Z}_\nu w)\,dS.$$

$$\delta \Pi_p = 0$$

付帯条件.

$$u = \bar{u}, \quad v = \bar{v}, \quad w = \bar{w} \quad ; \quad S_u\text{上}.$$

仮想仕事の原理から最小ポテンシャルエネルギの原理を導く。

仮想仕事の原理.

$$\underbrace{\iiint_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} + \tau_{xy}\delta\gamma_{xy})\,dV}_{\delta A}$$

$$-\iiint_V (\bar{X}\delta u + \bar{Y}\delta v + \bar{Z}\delta w)\,dV$$

$$-\iint_{S_\sigma} (\bar{X}_\nu \delta u + \bar{Y}_\nu \delta v + \bar{Z}_\nu \delta w)\,dS = 0 \qquad \text{①}$$

ひずみエネルギ関数. $A(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{zx}, \gamma_{xy})$

適合条件

$$\rightarrow A(u, v, w)$$

$$\therefore \iiint_V \delta A(u,v,w)\,dV - \iiint_V (\bar{X}\delta u + \bar{Y}\delta v + \bar{Z}\delta w)\,dV$$

$$-\iint_{S_\sigma} (\bar{X}_\nu \delta u + \bar{Y}_\nu \delta v + \bar{Z}_\nu \delta w)\,dS = 0$$

ここで
$$-\delta\overset{\text{ファイ}}{\Phi} = \bar{X}\delta u + \bar{Y}\delta v + \bar{Z}\delta w$$
$$-\delta\overset{\text{プサイ}}{\Psi} = \bar{X}_\nu \delta u + \bar{Y}_\nu \delta v + \bar{Z}_\nu \delta w$$

とおくと.

$$\iiint_V \delta A\,dV + \iiint_V \delta\Phi\,dV + \iint_{S_\sigma} \delta\Psi\,dS = 0$$

$$\therefore \quad \delta\Pi = 0$$

ただし
$$\Pi = \underbrace{\iiint_V A\,dV}_{\text{ひずみエネルギー}} + \underbrace{\iiint_V \Phi\,dV}_{\text{体積力のポテンシャル}} + \underbrace{\iint_{S_\sigma} \Psi\,dS}_{\text{表面力のポテンシャル}}$$

$\delta \Pi = 0$ --->

許容

among all the admissible displacements u, v and w

許される変位.

which satisfy the prescribed geometrical boundary

規定された幾何学的境界条件.

conditions, the actual displacements make the total potential energy stationary.

$\Downarrow$

Among all the admissible displacement functions, the actual displacements make the total potential energy an absolute minimum.

正解の変位 u, v, w. [正解の応力 $\sigma_x, \sigma_y, \cdots, \tau_{xy}$ とする]

$$\Pi(u^*, v^*, w^*) = \Pi(u+\delta u, v+\delta v, w+\delta w)$$

$$= \Pi(u, v, w) + \delta\Pi + \delta^2\Pi + \cdots\cdots$$

$$\Pi = \iiint_V \left[\sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \sigma_z \frac{\partial w}{\partial z} + \tau_{yz}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right.$$

$$\left. + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] dV - \iiint_V (\bar{X}u + \bar{Y}v + \bar{Z}w)\, dV$$

$$- \iint_{S_\sigma} (\bar{X}_\nu u + \bar{Y}_\nu v + \bar{Z}_\nu w)\, dS$$

$$\delta\Pi = \frac{\partial \Pi}{\partial u}\delta u + \frac{\partial \Pi}{\partial v}\delta v + \frac{\partial \Pi}{\partial w}\delta w$$?

$$= \iiint_V \left[\frac{\partial \sigma_x}{\partial u} \cdot \frac{\partial u}{\partial x} \right.$$

$$\delta^2 A + \delta A = A(u+du,\ v+dv,\ w+dw) - A(u, v, w)$$

とすれば,

$$\delta^2 A + \delta A = \frac{E\nu}{2(1+\nu)(1-2\nu)}\left\{2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\left(\frac{\partial du}{\partial x}+\frac{\partial dv}{\partial y}+\frac{\partial dw}{\partial z}\right)+\cancel{\left(\frac{\partial du}{\partial x}+\frac{\partial dv}{\partial y}+\frac{\partial dw}{\partial z}\right)^2}\right\}$$

$$+G\left[\cancel{\left(\frac{\partial du}{\partial x}\right)^2}+\cancel{\left(\frac{\partial dv}{\partial y}\right)^2}+\cancel{\left(\frac{\partial dw}{\partial z}\right)^2}+2\frac{\partial u}{\partial x}\cdot\frac{\partial du}{\partial x}+2\frac{\partial v}{\partial y}\cdot\frac{\partial dv}{\partial y}+2\frac{\partial w}{\partial z}\cdot\frac{\partial dw}{\partial z}\right]$$

$$+\frac{G}{2}\left[\cancel{\left(\frac{\partial dw}{\partial y}+\frac{\partial dv}{\partial z}\right)^2}+\cancel{\left(\frac{\partial du}{\partial z}+\frac{\partial dw}{\partial x}\right)^2}+\cancel{\left(\frac{\partial dv}{\partial x}+\frac{\partial du}{\partial y}\right)^2}\right.$$

$$+2\frac{\partial w}{\partial y}\frac{\partial dv}{\partial z}+2\frac{\partial v}{\partial z}\cdot\frac{\partial dw}{\partial y}+2\frac{\partial u}{\partial z}\cdot\frac{\partial dw}{\partial x}+2\frac{\partial w}{\partial x}\cdot\frac{\partial du}{\partial z}$$

$$\left.+2\frac{\partial v}{\partial x}\frac{\partial du}{\partial y}+2\frac{\partial u}{\partial y}\cdot\frac{\partial dv}{\partial x}\right]$$

高次項を省略すると,

$$\delta A = \left\{\frac{E\nu}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2G\frac{\partial u}{\partial x}\right\}\frac{\partial \delta u}{\partial x}$$

$$+\left\{\frac{E\nu}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2G\frac{\partial v}{\partial y}\right\}\frac{\partial \delta v}{\partial y}$$

$$+\left\{\frac{E\nu}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2G\frac{\partial w}{\partial z}\right\}\frac{\partial \delta w}{\partial z}$$

$$+G\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\left(\frac{\partial \delta v}{\partial z}+\frac{\partial \delta w}{\partial y}\right)$$

$$+G\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\left(\frac{\partial \delta w}{\partial x}+\frac{\partial \delta u}{\partial z}\right)$$

$$+G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial \delta u}{\partial y}+\frac{\partial \delta v}{\partial x}\right)$$

$$=\sigma_x\frac{\partial \delta u}{\partial x}+\sigma_y\frac{\partial \delta v}{\partial y}+\sigma_z\frac{\partial \delta w}{\partial z}+\tau_{yz}\left(\frac{\partial \delta v}{\partial z}+\frac{\partial \delta w}{\partial y}\right)+\tau_{zx}\left(\frac{\partial \delta w}{\partial x}+\frac{\partial \delta u}{\partial z}\right)$$

$$+\tau_{xy}\left(\frac{\partial \delta u}{\partial y}+\frac{\partial \delta v}{\partial x}\right)$$

p37.

$$A(u, v, w)$$

$$= \frac{E\nu}{2(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2$$

$$+ G\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$+ \frac{G}{2}\left[\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2\right]$$

$$A(u+du, v+dv, w+dw)$$

$$= \frac{E\nu}{2(1+\nu)(1-2\nu)}\left\{\frac{\partial(u+du)}{\partial x} + \frac{\partial(v+dv)}{\partial y} + \frac{\partial(w+dw)}{\partial z}\right\}$$

$$+ G\left[\left\{\frac{\partial(u+du)}{\partial x}\right\}^2 + \left\{\frac{\partial(v+dv)}{\partial y}\right\}^2 + \left\{\frac{\partial(w+dw)}{\partial z}\right\}^2\right]$$

$$+ \frac{G}{2}\left[\left\{\frac{\partial(w+dw)}{\partial y} + \frac{\partial(v+dv)}{\partial z}\right\}^2 + \left\{\frac{\partial(u+du)}{\partial z} + \frac{\partial(w+dw)}{\partial x}\right\}^2 + \left\{\frac{\partial(v+dv)}{\partial x} + \frac{\partial(u+du)}{\partial y}\right\}^2\right]$$

$$= \frac{E\nu}{2(1+\nu)(1-2\nu)}\left\{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \left(\frac{\partial du}{\partial x} + \frac{\partial dv}{\partial y} + \frac{\partial dw}{\partial z}\right)\right\}^2$$

$$+ G\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial du}{\partial x}\right)^2 + \left(\frac{\partial dv}{\partial y}\right)^2 + \left(\frac{\partial dw}{\partial z}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial du}{\partial x} + 2\frac{\partial v}{\partial y}\frac{\partial dv}{\partial y}\right.$$

$$\left. + 2\frac{\partial w}{\partial z}\cdot\frac{\partial dw}{\partial z}\right]$$

$$+ \frac{G}{2}\left[\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right.$$

$$+ \left(\frac{\partial dw}{\partial y}\right)^2 + \left(\frac{\partial dv}{\partial z}\right)^2 + \left(\frac{\partial du}{\partial z}\right)^2 + \left(\frac{\partial dw}{\partial x}\right)^2 + \left(\frac{\partial dv}{\partial x}\right)^2 + \left(\frac{\partial du}{\partial y}\right)^2$$

$$+ 2\left[\frac{\partial w}{\partial y}\cdot\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\cdot\frac{\partial dv}{\partial z} + \frac{\partial v}{\partial z}\cdot\frac{\partial dw}{\partial y} + \frac{\partial dv}{\partial z}\cdot\frac{\partial dw}{\partial y} + \frac{\partial w}{\partial y}\cdot\frac{\partial dw}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial dv}{\partial z}\right.$$

$$+ \frac{\partial u}{\partial z}\cdot\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial dw}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial du}{\partial z} + \frac{\partial du}{\partial z}\frac{\partial dw}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial du}{\partial z} + \frac{\partial w}{\partial x}\cdot\frac{\partial dw}{\partial x}$$

$$\left.\left. + \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial du}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial dv}{\partial x} + \frac{\partial du}{\partial y}\cdot\frac{\partial dv}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial du}{\partial y} + \frac{\partial v}{\partial x}\cdot\frac{\partial dv}{\partial x}\right]\right]$$

$$\delta^2 A = \frac{E\nu}{2(1+\nu)(1-2\nu)}\left(\frac{\partial\delta u}{\partial x}+\frac{\partial\delta v}{\partial y}+\frac{\partial\delta w}{\partial z}\right)^2$$

$$+\ G\left\{\left(\frac{\partial\delta u}{\partial x}\right)^2+\left(\frac{\partial\delta v}{\partial y}\right)^2+\left(\frac{\partial\delta w}{\partial z}\right)^2\right\}$$

$$+\frac{G}{2}\left\{\left(\frac{\partial\delta w}{\partial y}+\frac{\partial\delta v}{\partial z}\right)^2+\left(\frac{\partial\delta u}{\partial z}+\frac{\partial\delta w}{\partial x}\right)^2+\left(\frac{\partial\delta v}{\partial x}+\frac{\partial\delta u}{\partial y}\right)^2\right\}$$

$$= A(\delta u, \delta v, \delta w)$$

(4) *principle of minimum complementary energy.*

コンプリメンタリ・エネルギ関数. $B(\sigma_x, \sigma_y, \cdots, \tau_{xy})$

$$\Pi_c = \iiint_V B(\sigma_x, \sigma_y, \cdots, \tau_{xy})\,dx\,dy\,dz - \iint_{S_u} (\bar{u} X_\nu + \bar{v} Y_\nu + \bar{w} Z_\nu)\,dS$$

$$\delta \Pi_c = 0$$

付帯条件.

$$(1)\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + \bar{X} = 0 \\[2ex] \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{yz}}{\partial z} + \bar{Y} = 0 \\[2ex] \dfrac{\partial \tau_{zx}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + \bar{Z} = 0 \end{cases}$$

(2) $X_\nu = \overline{X_\nu}$, $Y_\nu = \overline{Y_\nu}$, $Z_\nu = \overline{Z_\nu}$; S_σ上.

◎ 補仮想仕事の原理から最小コンプリメンタリエネルギの原理を導く。

補仮想仕事の原理

$$\iiint_V (\varepsilon_x \delta\sigma_x + \varepsilon_y \delta\sigma_y + \varepsilon_z \delta\sigma_z + \gamma_{yz}\delta\tau_{yz} + \gamma_{zx}\delta\tau_{zx} + \gamma_{xy}\delta\tau_{xy})dV$$

$$- \iint_{S_u} (\bar{u}\,\delta X_\nu + \bar{v}\,\delta Y_\nu + \bar{w}\,\delta Z_\nu)dS = 0$$

$$\varepsilon = \varepsilon(\sigma)$$

コンプリメンタリエネルギ関数 $B = B(\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy})$

$$\therefore \iiint_V \delta B\, dV - \iint_{S_u} (\bar{u}\delta X_\nu + \bar{v}\delta Y_\nu + \bar{w}\delta Z_\nu)dS = 0$$

$$\therefore \quad \delta \Pi_c = 0$$

ただし $\Pi_c = \iiint_V B\,dV - \iint_{S_u} (\bar{u} X_\nu + \bar{v} Y_\nu + \bar{w} Z_\nu)dS$

全コンプリメンタリエネルギ　　表

The principle of minimum complementary energy :

Among all the sets of admissible stresses $\sigma_x, \sigma_y, \cdots$ and τ_{xy}, which satisfy the equations of equilibrium and the prescribed mechanical boundary conditions on S_σ, the set of actual stress components make the total complementary energy Π_c an absolute minimum

最小コンプリメンタリエネルギの原理

$$B=\frac{1}{2E}\left\{(\sigma_x+\sigma_y+\sigma_z)^2+2(1+\nu)\left(\tau_{yz}^2+\tau_{zx}^2+\tau_{xy}^2-\sigma_y\sigma_z-\sigma_z\sigma_x-\sigma_x\sigma_y\right)\right\}$$

$$B(\sigma_x+\delta\sigma_x,\ \sigma_y+\delta\sigma_y,\ \sigma_z+\delta\sigma_z,\ \tau_{yz}+\delta\tau_{yz},\ \tau_{zx}+\delta\tau_{zx},\ \tau_{xy}+\delta\tau_{xy})$$

$$=\frac{1}{2E}\Big[(\sigma_x+\delta\sigma_x+\sigma_y+\delta\sigma_y+\sigma_z+\delta\sigma_z)^2$$

$$+2(1+\nu)\Big\{(\tau_{yz}+\delta\tau_{yz})^2+(\tau_{zx}+\delta\tau_{zx})^2+(\tau_{xy}+\delta\tau_{xy})^2$$

$$-(\sigma_y+\delta\sigma_y)(\sigma_z+\delta\sigma_z)-(\sigma_z+\delta\sigma_z)(\sigma_x+\delta\sigma_x)-(\sigma_x+\delta\sigma_x)(\sigma_y+\delta\sigma_y)\Big\}\Big]$$

$$=\frac{1}{2E}\Big[\underline{(\sigma_x+\sigma_y+\sigma_z)^2}+\underline{(\delta\sigma_x+\delta\sigma_y+\delta\sigma_z)^2}+2(\sigma_x+\sigma_y+\sigma_z)(\delta\sigma_x+\delta\sigma_y+\delta\sigma_z)$$

$$+2(1+\nu)\Big\{\underline{\tau_{yz}^2+\tau_{zx}^2+\tau_{xy}^2-\sigma_y\sigma_z-\sigma_z\sigma_x-\sigma_x\sigma_y}$$

$$\underline{+\delta\tau_{yz}^2+\delta\tau_{zx}^2+\delta\tau_{xy}^2-\delta\sigma_y\delta\sigma_z-\delta\sigma_z\delta\sigma_x-\delta\sigma_x\delta\sigma_y}$$

$$2\tau_{yz}\delta\tau_{yz}+2\tau_{zx}\delta\tau_{zx}+2\tau_{xy}\delta\tau_{xy}$$

$$-\sigma_y\delta\sigma_z-\sigma_z\delta\sigma_y-\underline{\sigma_z\delta\sigma_x}-\underline{\sigma_x\delta\sigma_z}-\sigma_x\delta\sigma_y-\underline{\sigma_y\delta\sigma_x}\Big\}\Big]$$

$$\cancel{=\frac{1}{2E}}=B(\sigma_x,\sigma_y,\sigma_z,\tau_{yz},\tau_{zx},\tau_{xy})$$

$$+\frac{1}{E}\left\{(\sigma_x+\sigma_y+\sigma_z)-(1+\nu)(\sigma_z+\sigma_y)\right\}\delta\sigma_x$$

$$+\frac{1}{E}\left\{(\sigma_x+\sigma_y+\sigma_z)-(1+\nu)(\sigma_z+\sigma_x)\right\}\delta\sigma_y$$

$$+\frac{1}{E}\left\{(\sigma_x+\sigma_y+\sigma_z)-(1+\nu)(\sigma_x+\sigma_y)\right\}\delta\sigma_z$$

$$+\frac{2(1+\nu)}{E}\left\{\tau_{yz}\delta\tau_{yz}+\tau_{zx}\delta\tau_{zx}+\tau_{xy}\delta\tau_{xy}\right\}$$

$$+B(\delta\sigma_x,\delta\sigma_y,\delta\sigma_z,\delta\tau_{yz},\delta\tau_{zx},\delta\tau_{xy})$$

$$= B(\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy})$$

$$+ \varepsilon_x \delta\sigma_x + \varepsilon_y \delta\sigma_y + \varepsilon_z \delta\sigma_z + \gamma_{yz}\delta\tau_{yz} + \gamma_{zx}\delta\tau_{zx} + \gamma_{xy}\delta\tau_{xy}$$

$$+ B(\delta\sigma_x, \delta\sigma_y, \delta\sigma_z, \delta\tau_{yz}, \delta\tau_{zx}, \delta\tau_{xy})$$

$$\therefore \delta B = \varepsilon_x \delta\sigma_x + \varepsilon_y \delta\sigma_y + \varepsilon_z \delta\sigma_z + \gamma_{yz}\delta\tau_{yz} + \gamma_{zx}\delta\tau_{zx} + \gamma_{xy}\delta\tau_{xy}$$

$$\delta^2 B = B(\delta\sigma_x, \delta\sigma_y, \delta\sigma_z, \delta\tau_{yz}, \delta\tau_{zx}, \delta\tau_{xy})$$

$$\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\iint_S \sigma_x \delta u \overset{l\,dS}{dy\,dz} - \iiint \frac{\partial \sigma_x}{\partial x} \underline{\delta u\, dV} =$$

$$\iint_S \sigma_y \delta v \overset{m\,dS}{dx\,dz} - \iiint \frac{\partial \sigma_y}{\partial y} \delta v\, dV$$

$$\iint_S \sigma_z \delta w \overset{n\,dS}{dx\,dy} - \iiint \frac{\partial \sigma_z}{\partial z} \delta w\, dV$$

$$\iint \tau_{xy} \delta u \overset{m\,dS}{dx\,dz} - \iiint \frac{\partial \tau_{xy}}{\partial y} \underline{\delta u\, dV}$$

$$\iint \tau_{xy} \delta v \overset{l\,dS}{dy\,dz} - \iiint \frac{\partial \tau_{xy}}{\partial x} \delta v\, dV$$

$$\iint \tau_{yz} \delta w \overset{m\,dS}{dx\,dz} - \iiint \frac{\partial \tau_{yz}}{\partial y} \delta w\, dV$$

$$\iint \tau_{yz} \delta v \overset{n\,dS}{dx\,dy} - \iiint \frac{\partial \tau_{yz}}{\partial z} \delta v \cancel{dx\,dy}\, dV$$

$$\iint \tau_{zx} \delta w \overset{l\,dS}{dy\,dz} - \iiint \frac{\partial \tau_{zx}}{\partial x} \delta w\, dV$$

$$\iint \tau_{zx} \delta u \overset{n\,dS}{dx\,dy} - \iiint \frac{\partial \tau_{zx}}{\partial z} \underline{\delta u\, dV}$$

total

$$\iint_S \Big[\overbrace{(l\sigma_x + m\tau_{xy} + n\tau_{xz})}^{X_\nu}\delta u + \overbrace{(l\tau_{xy} + m\sigma_y + n\tau_{yz})}^{Y_\nu}\delta v$$

S_σ

$$+ \overbrace{(l\tau_{xz} + m\tau_{yz} + n\sigma_z)}^{Z_\nu}\delta w \Big] dS$$

$$- \iiint \Big[\overbrace{\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)}^{+X}\delta u$$

$$+ \overbrace{\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right)}^{+Y}\delta v$$

$$+ \overbrace{\left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right)}^{Z}\delta w \Big] dV$$

∵ S_u上 $\delta u = 0$, $\delta v = 0$, $\delta w = 0$

[(3.21) 右辺第1項の計算]

divergence theorem の証明

$$\iiint_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx} + \tau_{xy}\gamma_{xy}) dV$$

$$= \iiint_V \left[\sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \sigma_z \frac{\partial w}{\partial z} + \tau_{yz}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) + \tau_{zx}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] dV$$

$$\iiint_V \sigma_x \frac{\partial u}{\partial x} dV = \iint_S \sigma_x \partial u \overbrace{dy\,dz}^{l\,dS} - \iiint \frac{\partial \sigma_x}{\partial x} u \cancel{dy\,dz}\, dV$$

$$= \iint \sigma_y v \overbrace{dz\,dx}^{m\,dS} - \iiint \frac{\partial \sigma_y}{\partial y} v\, dV$$

$$= \iint \sigma_z w \overbrace{dx\,dy}^{n\,dS} - \iiint \frac{\partial \sigma_z}{\partial z} w\, dV$$

$$\iint \tau_{yz} \cdot w \overbrace{dz\,dx}^{m\,dS} - \iiint \frac{\partial \tau_{yz}}{\partial y} w\, dV$$

$$\iint \tau_{yz} v \overbrace{dx\,dy}^{n\,dS} - \iiint \frac{\partial \tau_{yz}}{\partial z} v\, dV$$

$$\iint \tau_{zx} \cdot u \overbrace{dx\,dy}^{n\,dS} - \iiint \frac{\partial \tau_{zx}}{\partial z} u\, dV$$

$$\iint \tau_{zx} \cdot w \overbrace{dy\,dz}^{l\,dS} - \iiint \frac{\partial \tau_{zx}}{\partial x} w\, dV$$

$$\iint \tau_{xy} \cdot v \overbrace{dy\,dz}^{l\,dS} - \iiint \frac{\partial \tau_{xy}}{\partial x} v\, dV$$

$$\iint \tau_{xy} \cdot u \cdot \overbrace{dx\,dz}^{m\,dS} - \iiint \frac{\partial \tau_{xy}}{\partial y} u\, dV$$

total.

$$\iint_S \Big[(\overbrace{l\sigma_x + m\tau_{xy} + n\tau_{zx}}^{X_\nu}) u + (\overbrace{l\tau_{xy} + m\sigma_y + n\tau_{yz}}^{Y_\nu}) v + (\overbrace{l\tau_{zx} + m\tau_{yz} + n\sigma_z}^{Z_\nu}) w \Big] dS$$

$$- \iiint \Big[(\overbrace{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}^{-X}) u + (\overbrace{\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}}^{-Y}) v$$ ← つり合い方程式より

$$+ (\overbrace{\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}}^{-Z}) w \Big] dV$$

P41 [問題1]

divergence theorem

principle of virtual work

① $(\varepsilon_x, \varepsilon_y, \cdots, \gamma_{xy} ; u, v, w) \rightarrow$ 微小ひずみ
微小変位 }.

② $X_\nu = \overline{X_\nu}$, $Y_\nu = \overline{Y_\nu}$, $Z_\nu = \overline{Z_\nu}$; S_σ上

③ $\delta u = \delta v = \delta w = 0$; S_u上.

principle of complementary virtual work.

① $(\sigma_x, \sigma_y, \cdots, \tau_{xy}) \longrightarrow$ 微小応力変分

② $u = \overline{u}$, $v = \overline{v}$, $w = \overline{w}$; S_u上.

③ $\delta X_\nu = \delta Y_\nu = \delta Z_\nu = 0$; S_σ上.

問題2.

文献(10) unit displacement method

Given the true stresses σ in a structure, the equilibrating force R at a given point and direction can be calculated from

$$1 \cdot R = \int_V \bar{\varepsilon} \sigma \, dV$$

where $\bar{\varepsilon}$ is a virtual strain system due to unit displacement in given direction.

$\parallel$

~~\~~ 仮想仕事の原理.

① 体積力は無視.

② $\delta\bar{u} = 0$, $\delta\bar{v} = 0$, $\delta\bar{w} = 0$; S_u上.

kinematically equivalent strains ignore equilibrium conditions
無視する

unit load method 文献 10)

Given the true strains ε in a structure,
the kinematically related displacement r at a given point and direction can be calculated from

$$1 \cdot r = \int_V \bar{\sigma}\, \varepsilon \, dV$$

where $\bar{\sigma}$ is a virtual or otherwise statically equivalent stress system due to unit load in given direction.

$\parallel$

補仮想仕事の原理.

Statically equivalent stresses ignore compatibility conditions
無視する 適合条件.

① 体積力は変わらない.

② S_u上で変位はしない. $\bar{u} = \bar{v} = \bar{w} = 0$

文献（11） dummy load method P24～P26

⇓

unit load method

幾何学的境界条件の変化.

$$\bar{u} + d\bar{u} \quad , \quad \bar{v} + d\bar{v} \,, \quad \bar{w} + d\bar{w} \qquad : \ S_u \text{上}.$$

$$dU = \iiint_V dA\, dx\, dy\, dz$$

$$= \iiint_V \Big[\sigma_x \frac{\partial du}{\partial x} + \sigma_y \frac{\partial dv}{\partial y} + \sigma_z \frac{\partial dw}{\partial z} + \tau_{yz}\Big(\frac{\partial dw}{\partial y} + \frac{\partial dv}{\partial z}\Big) + \tau_{zx}\Big(\frac{\partial du}{\partial z} + \frac{\partial dw}{\partial x}\Big)$$

$$+ \tau_{xy}\Big(\frac{\partial dv}{\partial x} + \frac{\partial du}{\partial y}\Big)\Big]\, dx\, dy\, dz.$$

$$= \iint \sigma_x\, du\, \overset{l\,dS}{dy\, dz} - \iiint \frac{\partial \sigma_x}{\partial x} \cdot du\, dV$$

$$+ \iint \sigma_y \cdot dv\, \overset{m\,dS}{dz\, dx} - \iiint \frac{\partial \sigma_y}{\partial y} \cdot dv\, dV$$

$$+ \iint \sigma_z\, dw\, \overset{m\,dS}{dx\, dy} - \iiint \frac{\partial \sigma_z}{\partial z} \cdot dw\, dV$$

$$+ \iint \tau_{yz}\, dw\, \overset{m\,dS}{dz\, dx} - \iiint \frac{\partial \tau_{yz}}{\partial y} \cdot dw\, dV$$

$$+ \iint \tau_{yz}\, dv\, \overset{m\,dS}{dx\, dy} - \iiint \cancel{\tau_{yz}} \cdot \frac{\partial \tau_{yz}}{\partial z}\, dv\, dV$$

$$+ \iint \tau_{zx} \cdot du \cdot \overset{m\,dS}{dx\, dy} - \iiint \frac{\partial \tau_{zx}}{\partial z} \cdot du\, dV$$

$$+ \iint \tau_{zx} \cdot dw\, \overset{l\,dS}{dy\, dz} - \iiint \frac{\partial \tau_{zx}}{\partial x}\, dw\, dV$$

$$+ \iint \tau_{xy} \cdot dv\, \overset{l\,dS}{dy\, dz} - \iiint \frac{\partial \tau_{xy}}{\partial x} \cdot dv\, dV$$

$$+ \iint \tau_{xy}\, du\, \overset{m\,dS}{dz\, dx} - \iiint \frac{\partial \tau_{xy}}{\partial y} \cdot du \cdot dV$$

$$= \iint_S \Big[\big(\overset{X_\nu = \bar{X}_\nu}{l\sigma_x + m\tau_{xy} + n\tau_{zx}}\big) du + \big(\overset{Y_\nu = \bar{Y}_\nu}{l\tau_{xy} + m\sigma_y + n\tau_{yz}}\big) dv + \big(\overset{Z_\nu = \bar{Z}_\nu}{l\tau_{zx} + m\tau_{yz} + n\sigma_z}\big) \underset{dw}{} \Big] dS$$

$$- \iiint_V \Big[\big(\overset{-\bar{X}}{\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}\big) du + \big(\overset{-\bar{Y}}{\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}}\big) dv + \big(\overset{-\bar{Z}}{\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}}\big) dw \Big] dV$$

力学的境界条件の変化. 3-7:2

$$dV = \iiint_V dB\,dx\,dy\,dz$$

$$= \iiint_V (\varepsilon_x d\sigma_x + \varepsilon_y d\sigma_y + \varepsilon_z d\sigma_z + \gamma_{yz} d\tau_{yz} + \gamma_{zx} d\tau_{zx} + \gamma_{xy} d\tau_{xy})\,dx\,dy\,dz$$

$$= \iiint_V \Big[\frac{\partial u}{\partial x} d\sigma_x + \frac{\partial v}{\partial y} d\sigma_y + \frac{\partial w}{\partial z} d\sigma_z + \Big(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\Big) d\tau_{yz} + \Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big) d\tau_{zx}$$

$$+ \Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\Big) d\tau_{xy} \Big]\,dx\,dy\,dz$$

$$= \iint_{S_\sigma} u\, d\sigma_x\, l\, dS - \iiint_V u \frac{\partial d\sigma_x}{\partial x}\,dx\,dy\,dz$$

$$+ \iint_{S_\sigma} v\, d\sigma_y\, m\, dS - \iiint_V v \frac{\partial d\sigma_y}{\partial y}\,dx\,dy\,dz$$

$$+ \iint_{S_\sigma} w\, d\sigma_z\, n\, dS - \iiint_V w \frac{\partial d\sigma_z}{\partial z}\,dx\,dy\,dz$$

$$+ \iint_{S_\sigma} w\, d\tau_{yz}\, m\, dS - \iiint_V w \frac{\partial d\tau_{yz}}{\partial y}\,dx\,dy\,dz$$

$$+ \iint_{S_\sigma} v\, d\tau_{yz}\, n\, dS - \iiint_V v \frac{\partial d\tau_{yz}}{\partial z}\,dx\,dy\,dz$$

$$+ \iint_{S_\sigma} u\, d\tau_{zx}\, n\, dS - \iiint_V u \frac{\partial d\tau_{zx}}{\partial z}\,dx\,dy\,dz$$

$$+ \iint_{S_\sigma} w\, d\tau_{zx}\, l\, dS - \iiint_V w \frac{\partial d\tau_{zx}}{\partial x}\,dx\,dy\,dz$$

$$+ \iint_{S_\sigma} v\, d\tau_{xy}\, l\, dS - \iiint_V v \frac{\partial d\tau_{xy}}{\partial x}\,dx\,dy\,dz$$

$$+ \iint_{S_\sigma} u\, d\tau_{xy}\, m\, dS - \iiint_V u \frac{\partial d\tau_{xy}}{\partial y}\,dx\,dy\,dz$$

$$= \iint_{S_\sigma} \Big[(d\sigma_x l + d\tau_{xy} m + d\tau_{zx} n) u + (d\tau_{xy} l + d\sigma_y m + d\tau_{yz} n) v$$

$$+ (d\tau_{zx} l + d\tau_{yz} m + d\sigma_z n) w \Big]\,dS$$

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問題2.

$$\Pi_a = V = \frac{1}{2}\cdot\frac{P_{12}^2 l}{A_0E} + 2\cdot\frac{1}{2}\cdot\frac{P_{13}^2}{A_0E}\cdot\frac{l}{\cos\alpha}$$

$$= \frac{1}{2}\frac{P_{12}^2 l}{A_0E} + 2\cdot\frac{1}{2}\cdot\frac{1}{A_0E}\cdot\left(\frac{\overline{Q}-P_{12}}{2\cos\alpha}\right)^2\frac{l}{\cos\alpha} \qquad (i)$$

$$\frac{\partial V}{\partial P_{12}} = \frac{\cancel{l}}{\cancel{A_0E}}P_{12} + \frac{1}{\cancel{A_0E}}\cdot\frac{\cancel{l}}{4\cos^3\alpha}\left(-2\overline{Q}+2P_{12}\right) = 0$$

$$1 + \frac{1}{2\cos^3\alpha}P_{12} = \frac{\overline{Q}}{2\cos^3\alpha}$$

$$\therefore P_{12} = \frac{1}{1+2\cos^3\alpha}\overline{Q} \qquad (ii)$$

(i) に代入

$$V = \frac{1}{2}\cdot\frac{l}{A_0E}\left(\frac{\overline{Q}}{1+2\cos^3\alpha}\right)^2 + 2\cdot\frac{1}{2}\cdot\frac{l}{A_0E}\cdot\frac{1}{4\cos^3\alpha}\left(\frac{2\cos^3\alpha}{1+2\cos^3\alpha}\right)^2\overline{Q}^2$$

$$= \left\{\frac{l}{2A_0E}\cdot\left(\frac{1}{1+2\cos^3\alpha}\right)^2 + \frac{l}{A_0E}\cdot\frac{\cos^3\alpha}{(1+2\cos^3\alpha)^2}\right\}\overline{Q}^2$$

$$= \frac{l}{2A_0E}\cdot\frac{1+2\cos^3\alpha}{(1+2\cos^3\alpha)^2}\overline{Q}^2 = \frac{l}{2A_0E}\cdot\frac{1}{1+2\cos^3\alpha}\overline{Q}^2$$

$$u_1 = \frac{\partial V(\overline{Q})}{\partial\overline{Q}} = \frac{l}{A_0E}\cdot\frac{1}{1+2\cos^3\alpha}\overline{Q} = \frac{l\overline{Q}}{A_0E(1+2\cos^3\alpha)} \qquad (iii)$$

$= (1,10)$式

4. 変分原理に基礎をおく近似解法

4-1. 仮想仕事の原理に基礎をおく近似解法

二次元弾性問題における仮想仕事の原理.

$$\iint_S (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy}) t\,dS - \int_{C_\sigma} (\overline{X_\nu}\delta u + \overline{Y_\nu}\delta v) t\,ds = 0$$

$$\iint_S (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy}) t\,dx\,dy - \int_{C_\sigma} (\overline{X_\nu}\delta u + \overline{Y_\nu}\delta v) t\,ds = 0 \qquad (4.1)$$

t : 板厚.

変位 u, v を次のように仮定する。すなわち.

$$u(x, y) = u_0(x, y) + \sum_{r=1}^{m} a_r u_r(x, y)$$

$$v(x, y) = v_0(x, y) + \sum_{r=1}^{m} b_r v_r(x, y) \qquad (4.2)$$

幾何学的境界条件. C_u上 ; $u_0 = \overline{u}$, $v_0 = \overline{v}$, $u_r = 0$, $v_r = 0$

$r = 1, 2, 3, \cdots, n$ (4.3)

仮想変位.

$$\delta u = \sum_{r=1}^{m} \delta a_r u_r$$

$$\delta v = \sum_{r=1}^{m} \delta b_r v_r \qquad (4.4)$$

$$\sigma_x \delta\varepsilon_x = \sigma_x \frac{\partial \delta u}{\partial x} = \sigma_x \sum_{r=1}^{m} \delta a_r \frac{\partial u_r}{\partial x}$$

$$\sigma_y \delta\varepsilon_y = \sigma_y \frac{\partial \delta v}{\partial y} = \sigma_y \sum_{r=1}^{m} \delta b_r \frac{\partial v_r}{\partial y}$$

$$\tau_{xy}\delta\gamma_{xy} = \tau_{xy}\left(\frac{\partial \delta v}{\partial x} + \frac{\partial \delta u}{\partial y}\right) = \tau_{xy}\sum_{r=1}^{m}\left(\delta a_r \frac{\partial u_r}{\partial y} + \delta b_r \frac{\partial v_r}{\partial x}\right)$$

(4,1)式に代入する．

$$\iint_S \sum_{r=1}^{m} \left(\sigma_x \frac{\partial u_r}{\partial x} + \tau_{xy} \frac{\partial u_r}{\partial y} \right) \delta a_r t\, dx dy - \int_{C_\sigma} \sum_{r=1}^{m} \overline{X_\nu} u_r \delta a_r t\, ds$$

$$+ \iint_S \sum_{r=1}^{m} \left(\tau_{xy} \frac{\partial v_r}{\partial x} + \sigma_y \frac{\partial v_r}{\partial y} \right) \delta b_r t\, dx dy - \int_{C_\sigma} \sum_{r=1}^{m} \overline{Y_\nu} v_r \delta b_r t\, ds = 0$$

ここで

$$\left.\begin{aligned} &\iint_S \left(\sigma_x \frac{\partial u_r}{\partial x} + \tau_{xy} \frac{\partial u_r}{\partial y} \right) t\, dx dy - \int_{C_\sigma} \overline{X_\nu} u_r t\, ds = L_r \\ &\iint_S \left(\tau_{xy} \frac{\partial v_r}{\partial x} + \sigma_y \frac{\partial v_r}{\partial y} \right) t\, dx dy - \int_{C_\sigma} \overline{Y_\nu} v_r t\, ds = M_r \end{aligned}\right\} \quad (4,6)$$

とおけば，

$$\sum_{r=1}^{m} \left(L_r \delta a_r + M_r \delta b_r \right) = 0 \qquad (4,5)$$

である。 δa_r, δb_r は任意であるから，

$$L_r = 0 , \quad M_r = 0 \quad ; r = 1, 2, 3, \cdots, n \qquad (4,7)$$

ただし，

$$L_r = \int \sigma_x u_r t \underset{}{\overset{l\,ds}{\overset{\shortparallel}{dy}}} + \int \tau_{xy} u_r t \overset{-m\,ds}{\overset{|}{dx}}$$

$$- \iint_S \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) u_r t\, dx dy$$

$$- \int_{C_\sigma} \overline{X_\nu} u_r t\, ds$$

$$= - \iint_S \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) u_r t\, dx dy + \int_{C_\sigma} (\sigma_x l + \tau_{xy} m) u_r t\, ds - \int_{C_\sigma} \overline{X_\nu} u_r t\, ds$$

$$= - \iint_S \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) u_r t\, dx dy + \int_{C_\sigma} (X_\nu - \overline{X_\nu}) u_r t\, ds$$

同様に $M_r = -\iint_S \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) v_r t\, dx\, dy + \int_{C_\sigma} (Y_\nu - \overline{Y_\nu}) v_r t\, ds$

次に応力 σ_x, σ_y, τ_{xy} を u_0, v_0, u_r, v_r, a_r, b_r で表わす。

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + \sum_{r=1}^{m} a_r \frac{\partial u_r}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} + \sum_{r=1}^{m} b_r \frac{\partial v_r}{\partial y}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial v_0}{\partial x} + \sum_{r=1}^{m} b_r \frac{\partial v_r}{\partial x} + \frac{\partial u_0}{\partial y} + \sum_{r=1}^{m} a_r \frac{\partial u_r}{\partial y}$$

$$\sigma_x = \frac{E}{(1-\nu^2)} (\varepsilon_x + \nu \varepsilon_y) = \frac{E}{1-\nu^2} \left[\frac{\partial u_0}{\partial x} + \sum_{r=1}^{m} a_r \frac{\partial u_r}{\partial x} + \nu \left(\frac{\partial v_0}{\partial y} + \sum_{r=1}^{m} b_r \frac{\partial v_r}{\partial y} \right) \right]$$

$$\sigma_y = \frac{E}{1-\nu^2} (\varepsilon_y + \nu \varepsilon_x) = \frac{E}{1-\nu^2} \left[\frac{\partial v_0}{\partial y} + \sum_{r=1}^{m} b_r \frac{\partial v_r}{\partial y} + \nu \left(\frac{\partial u_0}{\partial x} + \sum_{r=1}^{m} a_r \frac{\partial u_r}{\partial x} \right) \right]$$

$$\tau_{xy} = G \gamma_{xy} = G \left[\frac{\partial u_0}{\partial y} + \sum_{r=1}^{m} a_r \frac{\partial u_r}{\partial y} + \frac{\partial v_0}{\partial x} + \sum_{r=1}^{m} b_r \frac{\partial v_r}{\partial x} \right] \tag{4.10}$$

(4.6)式 に (4.10)式 を代入する

$$L_r = \iint_S \left(\sigma_x \frac{\partial u_r}{\partial x} + \tau_{xy} \frac{\partial u_r}{\partial y} \right) t\, dx\, dy - \int_{C_\sigma} \overline{X_\nu} u_r t\, ds$$

釣合い方程式はa_rによらず満足される。

$$\frac{\partial \sigma_x}{\partial x} = \frac{\partial^3 F_0}{\partial x \partial y^2} + \sum_{r=1}^{n} a_r \frac{\partial^3 F_r}{\partial x \partial y^2}$$

$$\frac{\partial \tau_{xy}}{\partial y} = -\frac{\partial^3 F_0}{\partial x \partial y^2} - \sum_{r=1}^{n} a_r \frac{\partial^3 F_r}{\partial x \partial y^2}$$

$$\therefore \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} = -\frac{\partial^3 F_0}{\partial x^2 \partial y} - \sum_{r=1}^{n} a_r \frac{\partial^3 F_r}{\partial x^2 \partial y}$$

$$\frac{\partial \sigma_y}{\partial y} = +\frac{\partial^3 F_0}{\partial x^2 \partial y} + \sum_{r=1}^{n} a_r \frac{\partial^3 F_r}{\partial x^2 \partial y}$$

$$\therefore \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

S_σ上の境界条件はa_rによらず満足される。

$$\sigma_x = \frac{\partial}{\partial y}\left(\frac{\partial F_0}{\partial y} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial y}\right)$$

$$\tau_{xy} = -\frac{\partial}{\partial x}\left(\frac{\partial F_0}{\partial y} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial y}\right)$$

$$l = \frac{dy}{ds} \qquad m = -\frac{dx}{ds}$$

$$l\sigma_x + m\tau_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial F_0}{\partial y} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial y}\right)\cdot\frac{dy}{ds} + \frac{\partial}{\partial x}\left(\frac{\partial F_0}{\partial y} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial y}\right)\cdot\frac{dx}{ds}$$

$$= \frac{d}{ds}\left(\frac{\partial F_0}{\partial y} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial y}\right) = \overline{X}_\nu$$

$$\sigma_y = \frac{\partial}{\partial x}\left(\frac{\partial F_0}{\partial x} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial x}\right)$$

$$\tau_{xy} = -\frac{\partial}{\partial y}\left(\frac{\partial F_0}{\partial x} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial x}\right)$$

$$l\tau_{xy} + m\sigma_y = -\frac{\partial}{\partial y}\left(\frac{\partial F_0}{\partial x} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial x}\right)\cdot\frac{\partial y}{\partial s} - \frac{\partial}{\partial x}\left(\frac{\partial F_0}{\partial x} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial x}\right)\cdot\frac{dx}{ds}$$

$$= -\frac{d}{ds}\left(\frac{\partial F_0}{\partial x} + \sum_{r=1}^{n} a_r \frac{\partial F_r}{\partial x}\right) = \overline{Y}_\nu$$

応力変分について.

$$\frac{\partial \delta \sigma_x}{\partial x} = \sum_{r=1}^{n} \frac{\partial^2 F_r}{\partial x \partial y^2} \delta a_r$$

$$\frac{\partial \delta \tau_{xy}}{\partial y} = -\sum_{r=1}^{n} \frac{\partial^3 F_r}{\partial x \partial y^2} \delta a_r$$

$$\therefore \frac{\partial \delta \sigma_x}{\partial x} + \frac{\partial \delta \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \delta \tau_{xy}}{\partial x} = -\sum_{r=1}^{n} \frac{\partial^3 F_r}{\partial x^2 \partial y} \delta a_r$$

$$\frac{\partial \delta \sigma_y}{\partial y} = \sum_{r=1}^{n} \frac{\partial^3 F_r}{\partial x^2 \partial y} \delta a_r$$

$$\therefore \frac{\partial \delta \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$

$$\delta \sigma_x l = \frac{\partial}{\partial y}\left(\sum_{r=1}^{n} \frac{\partial F_r}{\partial y} \delta a_r\right) \cdot \frac{dy}{ds}$$

$$\delta \tau_{xy} m = +\frac{\partial}{\partial x}\left(\sum_{r=1}^{n} \frac{\partial F_r}{\partial y} \delta a_r\right) \cdot \frac{dx}{ds}$$

$$\therefore \delta \sigma_x l + \delta \tau_{xy} m = \frac{d}{ds}\left(\sum_{r=1}^{n} \frac{\partial F_r}{\partial y} \delta a_r\right) = 0$$

同様に

$$\delta \tau_{xy} l + \delta \sigma_y m = 0 .$$

$$= -\frac{d}{ds}\left(\sum_{r=1}^{n} \frac{\partial F_r}{\partial x} \delta a_r\right)$$

応力変分を補仮想仕事の原理に代入すると、

$$\sum_{r=1}^{m}\left[\iint_S \left(\varepsilon_x \frac{\partial^2 F_r}{\partial y^2} + \varepsilon_y \frac{\partial^2 F_r}{\partial x^2} - \gamma_{xy}\frac{\partial^2 F_r}{\partial x \partial y}\right) t\,dx\,dy - \int_{C_u}\left\{\bar{u}\frac{d}{ds}\left(\frac{\partial F_r}{\partial y}\right) - \bar{v}\frac{d}{ds}\left(\frac{\partial F_r}{\partial x}\right)\right\} t\,ds\right]\delta a_r = 0.$$

δa_r は任意である。

$$\therefore \iint_S \left(\varepsilon_x \frac{\partial^2 F_r}{\partial y^2} + \varepsilon_y \frac{\partial^2 F_r}{\partial x^2} - \gamma_{xy}\frac{\partial^2 F_r}{\partial x \partial y}\right) t\,dx\,dy - \int_{C_u}\left\{\bar{u}\frac{d}{ds}\left(\frac{\partial F_r}{\partial y}\right) - \bar{v}\frac{d}{ds}\left(\frac{\partial F_r}{\partial x}\right)\right\} t\,ds = 0.$$

ここで、

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = \frac{1}{E}\left[\frac{\partial^2 F_0}{\partial y^2} + \sum_{r=1}^{n} a_r \frac{\partial^2 F_r}{\partial y^2} - \nu\left(\frac{\partial^2 F_0}{\partial x^2} + \sum_{r=1}^{m} a_r \frac{\partial^2 F_r}{\partial x^2}\right)\right]$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1}{E}\left[\frac{\partial^2 F_0}{\partial x^2} + \sum_{r=1}^{n} a_r \frac{\partial^2 F_r}{\partial x^2} - \nu\left(\frac{\partial^2 F_0}{\partial y^2} + \sum_{r=1}^{n} a_r \frac{\partial^2 F_r}{\partial y^2}\right)\right]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = -\frac{1}{G}\left[\frac{\partial^2 F_0}{\partial x \partial y} + \sum_{r=1}^{m} a_r \frac{\partial^2 F_r}{\partial x \partial y}\right]$$

4.22式

$$L_r=\iint_S\left(\varepsilon_x\frac{\partial^2F_r}{\partial y^2}+\varepsilon_y\frac{\partial^2F_r}{\partial x^2}-\gamma_{xy}\frac{\partial^2F_r}{\partial x\partial y}\right)t\,dx\,dy$$

$$\iint_S\varepsilon_x\frac{\partial^2F_r}{\partial y^2}t\,dx\,dy=\int\varepsilon_x\frac{\partial F_r}{\partial y}t\,dx-\iint_S\frac{\partial\varepsilon_x}{\partial y}\cdot\frac{\partial F_r}{\partial y}t\,dx\,dy$$

$$\iint\qquad=\int_C\varepsilon_x\frac{\partial F_r}{\partial y}t\underbrace{dx}_{-mds}-\int_C\frac{\partial\varepsilon_x}{\partial y}F_r t\underbrace{dx}_{-mds.}+\iint_S\frac{\partial^2\varepsilon_x}{\partial y^2}F_r t\,dx\,dy$$

$$\iint_S\varepsilon_y\frac{\partial^2F_r}{\partial x^2}t\,dx\,dy=\int_C\varepsilon_y\frac{\partial F_r}{\partial x}t\underbrace{dy}_{lds}-\int_C\frac{\partial\varepsilon_y}{\partial x}\cdot F_r t\underbrace{dy}_{lds}+\iint_S\frac{\partial^2\varepsilon_y}{\partial x^2}F_r t\,dx\,dy$$

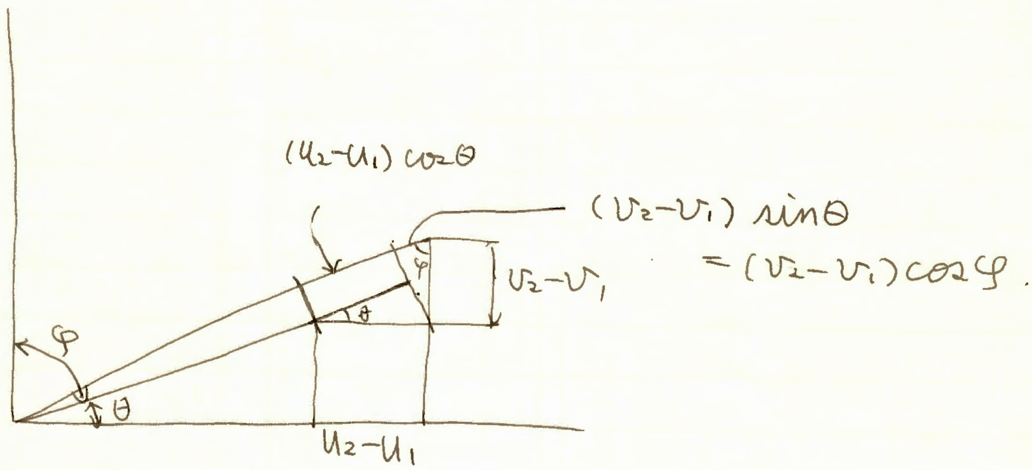
$$\iint_S\gamma_{xy}\frac{\partial^2F_r}{\partial x\partial y}t\,dx\,dy=\int\gamma_{xy}\frac{\partial F_r}{\partial x}t\,dx-\iint_S\frac{\partial\gamma_{xy}}{\partial y}\cdot\frac{\partial F_r}{\partial x}t\,dx\,dy$$

$$=\int\gamma_{xy}\frac{\partial F_r}{\partial x}t\underbrace{dx}_{-mds}-\int\frac{\partial\gamma_{xy}}{\partial y}\cdot F_r t\underbrace{dy}_{\cancel{ms}\ ldx.}+\iint_S\frac{\partial^2\gamma_{xy}}{\partial x\partial y}F_r t\,dx\,dy$$

$$\therefore L_r=\iint_S\left(\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\right)F_r t\,dx\,dy$$

$$\delta = (u_2 - u_1)\lambda + (v_2 - v_1)\mu + (w_2 - w_1)\nu$$

二次元 $\quad \delta = (u_2 - u_1)\lambda + (v_2 - v_1)\mu .$



$$\therefore \quad \delta = (u_2 - u_1)\cos\theta + (v_2 - v_1)\cos\varphi .$$

$$U = \frac{1}{2}\frac{EA_0}{l}\delta_1^2$$

$$= \frac{1}{2}\cdot\frac{EA_0}{l}\left\{(u_2-u_1)\lambda + (v_2-v_1)\mu + (w_2-w_1)\nu\right\}^2$$

$$F_{x1} = \frac{\partial U}{\partial u_1} = \frac{1}{2}\cdot\frac{EA_0}{l}\left\{-2(u_2-u_1)\lambda^2 + 2(v_1-v_2)\lambda\mu + 2(w_1-w_2)\lambda\nu\right\}$$

$$= \frac{EA_0}{l}\left(\lambda^2 u_1 + \lambda\mu v_1 + \lambda\nu w_1 - \lambda^2 u_2 - \lambda\mu v_2 - \lambda\nu w_2\right)$$

$$F_{y1} = \frac{\partial U}{\partial v_1} = \frac{1}{2}\cdot\frac{EA_0}{l}\left\{-2(v_2-v_1)\mu^2 + 2(u_1-u_2)\lambda\mu + 2(w_1-w_2)\mu\nu\right\}$$

$$= \frac{EA_0}{l}\left(\lambda\mu u_1 + \mu^2 v_1 + \mu\nu w_1 - \lambda\mu u_2 - \mu^2 v_2 - \mu\nu w_2\right)$$

$\{(u_2-u_1)\mu + (v_2-v_1)\mu + (w_2-w_1)\nu\}(-\lambda)$

$$F_{z1} = \frac{\partial U}{\partial w_1} = \frac{1}{2}\cdot\frac{EA_0}{l}\left\{-2(w_2-w_1)\nu^2 + 2(u_1-u_2)\lambda\nu + 2(v_1-v_2)\mu\nu\right\}$$

$$= \frac{EA_0}{l}\left(\lambda\nu u_1 + \mu\nu v_1 + \nu^2 w_1 - \lambda\nu u_2 - \mu\nu v_2 - \nu^2 w_2\right)$$

$$F_{x2} = \frac{\partial U}{\partial u_2} = \frac{1}{2}\cdot\frac{EA_0}{l}\left\{2(u_2-u_1)\lambda^2 + 2(v_2-v_1)\lambda\mu + 2(w_2-w_1)\lambda\nu\right\}$$

$$= \frac{EA_0}{l}\left(-\lambda^2 u_1 - \lambda\mu v_1 - \lambda\nu w_1 + \lambda^2 u_2 + \lambda\mu v_2 + \lambda\nu w_2\right)$$

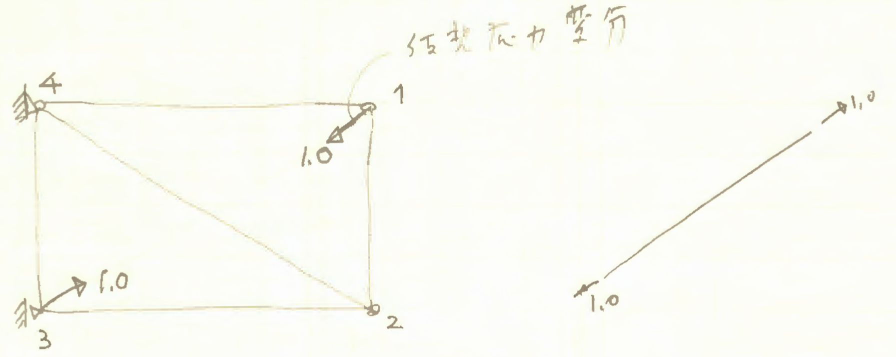
$$F_{y2} = \frac{\partial U}{\partial v_2} = \frac{1}{2}\cdot\frac{EA_0}{l}\left\{2(v_2-v_1)\mu^2 + 2(u_2-u_1)\lambda\mu + 2(w_2-w_1)\mu\nu\right\}$$

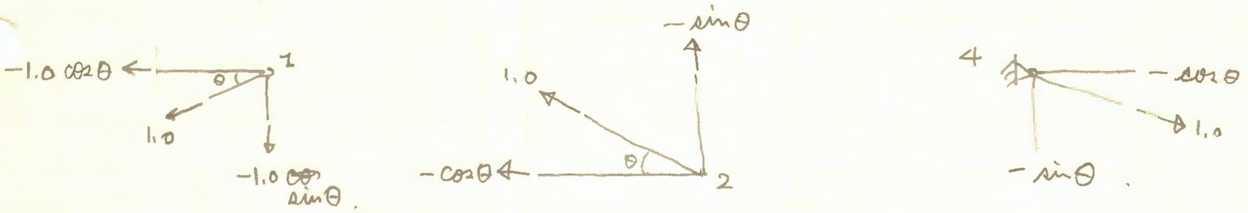
$$= \frac{EA_0}{l}\left(-\lambda\mu u_1 - \mu^2 v_1 - \mu\nu w_1 + \lambda\mu u_2 + \mu^2 v_2 + \mu\nu w_2\right)$$

$$F_{z2} = \frac{\partial U}{\partial w_2} = \frac{1}{2}\cdot\frac{EA_0}{l}\left\{2(w_2-w_1)\nu^2 + 2(u_2-u_1)\lambda\nu + 2(v_2-v_1)\mu\nu\right\}$$

$$= \frac{EA_0}{l}\left(-\lambda\nu u_1 - \mu\nu v_1 - \nu^2 w_1 + \lambda\nu u_2 + \mu\nu v_2 + \nu^2 w_2\right)$$

問2





	伸び	$\bar{S}$	$\delta \cdot \bar{S}$
12	δ_{12}	$-\sin\theta$	$-\delta_{12}\sin\theta$
~~13~~	~~δ_{13}~~ δ_{13}	1.0	δ_{13}.
14	δ_{14}	$-\cos\theta$	$-\delta_{14}\cos\theta$
23	δ_{23}	$-\cos\theta$	$-\delta_{23}\cos\theta$
24	δ_{24}	1.0	δ_{24}
34	δ_{34}	$-\sin\theta$	$-\delta_{34}\sin\theta$

$\cancel{\sum \delta \cdot \bar{S} = 1.0 \cdot \delta_{13}}$. $\sum \delta \cdot \bar{S} = 0$

$\cancel{\therefore\ \delta_{13} =}$ $\therefore\ -(\delta_{12}+\delta_{34})\sin\theta - (\delta_{14}+\delta_{23})\cos\theta$
$+\delta_{13}+\delta_{24}=0$

(問3)

(5,28)

$$\delta_{12} = v_1 - v_2 \quad ①$$

$$\delta_{13} = u_1 \cos\theta + (v_1 - v_3)\sin\theta \quad ②$$

$$\delta_{14} = u_1 \quad ③$$

$$\delta_{23} = u_2 \quad ④$$

$$\delta_{24} = u_2 \cos\theta - v_2 \sin\theta \quad ⑤$$

$$\delta_{34} = -v_3 \quad ⑥$$

③より $u_1 = \delta_{14}$

④より $u_2 = \delta_{23}$

⑥より $v_3 = -\delta_{34}$

⑤より

$$\begin{aligned} v_2 &= \cancel{\cot\theta}\,(u_2 \cos\theta - \delta_{24}) \\ &= \cancel{\cot\theta}\,(\delta_{23}\cos\theta - \delta_{24}) \\ &= \cot\theta\,\delta_{23} - \delta_{24}\,\mathrm{cosec}\,\theta \end{aligned}$$

①より

$$\begin{aligned} v_1 &= \delta_{12} + v_2 \\ &= \delta_{12} + \cot\theta\,\delta_{23} - \delta_{24}\,\mathrm{cosec}\,\theta \end{aligned}$$

$$\frac{\partial V}{\partial \bar{P}_i} = \Delta_i$$

$$\partial V = \Delta_i \cdot \partial P_i$$

$$\{\delta\}^T = \{\delta_{12}, \delta_{13}, \delta_{14}, \delta_{23}, \delta_{24}, \delta_{34}\}$$

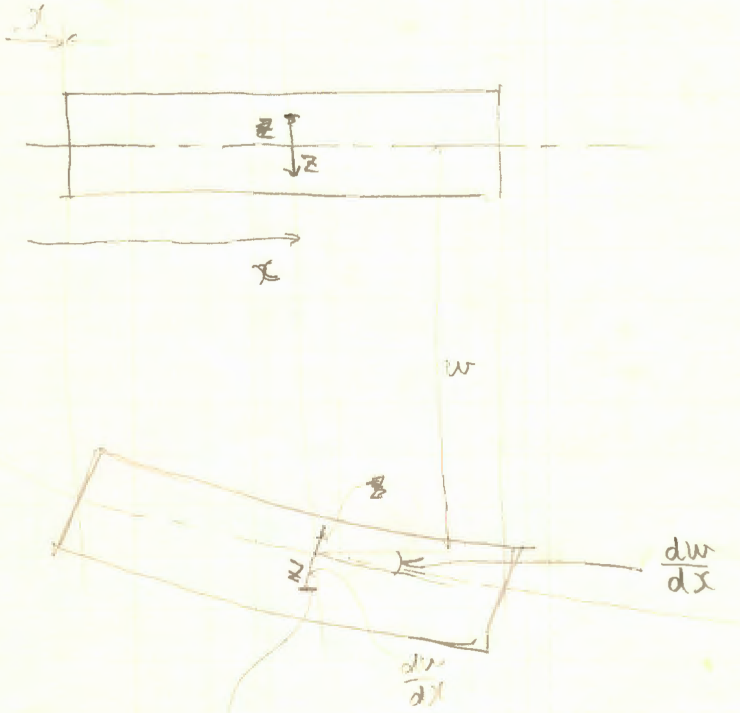
$$\{dP\}^T = \{dP_{12}, dP_{13}, dP_{14}, dP_{23}, dP_{24}, dP_{34}\}$$

$$\{\delta\}^T \{dP\} = \{dV\}$$

$$\{d\bar{X}\}^T = [d\bar{X}_1, d\bar{X}_2, d\bar{Y}_1, d\bar{Y}_2, d\bar{Y}_3].$$

$$\{\delta_*\}^T = [u_1, u_2, v_1, v_2, v_3].$$

$$\frac{\partial P}{\partial \bar{X}_i} = [\alpha]_i$$



$$u = -z\frac{dw}{dx} \qquad (5,36)$$

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \qquad (5,37)$$

$$\sigma_x = E\,\varepsilon_x = -Ez\frac{\partial^2 w}{\partial x^2} \qquad (5,38)$$

$$M = \iint \sigma_x z\,dy\,dz = -\iint Ez^2\frac{\partial^2 w}{\partial x^2}dy\,dz = -EI\frac{\partial^2 w}{\partial x^2}$$

$$I = \iint z^2\,dy\,dz$$

P62

(問B)

$$\Pi = \frac{1}{2}\int_0^l EI(w'')^2 dx - \int_0^l \bar{p} w dx + \bar{M} w'(l) - \bar{F} w(l)$$

$$\delta\Pi = \int_0^l EI w'' \delta w'' dx - \int_0^l \bar{p}\delta w dx + \bar{M}\delta w'(l) - \bar{F}\delta w(l)$$

ここで $\int_0^l EI w'' \delta w'' dx$

$$= \left[EI w'' \delta w'\right]_0^l - \int_0^l (EI w'')' \delta w' dx$$

$$= \left[EI w'' \delta w'\right]_0^l - \left[(EI w'')' \delta w\right]_0^l + \int_0^l (EI w'')'' \delta w dx$$

故に

$$\delta\Pi = \left[EI w'' \delta w'\right]_0^l - \left[(EI w'')' \delta w\right]_0^l + \int_0^l (EI w'')'' \delta w dx$$

$$- \int_0^l \bar{p}\delta w dx + \bar{M}\delta w'(l) - \bar{F}\delta w(l) = 0$$

$x=0$ において $w=0$ $w'=0$

$$\delta\Pi = \int_0^l \left[(EI w'')'' - \bar{p}\right]\delta w dx + \left[EI w''(l) + \bar{M}\right]\delta w'(l)$$

$$- \left[\{EI w''(l)\}' + \bar{F}\right]\delta w(l)$$

$$\underbrace{- EI w''(0)\cdot\delta w'(0)}_{0} + \underbrace{\{EI w''(0)\}'\delta w(0)}_{0}$$ 付帯条件

$$= \int_0^l \left[(EI w'')'' - \bar{p}\right]\delta w dx + \left[EI w''(l) + \bar{M}\right]\delta w'(l)$$

$$- \left[\{EI w''(l)\}' + \bar{F}\right]\delta w(l)$$

ここで　$x=\ell$ において　δw , $\delta w'$ は任意である。
$x=0$ をのぞいて　δw は任意

故に　$(EIw'')'' - \overline{P} = 0$ ①

$EIw''(\ell) + \overline{M} = 0$ ②

$\{EIw''(\ell)\}' + \overline{F} = 0$ ③

① より
$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) = \overline{P}$$

② より　$x=\ell$ において
$$-EIw'' = \overline{M}$$

③ より　$x=\ell$ において
$$-(EIw'')' = \overline{F}$$

P63.
問4

$$\iiint \sigma_x \delta \varepsilon_x \, dx\,dy\,dz - \int_0^l \bar{p}\,\delta w\,dx + \bar{M}\,\delta w'(l) - \bar{F}\,\delta w(l) = 0 \quad —①$$

$$\delta \varepsilon_x = -z\frac{d^2 \delta w}{dx^2} = -z\,\delta w''$$

$$\iiint \sigma_x \delta \varepsilon_x \, dx\,dy\,dz = -\iiint \sigma_x z\,\delta w''\,dx\,dy\,dz$$

$$= -\int_0^l \left[\iint \sigma_x z\,dy\,dz\right] \delta w''\,dx$$

$$= -\int_0^l M\,\delta w''\,dx$$

$$= -\left[M\,\delta w'\right]_0^l + \int_0^l M'\,\delta w'\,dx$$

$$= -\left[M\,\delta w'\right]_0^l + \left[M'\,\delta w\right]_0^l - \int_0^l M''\,\delta w\,dx$$

$\delta w(0) = \delta w'(0) = 0$ より、

$$= -M(l)\cdot\delta w'(l) + M'(l)\cdot\delta w(l) - \int_0^l M''\,\delta w\,dx$$

① に代入して

$$-M(l)\cdot\delta w'(l) + M'(l)\,\delta w(l) - \int_0^l M''\,\delta w\,dx$$

$$-\int_0^l \bar{p}\,\delta w\,dx + \bar{M}\,\delta w'(l) - \bar{F}\,\delta w(l) = 0$$

$$\therefore\ -\int_0^l (M'' + \bar{p})\,\delta w\,dx + (\bar{M} - M(l))\,\delta w'(l)$$

$$+ \left[M'(l) - \bar{F}\right] \delta w(l) = 0$$

$$\frac{\partial \Pi_c}{\partial P_{13}} = \sum \frac{P\ell}{A \cdot E} \cdot \frac{\partial P}{\partial P_{13}}$$

$$= -\sin\theta\,\delta_{12} + \delta_{13} - \cos\theta\,\delta_{14} - \cos\theta\,\delta_{23} + \delta_{24} - \sin\theta\,\delta_{34}$$

$$= -(\delta_{12} + \delta_{34})\sin\theta - (\delta_{14} + \delta_{23})\cos\theta + \delta_{13} + \delta_{24} = 0$$

[問1]

$$\delta_{12} = (u_2 - u_1)\lambda + (v_2 - v_1)\mu = -(v_2 - v_1)$$
$$\delta_{13} = (u_3 - u_1)\lambda + (v_3 - v_1)\mu = +u_1\cos\theta + (v_3 - v_1)\sin\theta$$
$$\delta_{14} = (u_4 - u_1)\lambda + (v_4 - v_1)\mu = +u_1$$
$$\delta_{23} = (u_3 - u_2)\lambda + (v_3 - v_2)\mu = +u_2$$
$$\delta_{24} = (u_4 - u_2)\lambda + (v_4 - v_2)\mu \quad + u_2\cos\theta - v_2\sin\theta$$
$$\delta_{34} = (u_4 - u_3)\lambda + (v_4 - v_3)\mu = -v_3$$

$\therefore$

$$\delta_{12} = v_1 - v_2 \quad ①$$
$$\delta_{13} = u_1\cos\theta + (v_1 - v_3)\sin\theta \quad ②$$
$$\delta_{14} = u_1 \quad ③$$
$$\delta_{23} = u_2 \quad ④ \qquad (5,28)$$
$$\delta_{24} = u_2\cos\theta - v_2\sin\theta \quad ⑤$$
$$\delta_{34} = -v_3 \quad ⑥$$

③，⑥を②に代入

$$\delta_{13} = \delta_{14}\cos\theta + \delta_{34}\sin\theta + v_1\sin\theta \quad ⑦$$

④を⑤に代入

$$\delta_{24} = \delta_{23}\cos\theta - v_2\sin\theta \quad ⑧$$
$$\delta_{12} = v_1 - v_2$$

①を⑧に代入

$$\delta_{24} = \delta_{23}\cos\theta + (\delta_{12} - v_1)\sin\theta$$
$$= \delta_{23}\cos\theta + \delta_{12}\sin\theta - v_1\sin\theta \quad ⑨$$

⑦＋⑨ 辺々

$$\delta_{13} + \delta_{24} = (\delta_{14} + \delta_{23})\cos\theta + (\delta_{12} + \delta_{34})\sin\theta$$

$$\therefore -(\delta_{12} + \delta_{34})\sin\theta - (\delta_{14} + \delta_{23})\cos\theta + \delta_{13} + \delta_{24} = 0$$

(5.65) → (5.66) の計算.

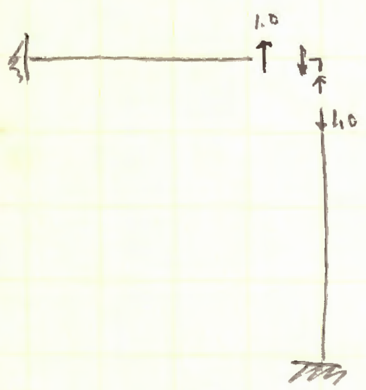
$$\frac{\partial \Pi_c}{\partial N_{12}} = \frac{\partial V_{12}}{\partial N_{12}} + \frac{\partial V_{23}}{\partial F_{23}} \cdot \frac{\partial F_{23}}{\partial N_{12}} = \delta_{12}^{N} + \delta_{23}^{F} = 0$$

$$\frac{\partial \Pi_c}{\partial F_{12}} = \frac{\partial V_{12}}{\partial F_{12}} + \frac{\partial V_{23}}{\partial N_{23}} \cdot \frac{\partial N_{23}}{\partial F_{12}} = \delta_{12}^{F} - \delta_{23}^{N} = 0$$

$$\frac{\partial \Pi_c}{\partial M_{12}} = \frac{\partial V_{12}}{\partial M_{12}} + \frac{\partial V_{23}}{\partial M_{23}} \cdot \frac{\partial M_{23}}{\partial M_{12}} = \delta_{12}^{M} - \delta_{23}^{M} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & +1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \delta_{12}^{N} \\ \delta_{12}^{F} \\ \delta_{12}^{M} \\ \delta_{23}^{N} \\ \delta_{23}^{F} \\ \delta_{23}^{M} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

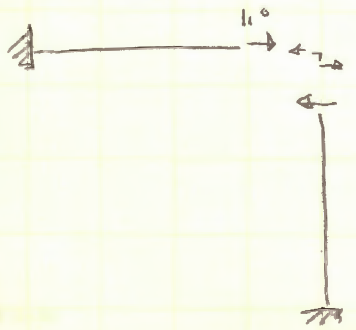
補仮想仕事から求める。（問題2.）



→

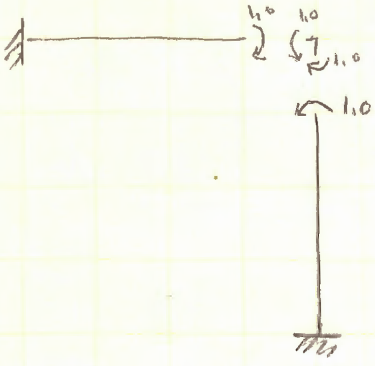
$$\delta_{12}^{F} \times 1.0 - \delta_{23}^{N} \times 1.0 = 0$$

$$\therefore \delta_{12}^{F} - \delta_{23}^{N} = 0$$



$$1.0 \times \delta_{12}^{N} + 1.0 \times \delta_{23}^{F} = 0$$

$$\therefore \delta_{12}^{N} + \delta_{23}^{F} = 0$$



$$1{,}0 \times \delta_{12}^{M} - 1{,}0 \times \delta_{23}^{M} = 0$$

$$\therefore \delta_{12}^{M} - \delta_{23}^{M} = 0$$

$$Z_1 = EIw'''(0)$$

$$= EI\left(\frac{12}{l^3}w_1 - \frac{6}{l^2}\theta_1 - \frac{12}{l^3}w_2 - \frac{6}{l^2}\theta_2\right)$$

$$= \frac{EI}{l^3}(12w_1 - 6l\theta_1 - 12w_2 - 6l\theta_2)$$

$$Z_2 = -EIw'''(l)$$

$$= EI\left(-\frac{12}{l^3}w_1 + \frac{6}{l^2}\theta_1 + \frac{12}{l^3}w_2 + \frac{6}{l^2}\theta_2\right)$$

$$= \frac{EI}{l^3}(-12w_1 + 6l\theta_1 + 12w_2 + 6l\theta_2)$$

$$M_1 = EIw''(0)$$

$$= EI\left(-\frac{6}{l^2}w_1 + \frac{4}{l}\theta_1 + \frac{6}{l^2}w_2 + \frac{2}{l}\theta_2\right)$$

$$= \frac{EI}{l^3}(-6lw_1 + 4l^2\theta_1 + 6lw_2 + 2l^2\theta_2)$$

$$M_2 = -EIw''(l)$$

$$= -EI\left[\left\{-\frac{6}{l^2} + \frac{12}{l^2}\right\}w_1 + \left\{\frac{4}{l} - \frac{6}{l}\right\}\theta_1 + \left(\frac{6}{l^2} - \frac{12}{l^2}\right)w_2 + \left(\frac{2}{l} - \frac{6}{l}\right)\theta_2\right]$$

$$= EI\left(-\frac{6}{l^2}w_1 + \frac{2}{l}\theta_1 + \frac{6}{l^2}w_2 + \frac{4}{l}\theta_2\right)$$

$$= \frac{EI}{l^3}(-6lw_1 + 2l^2\theta_1 + 6lw_2 + 4l^2\theta_2) \longrightarrow (5.48)$$

$$EIw'''' = 0$$

$$w'''' = 0$$

$$w = C_0 + C_1 x + C_2 x^2 + C_3 x^3$$
$$w' = C_1 + 2C_2 x + 3C_3 x^2$$

$x = 0 \longrightarrow$ $w = w_1$ ①
$w' = -\theta_1$ ②

$x = l \longrightarrow$ $w = w_2$ ③
$w' = -\theta_2$ ④

① より $C_0 = w_1$
② より $C_1 = -\theta_1$

$$w = w_1 - \theta_1 x + C_2 x^2 + C_3 x^3$$

$$w' = -\theta_1 + 2C_2 x + 3C_3 x^2$$

③ より $w_1 - \theta_1 l + C_2 l^2 + C_3 l^3 = w_2$ ⑤

④ より $-\theta_1 + 2C_2 l + 3C_3 l^2 = -\theta_2$ ⑥

⑤ × 2 $2w_1 - 2\theta_1 l + 2C_2 l^2 + 2C_3 l^3 = 2w_2$

⑥ × l $-\theta_1 l + 2C_2 l^2 + 3C_3 l^3 = -\theta_2 l$ −

$$2w_1 - 2\theta_1 l + \theta_1 l - C_3 l^3 = 2w_2 + \theta_2 l$$

$$C_3 = \frac{2w_1 - 2w_2 + l\theta_1 - l\theta_2}{l^3}$$

$$= \frac{2}{l^3} w_1 - \frac{2}{l^3} w_2 + \frac{1}{l^2}\theta_1 - \frac{1}{l^2}\theta_2$$

⑤ × 3 $\quad 3w_1 - 3l\theta_1 + 3l^2C_2 + 3l^3C_3 = 3w_2$

⑥ × l $\quad -l\theta_1 + 2l^2C_2 \quad 3l^3C_3 = -l\theta_2$ (−

$$3w_1 - 3l\theta_1 + l\theta_1 + l^2C_2 = 3w_2 + l\theta_2$$

$$C_2 = \frac{-3w_1 + 3w_2 + 2l\theta_1 + l\theta_2}{l^2}$$

$$= -\frac{3}{l^2}w_1 + \frac{3}{l^2}w_2 + \frac{2}{l}\theta_1 + \frac{1}{l}\theta_2$$

$$w = w_1 - \theta_1 x + \left(-\frac{3}{l^2}w_1 + \frac{3}{l^2}w_2 + \frac{2}{l}\theta_1 + \frac{1}{l}\theta_2\right)x^2$$

$$+\left(\frac{2}{l^3}w_1 - \frac{2}{l^3}w_2 - \frac{1}{l^2}\theta_1 - \frac{1}{l^2}\theta_2\right)x^3$$

$$= \left[1 - 3\left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)^3\right]w_1 - \left[\frac{x}{l} - 2\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3\right]\theta_1 l$$

$$+\left[3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3\right]w_2 - \left[-\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3\right]\theta_2 l$$

$$w' = \left[-6\frac{x}{l}\frac{1}{l} + 6\left(\frac{x}{l}\right)^2\cdot\frac{1}{l}\right]w_1 - \left[\frac{1}{l} - 4\left(\frac{x}{l}\right)\cdot\frac{1}{l} + 3\left(\frac{x}{l}\right)^2\cdot\frac{1}{l}\right]\theta_1 l$$

$$+\left[6\left(\frac{x}{l}\right)\cdot\frac{1}{l} - 6\left(\frac{x}{l}\right)^2\cdot\frac{1}{l}\right]w_2 - \left[-2\left(\frac{x}{l}\right)\cdot\frac{1}{l} + 3\left(\frac{x}{l}\right)^2\cdot\frac{1}{l}\right]\theta_2 l$$

$$= \left[-\frac{6}{l^2}x + \frac{6}{l^3}x^2\right]w_1 - \left[1 - \frac{4}{l}x + \frac{3}{l^2}x^2\right]\theta_1$$

$$+\left[\frac{6}{l^2}x - \frac{6}{l^3}x^2\right]w_2 - \left[-\frac{2}{l}x + \frac{3}{l^2}x^2\right]\theta_2$$

$$w'' = \left[-\frac{6}{l^2} + \frac{12}{l^3}x\right]w_1 - \left[-\frac{4}{l} + \frac{6}{l^2}x\right]\theta_1$$

$$+\left[\frac{6}{l^2} - \frac{12}{l^3}x\right]w_2 - \left[-\frac{2}{l} + \frac{6}{l^2}x\right]\theta_2$$

$$w''' = \frac{12}{l^3}w_1 - \frac{6}{l^2}\theta_1 - \frac{12}{l^3}w_2 - \frac{6}{l^2}\theta_2$$

$$U = \frac{1}{2}\int_0^l EI(w'')^2 dx$$

$$w'' = \left(-\frac{6}{l^2} + \frac{12}{l^3}x\right)w_1 + \left(\frac{4}{l} - \frac{6}{l^2}x\right)\theta_1$$
$$+ \left(\frac{6}{l^2} - \frac{12}{l^3}x\right)w_2 + \left(\frac{2}{l} - \frac{6}{l^2}x\right)\theta_2$$

$$= \left(-\frac{6}{l^2}w_1 + \frac{4}{l}\theta_1 + \frac{6}{l^2}w_2 + \frac{2}{l}\theta_2\right)$$
$$+ \left(\frac{12}{l^3}w_1 - \frac{6}{l^2}\theta_1 - \frac{12}{l^3}w_2 - \frac{6}{l^2}\theta_2\right)x$$
$$= \left\{\frac{6}{l^2}(w_2 - w_1) + \frac{2}{l}(2\theta_1 + \theta_2)\right\}$$
$$+ \left\{-\frac{12}{l^3}(w_2 - w_1) - \frac{6}{l^2}(\theta_1 + \theta_2)\right\}x$$

$$(w'')^2 = \frac{36}{l^4}(w_2 - w_1)^2 + \frac{4}{l^2}(2\theta_1 + \theta_2)^2 + \frac{24}{l^3}(w_2 - w_1)(2\theta_1 + \theta_2)$$
$$+ 2\left\{\frac{6}{l^2}(w_2 - w_1) + \frac{2}{l}(2\theta_1 + \theta_2)\right\}\left\{-\frac{12}{l^3}(w_2 - w_1) - \frac{6}{l^2}(\theta_1 + \theta_2)\right\}x$$
$$+ \left\{\frac{144}{l^6}(w_2 - w_1)^2 + \frac{144}{l^5}(w_2 - w_1)(\theta_1 + \theta_2) + \frac{36}{l^4}(\theta_1 + \theta_2)^2\right\}x^2$$

$$\int_0^l (w'')^2 dx = \frac{36}{l^3}(w_2 - w_1)^2 + \frac{4}{l}(2\theta_1 + \theta_2)^2 + \frac{24}{l^2}(w_2 - w_1)(2\theta_1 + \theta_2)$$
$$- \left\{\frac{6}{l}(w_2 - w_1) + 2(2\theta_1 + \theta_2)\right\}\left\{\frac{12}{l^2}(w_2 - w_1) + \frac{6}{l}(\theta_1 + \theta_2)\right\}$$
$$+ \left\{\frac{48}{l^3}(w_2 - w_1)^2 + \frac{48}{l^2}(w_2 - w_1)(\theta_1 + \theta_2) + \frac{12}{l}(\theta_1 + \theta_2)^2\right\}$$

$$\therefore \frac{U}{EI} = \frac{18}{l^3}(w_2-w_1)^2 + \frac{2}{l}(2\theta_1+\theta_2)^2 + \frac{12}{l^2}(w_2-w_1)(2\theta_1+\theta_2)$$

$$- \frac{36}{l^3}(w_2-w_1)^2 - \frac{18}{l^2}\cancel{(w_2-w_1)(\theta_1+\theta_2)} - \frac{12}{l^2}(w_2-w_1)(2\theta_1+\theta_2)$$

$$- \frac{6}{l}(\theta_1+\theta_2)(2\theta_1+\theta_2)$$

$$+ \frac{24}{l^3}(w_2-w_1)^2 + \frac{24}{l^2}\cancel{(w_2-w_1)(\theta_1+\theta_2)} + \frac{6}{l}(\theta_1+\theta_2)^2$$

$$= \frac{6}{l^3}(w_2-w_1)^2 + \frac{6}{l^2}(w_2-w_1)(\theta_1+\theta_2)$$

$$+ \frac{6}{l}(\theta_1+\theta_2)^2 - \frac{6}{l}(\theta_1+\theta_2)(2\theta_1+\theta_2) + \frac{2}{l}(2\theta_1+\theta_2)^2$$

$$\frac{\partial}{\partial w_1}\left(\frac{U}{EI}\right) = -\frac{12}{l^3}(w_2-w_1) - \frac{6}{l^2}(\theta_1+\theta_2)$$

$$= \frac{1}{l^3}\left(12w_1 - 6l\theta_1 - 12w_2 - 6l\theta_2\right)$$

$$\frac{\partial}{\partial \theta_1}\left(\frac{U}{EI}\right) = \frac{6}{l^2}(w_2-w_1) + \frac{12}{l}(\theta_1+\theta_2) - \frac{6}{l}(4\theta_1+3\theta_2) + \frac{8}{l}(2\theta_1+\theta_2)$$

$$= \frac{6}{l^2}(w_2-w_1) + \frac{4}{l}\theta_1 + \frac{2}{l}\theta_2$$

$$= \frac{1}{l^3}\left(-6lw_1 + 4l^2\theta_1 + 6lw_2 + 2l^2\theta_2\right)$$

$$\frac{\partial}{\partial w_2}\left(\frac{U}{EI}\right) = \frac{12}{l^3}(w_2-w_1) + \frac{6}{l^2}(\theta_1+\theta_2)$$

$$= \frac{1}{l^3}\left(-12w_1 + 6l\theta_1 + 12w_2 + 6l\theta_2\right)$$

$$\frac{\partial}{\partial\theta_2}\left(\frac{U}{EI}\right) = \frac{6}{l^2}(\omega_2-\omega_1)+\frac{12}{l}(\theta_1+\theta_2)-\frac{6}{l}(2\theta_2+3\theta_1)+\frac{4}{l}(2\theta_1+\theta_2)$$

$$= \frac{6}{l^2}(\omega_2-\omega_1)+\frac{2}{l}\theta_1+\frac{4}{l}\theta_2$$

$$= \frac{1}{l^3}(-6l\omega_1+2l^2\theta_1+6l\omega_2+4l^2\theta_2)$$

$$M = \begin{cases} \bar{M} - (l - x)\bar{F} & l_Q \leqq x \leqq l \\ \bar{M} - (l-x)\bar{F} - (l_Q - x)\bar{Q} & 0 \leqq x \leqq l_Q \end{cases}$$

$$N = \bar{N} \qquad 0 \leqq x \leqq l$$

$$V = \frac{1}{2}\iiint \frac{\sigma_x^2}{E}\, dx\, dy\, dz$$

$$= \frac{1}{2}\iiint \left\{ \left(\frac{M}{I} z\right)^2 \cdot \frac{1}{E} + \left(\frac{N}{A_0}\right)^2 \frac{1}{E} \right\} dx\, dy\, dz$$

$$= \frac{1}{2}\iiint \frac{M^2}{EI^2} z^2\, dx\, dy\, dz + \frac{1}{2}\iiint \frac{N^2}{A_0^2 E}\, dx\, dy\, dz$$

$$= \frac{1}{2}\int_0^l \frac{M^2}{EI}\, dx + \frac{1}{2}\int_0^l \frac{N^2}{EA_0}\, dx$$

$$= \frac{1}{2EI}\int_0^{l_Q} \left\{ \bar{M} - l\bar{F} - l_Q\bar{Q} + (\bar{F} + \bar{Q})x \right\}^2 dx$$

$$+ \frac{1}{2EI}\int_{l_Q}^{l} \left\{ \bar{M} - l\bar{F} + \bar{F}x \right\}^2 dx$$

$$+ \frac{1}{2EA_0}\int_0^l \bar{N}^2\, dx$$

$$= \frac{1}{2EI}\left[(\bar{M} - l\bar{F} - l_Q\bar{Q})^2 x + \cancel{2}\,1\,(\bar{M} - l\bar{F} - l_Q\bar{Q})(\bar{F} + \bar{Q})x^2 + \frac{1}{3}(\bar{F} + \bar{Q})^2 x^3 \right]_0^l$$

$$+ \frac{1}{2EI}\left[(\bar{M} - l\bar{F})^2 x + (\bar{M} - l\bar{F})\bar{F}x^2 + \frac{1}{3}\bar{F}^2 x^3 \right]_{l_Q}^{l}$$

$$+ \frac{1}{2EA_0}\left[\cancel{3}\,1\,\bar{M}^2 x \right]_0^l$$

$$= \frac{1}{2EI}\left[(\bar{M} - \ell\bar{F} - \ell_a\bar{Q})^2 \ell + (\bar{M} - \ell\bar{F} - \ell_a\bar{Q})(\bar{F} + \bar{Q})\ell^2 + \frac{1}{3}(\bar{F} + \bar{Q})^2 \ell^3 \right]$$

$$+ \frac{1}{2EI}\left[(\bar{M} - \ell\bar{F})^2 (\ell - \ell_a) + (\bar{M} - \ell\bar{F})\bar{F}(\ell^2 - \ell_a^2) + \frac{1}{3}\bar{F}^2 (\ell^3 - \ell_a^3) \right]$$

$$+ \frac{\ell \bar{N}^2}{2EA_a}$$

$$\delta^N = \frac{\partial V}{\partial \bar{N}} = \frac{\ell}{EA_a}\bar{N}$$

$$\delta^F = \frac{\partial V}{\partial \bar{F}} = \frac{1}{2EI}\Big[-2\ell^2(\bar{M} - \ell\bar{F} - \ell_a\bar{Q}) - \ell(\bar{F} + \bar{Q})\ell^2 + (\bar{M} - \ell\bar{F} - \ell_a\bar{Q})\ell^2$$

$$+ \frac{2}{3}(\bar{F} + \bar{Q})\ell^3 - 2\ell(\bar{M} - \ell\bar{F})(\ell - \ell_a)$$

$$+ (\bar{M} - \ell\bar{F})(\ell^2 - \ell_a^2) - \ell\bar{F}(\ell^2 - \ell_a^2) + \frac{2}{3}\bar{F}(\ell^3 - \ell_a^3) \Big]$$

$$= \frac{1}{2EI}\Big\{ \Big[\cancel{2\ell^3} - \cancel{\ell^3} - \cancel{\ell^3} + \frac{2}{3}\ell^3 + 2\ell^2(\ell - \ell_a) - \ell(\ell^2 - \ell_a^2) - \ell(\ell^2 - \ell_a^2)$$

$$+ \frac{2}{3}(\ell^3 - \ell_a^3) \Big] \bar{F}$$

$$+ \left[-2\ell^2 + \ell^2 - 2\ell(\ell - \ell_a) + \ell^2 - \ell_a^2 \right] \bar{M}$$

$$+ \left[2\ell^2\ell_a - \ell^3 - \ell_a\ell^2 + \frac{2}{3}\ell^3 \right] \bar{Q} \Big\}$$

$$= \frac{1}{2EI}\Big\{$$

5.58式

(b).

$$\int_0^{l_a} w''(l-x)\,dx + \int_{l_a}^{l} w''(l-x)\,dx$$

$$= \left[w'(l-x)\right]_0^{l_a} + \int_0^{l_a} w'\,dx + \left[w'(l-x)\right]_{l_a}^{l} + \int_{l_a}^{l} w'\,dx$$

$$= w'_{(l_a)}(l-l_a) - w'_{(0)} l + w(l_a) - w(0) + \cancel{w(l)}\, w'(l_a)(l_a - l) + w(l) - w(l_a)$$

$$w'(\cancel{l_a}) \times l - w'(\cancel{l_a}) \times l_a - w'(0)\,l + w(l_a) - w(0) + \cancel{w'(l_a)\,l_a} - \cancel{w'(l_a)\cdot l}$$
$$+ w(l) - w(\cancel{l_a})$$

$$= w(l) - w(0) - w'(0)\,l.$$

(d) $$\int_0^{l_a} w''(l_a - x)\,dx$$

$$= \left[w'(l_a - x)\right]_0^{l_a} + \int_0^{l_a} w'\,dx$$

$$= -w'(l_a)\,l_a + w(l_a) - w(0)$$

//

THE PRINCIPLE OF COMPLEMENTARY VIRTUAL WORK.

The sum of the complementary virtual work done in arbitrary infinitesimal virtual stress variations satisfying the equations of equilibrium and prescribed mechanical boundary conditions is zero.

$$\iiint_V (\varepsilon_x \delta\sigma_x + \varepsilon_y \delta\sigma_y + \varepsilon_z \delta\sigma_z + \gamma_{yz}\delta\tau_{yz} + \gamma_{zx}\delta\tau_{zx} + \gamma_{xy}\delta\tau_{xy})\,dV$$

$$-\iint_{S_u} (\bar{u}\,\delta X_\nu + \bar{v}\,\delta Y_\nu + \bar{w}\,\delta Z_\nu)\,dS = 0$$

付帯条件

(1) 釣り合い条件．

$$\frac{\partial \delta\sigma_x}{\partial x} + \frac{\partial \delta\tau_{xy}}{\partial y} + \frac{\partial \delta\tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \delta\tau_{xy}}{\partial x} + \frac{\partial \delta\sigma_y}{\partial y} + \frac{\partial \delta\tau_{yz}}{\partial z} = 0$$

$$\frac{\partial \delta\tau_{zx}}{\partial x} + \frac{\partial \delta\tau_{yz}}{\partial y} + \frac{\partial \delta\sigma_z}{\partial z} = 0$$

(2) 力学的境界条件　S_σ上；　$\delta X_\nu = 0$，$\delta Y_\nu = 0$，$\delta Z_\nu = 0$

⇒ ① 変位の適合条件および幾何学的境界条件と等価。

② 応力ひずみ関係にはまったく無関係に成立する。

THE PRINCIPLE OF MINIMUM POTENTIAL ENERGY.

Among all the admissible displacements u, v and w which satisfy the prescribed geometrical boundary conditions, the actual displacements make the total potential energy stationary.

(an absolute minimum.)

$$\Pi_p = \iiint_V A(u,v,w)\,dV - \iiint_V (\bar{X}u + \bar{Y}v + \bar{Z}w)\,dV$$

$$- \iint_{S_\sigma} (\bar{X}_\nu u + \bar{Y}_\nu v + \bar{Z}_\nu w)\,dS$$

$$\delta \Pi_p = 0 \quad (\text{or} \quad \Pi_{p\,\text{admissible}} \geq \Pi_{p\,\text{actual}})$$

付帯条件 (1) 幾何学的境界条件　S_u上　$u=\bar{u}$, $v=\bar{v}$, $w=\bar{w}$

(2) ひずみエネルギ関数が一義的に決まる。

$\Longrightarrow$ ひずみエネルギ関数が存在すれば、仮想仕事の原理と等価である。

THE PRINCIPLE OF MINIMUM COMPLEMENTARY ENERGY.

Among all the sets of admissible stresses $\sigma_x, \sigma_y, \cdots$ and τ_{xy}, which satisfy the equations of equilibrium and the prescribed mechanical boundary conditions on S_σ, the set of actual stress components makes the total complementary energy Π_c ~~to~~ an absolute minimum.

$$\Pi_c = \iiint_V B(\sigma_x, \sigma_y, \cdots, \tau_{xy})\, dV - \iint_{S_u} (\bar{u} X_\nu + \bar{v} Y_\nu + \bar{w} Z_\nu)\, dS$$

$$\delta \Pi_c = 0 \quad (\text{or } \Pi_c^* \geq \Pi_c \;)$$

付帯条件　(1) 釣合い条件

(2) 力学的境界条件.

(3) コンプリメンタリ・エネルギ関数の存在。

⇒ コンプリメンタリエネルギ関数の存在が保証されれば、補仮想仕事の原理と等価である。

THE PRINCIPLE OF VIRTUAL WORK

The sum of virtual work done by the ~~inertia~~ internal and external forces in arbitrary infinitesimal virtual displacements satisfying the prescribed geometrical boundary conditions is zero

$$\iiint_V (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx} + \tau_{xy}\delta\gamma_{xy})\,dV$$

$$- \iiint_V (\bar{X}\delta u + \bar{Y}\delta v + \bar{Z}\delta w)\,dV - \iiint_{S_\sigma} (\bar{X}_\nu \delta u + \bar{Y}_\nu \delta v + \bar{Z}_\nu \delta w)\,dS = 0$$

付帯条件

(1) ひずみ変位関係式

(2) 幾何学的境界条件　S_u上； $\delta u = 0$, $\delta v = 0$, $\delta w = 0$

⇒ ① 釣り合い条件 および 力学的境界条件 と等価。

② 応力・ひずみ関係には まったく無関係に成立する。

$$\delta = (u_2-u_1)\lambda + (v_2-v_1)\mu + (w_2-w_1)\nu$$

$$U = \frac{1}{2}\frac{EA_0}{l}\delta^2 = \frac{1}{2}\frac{EA_0}{l}$$

$$\Big\{ (u_2-u_1)^2\lambda^2 + (v_2-v_1)^2\mu^2 + (w_2-w_1)^2\nu^2$$

$$+ 2(u_2-u_1)(v_2-v_1)\lambda\mu$$

$$+ 2(v_2-v_1)(w_2-w_1)\mu\nu$$

$$+ 2(w_2-w_1)(u_2-u_1)\nu\lambda \Big\}$$

$\times \frac{EA_0}{l}$

$$F_{x1} = \frac{\partial U}{\partial u_1} = (u_1-u_2)\lambda^2 - (v_2-v_1)\lambda\mu - (w_2-w_1)\nu\lambda.$$

$$= \lambda^2 u_1 - \lambda^2 u_2 + \lambda\mu v_1 - \lambda\mu v_2 + \nu\lambda w_1 - \nu\lambda w_2$$

$$F_{y1} = \frac{\partial U}{\partial v_1} = (v_1-v_2)\mu^2 - (u_2-u_1)\lambda\mu - (w_2-w_1)\mu\nu$$

$$= \lambda\mu u_1 - \lambda\mu u_2 + \mu^2 v_1 - \mu^2 v_2 + \mu\nu w_1 - \mu\nu w_2$$

$$F_{z1} = \frac{\partial U}{\partial w_1} = (w_1-w_2)\nu^2 - (v_2-v_1)\mu\nu - (u_2-u_1)\nu\lambda.$$

$$= \nu\lambda u_1 - \nu\lambda u_2 + \mu\nu v_1 - \mu\nu v^2 + \nu^2 w_1 - \nu^2 w_2$$

$$F_{x2} = \frac{\partial U}{\partial u_2} = (u_2-u_1)\lambda^2 + (v_2-v_1)\lambda\mu + (w_2-w_1)\nu\lambda.$$

$$F_{y2} = \frac{\partial U}{\partial v_2} = (v_2-v_1)\mu^2 + (u_2-u_1)\lambda\mu + (w_2-w_1)\mu\nu$$

$$F_{z2} = \frac{\partial U}{\partial w_2} = (w_2-w_1)\nu^2 + (v_2-v_1)\mu\nu + (u_2-u_1)\nu\lambda.$$

4-2 補仮想仕事の原理に基づく解法

$$\iint_S (\varepsilon_x \delta\sigma_x + \varepsilon_y \delta\sigma_y + \gamma_{xy}\delta\tau_{xy}) t\,dx\,dy - \int_{C_u} (\bar{u}\,\delta X_\nu + \bar{v}\,\delta Y_\nu) t\,ds = 0 \tag{4.12}$$

Airy の stress function

$$\sigma_x = \frac{\partial^2 F}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{2.34}$$

$$X_\nu = \sigma_x \cdot l + \tau_{xy} \cdot m = \frac{\partial^2 F}{\partial y^2} \cdot \frac{dy}{ds} + \frac{\partial^2 F}{\partial x \partial y} \cdot \frac{dx}{ds} = \frac{d}{ds}\left(\frac{\partial F}{\partial y}\right)$$

$$Y_\nu = \tau_{xy} l + \sigma_y m = -\frac{\partial^2 F}{\partial x \partial y} \cdot \frac{dy}{ds} - \frac{\partial^2 F}{\partial x^2} \cdot \frac{dx}{ds} = -\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right)$$

仮定. $F(x, y) = F_0(x, y) + \sum_{r=1}^{n} a_r F_r(x, y)$

$$\frac{d}{ds}\left(\frac{\partial F_0}{\partial y}\right) = \overline{X_\nu}, \quad -\frac{d}{ds}\left(\frac{\partial F_0}{\partial x}\right) = \overline{Y_\nu} \tag{4.14}$$

$$\frac{d}{ds}\left(\frac{\partial F_r}{\partial y}\right) = 0, \quad -\frac{d}{ds}\left(\frac{\partial F_r}{\partial x}\right) = 0 \quad ; \quad r = 1, 2, \cdots, n$$

; C_σ上において、

$$\left.\begin{aligned}
\sigma_x &= \frac{\partial^2 F_0}{\partial y^2} + \sum_{r=1}^{n} a_r \frac{\partial^2 F_r}{\partial y^2} \\
\sigma_y &= \frac{\partial^2 F_0}{\partial x^2} + \sum_{r=1}^{n} a_r \frac{\partial^2 F_r}{\partial x^2} \\
\tau_{xy} &= -\frac{\partial^2 F_0}{\partial x \partial y} + \sum_{r=1}^{n} a_r \frac{\partial^2 F_r}{\partial x \partial y}
\end{aligned}\right| \tag{4.15}$$

Chapter 4 近似解法

4-1 仮想仕事の原理に基礎をおく.

［二次元］（体積力なし）

$$\iint_S (\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy}) \cancel{ds}\, t\, dx\, dy$$

$$- \int_{C_\sigma} (\bar{X}_\nu \delta u + \bar{Y}_\nu \delta v) t\, ds = 0 \qquad \text{—— (※)}$$

変位関数の仮定.

$$u(x, y) = u_0(x, y) + \sum_{r=1}^{n} a_r u_r(x, y)$$

$$v(x, y) = v_0(x, y) + \sum_{r=1}^{n} b_r v_r(x, y)$$

(C_u上 ; $u_0 = \bar{u}$, $v_0 = \bar{v}$, $u_r = 0$, $v_r = 0$)

→ 幾何学的境界条件を満たす。

$$\delta u = \sum_{r=1}^{n} \delta a_r u_r \quad , \quad \delta v = \sum_{r=1}^{n} \delta b_r v_r$$

$$(※) \longrightarrow \iint_S \left(\sigma_x \cdot \frac{\partial \delta u}{\partial x} + \sigma_y \frac{\partial \delta v}{\partial y} + \tau_{xy}\left(\frac{\partial \delta v}{\partial x} + \frac{\partial \delta u}{\partial y}\right) \right) t\, dx\, dy$$

$$- \int_{C_\sigma} \left(\bar{X}_\nu \cdot \sum_{r=1}^{n} \delta a_r u_r + \bar{Y}_\nu \sum_{r=1}^{n} \delta b_r v_r \right) t\, dS = 0$$

$$\iint_S \left[\sigma_x \sum_{r=1}^{n} \delta a_r \frac{\partial u_r}{\partial x} + \sigma_y \sum_{r=1}^{n} \delta b_r \frac{\partial v_r}{\partial y} + \tau_{xy} \sum_{r=1}^{n} \left(\delta b_r \frac{\partial v_r}{\partial x} + \delta a_r \frac{\partial u_r}{\partial y} \right) \right] t\,dx\,dy$$

$$- \int_{C_\sigma} \left[\overline{X_\nu} \sum_{r=1}^{n} \delta a_r u_r + \overline{Y_\nu} \sum_{r=1}^{n} \delta b_r v_r \right] t\,ds = 0$$

$$\sum_{r=1}^{n} \iint_S \left[\sigma_x \frac{\partial u_r}{\partial x} + \tau_{xy} \frac{\partial u_r}{\partial y} \right] \delta a_r\, t\,dx\,dy - \sum_{r=1}^{n} \int_{C_\sigma} \overline{X_\nu}\, u_r\, \delta a_r\, t\,ds$$

$$+ \sum_{r=1}^{n} \iint_S \left[\tau_{xy} \frac{\partial v_r}{\partial x} + \sigma_y \frac{\partial v_r}{\partial y} \right] \delta b_r\, t\,dx\,dy - \sum_{r=1}^{n} \int_{C_\sigma} \overline{Y_\nu}\, v_r\, \delta b_r\, t\,ds$$

$$= 0$$

$$\therefore \quad \sum_{r=1}^{n} \left(L_r \delta a_r + M_r \delta b_r \right) = 0$$

ただし

$$\left.\begin{aligned} L_r &= \iint_S \left[\sigma_x \frac{\partial u_r}{\partial x} + \tau_{xy} \frac{\partial u_r}{\partial y} \right] t\,dx\,dy - \int_{C_\sigma} \overline{X_\nu}\, u_r\, t\,ds \\ M_r &= \iint_S \left[\tau_{xy} \frac{\partial v_r}{\partial x} + \sigma_y \frac{\partial v_r}{\partial y} \right] t\,dx\,dy - \int_{C_\sigma} \overline{Y_\nu}\, v_r\, t\,ds \end{aligned}\right\} (※)$$

δa_r, δb_r は任意

$$\therefore \quad L_r = 0, \quad M_r = 0, \quad r = 1, 2, \cdots, n$$

※において部分積分をする.

$$L_r = \int \sigma_x u_r t \overset{l\,dS}{dy} + \int \tau_{xy} u_r t \overset{m\,dS}{dx}$$

$$- \iint_S \frac{\partial \sigma_x}{\partial x} \cdot u_r t \, dx dy - \iint_S \frac{\partial \tau_{xy}}{\partial y} \cdot u_r t \, dx dy$$

$$- \int_{C_\sigma} \overline{X_\nu} u_r t \, dS$$

$$= - \iint_S \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) u_r t \, dx dy$$

$$+ \int_{C_\sigma} (\overset{X_\nu}{\sigma_x l + \tau_{xy} m}) u_r t \, dS - \int_{C_\sigma} \overline{X_\nu} u_r t \, dS$$

∵ C_u上で $u_r = 0$

$$\therefore L_r = - \iint_S \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right) u_r t \, dx dy + \int_{C_\sigma} (X_\nu - \overline{X_\nu}) u_r t \, dS$$

同様に

$$M_r = - \iint_S \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right) v_r t \, dx dy - \int_{C_\sigma} (Y_\nu - \overline{Y_\nu}) v_r t \, dS$$

$$\sigma_x = \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right]$$

$$= \frac{E}{1-\nu^2}\left[\frac{\partial u_0}{\partial x} + \sum_{r=1}^{n} a_r \frac{\partial u_r}{\partial x} + \nu\left(\frac{\partial v_0}{\partial y} + \sum_{r=1}^{n} b_r \frac{\partial v_r}{\partial y}\right)\right]$$

$$\sigma_y = \frac{E}{1-\nu^2}\left[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right]$$

$$= \frac{E}{1-\nu^2}\left[\frac{\partial v_0}{\partial y} + \sum_{r=1}^{n} b_r \frac{\partial v_r}{\partial y} + \nu\left(\frac{\partial u_0}{\partial x} + \sum_{r=1}^{n} a_r \frac{\partial u_r}{\partial x}\right)\right] \qquad (4.10)$$

$$\tau_{xy} = G\cdot\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$= G\left[\frac{\partial v_0}{\partial x} + \sum_{r=1}^{n} b_r \frac{\partial v_r}{\partial x} + \frac{\partial u_0}{\partial y} + \sum_{r=1}^{n} a_r \frac{\partial u_r}{\partial y}\right]$$

→ σ_x, σ_y, τ_{xy} を (4.6) に代入.

→ a_r, b_r に関する $2n$ 個の連立一次方程式を解く。