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NONLINEAR DYNAMIC ANALYSIS OF FRAME STRUCTURES

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Abstract—A geometrically nonlinear dynamic analysis method is presented for frames which may be subjected to finite rotations in three-dimensional space. The proposed method is based on the static geometrically nonlinear analysis method reported by Yoshida *et al.*, in which the governing incremental equilibrium equation is represented by the coordinates after the deformation themselves rather than conventional displacements. The governing dynamic equilibrium equation for each element is obtained from the static equation by adding the inertia term. In the solution procedure, a modified Steffensen's iteration process is introduced and combined with the two-step approximation and iterative correction solution procedure developed for static analysis. A numerical example of a curved cantilever beam under lateral loads indicates the effectiveness of the proposed method in cases with three-dimensional finite rotations. Forced vibration analyses of a two-hinged shallow arch are conducted under centrally concentrated loading with several loading amplitudes. The resulting dynamic buckling load is compared with that given by Gregory and Plaut in 1982, who used Galerkin method, and shows good agreement.

1. INTRODUCTION

A dynamic load may cause an instability of a structure, even if the structure remains stable under a static load of the same magnitude as the dynamic one. Such a phenomenon may occur within the elastic range, so that the maximum displacement response as a function of the magnitude of loading abruptly increases at some point with respect to the magnitude. When this kind of dynamic instability problem is to be solved, geometrical nonlinearity must be considered in the dynamic response analysis in the same manner as in the static instability analysis. Under certain conditions of structural dimensions, loadings, and so on, geometrical nonlinearity must be considered also in dynamic analysis even if no dynamic instability is expected.

To date, studies conducted on dynamic frame analysis method with geometric nonlinearity are mainly concerned with plane frames; such studies dealing with three-dimensional frame structures are seldom found in the literature. One of the reasons for this is that dynamic stability analyses themselves are in general under development, although parametric resonance problems have been well discussed. Another reason may be that even static analysis is difficult for problems with finite rotations in three-dimensional space.

In this paper, a dynamic response analysis method is presented which can deal with frames with finite rotations in the three-dimensional space, and hence dynamic instability of space frames.

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First, an incremental equation of motion is developed. The formulation is an extension of that of the three-dimensional geometrically nonlinear static analysis procedure developed by one of the authors with Yoshida and Matsuda [1, 2]. The static procedure, referred hereafter as the YMM method after the developer's names, is based on the equilibrium equations expressed by coordinates after deformations. Then, the validity of the YMM method for static problems with finite rotations in three-dimensional space is demonstrated using numerical examples. Here effective predictive–corrective solution procedure based on Steffensen's iteration scheme is also illustrated. Finally, the applicability of the equation of motion developed herein is verified through a numerical example.

2. EXTENSION OF THE YMM METHOD TO DYNAMIC PROBLEMS

The basic incremental equilibrium equation of an element in the YMM method is given as follows [1, 2] (see Appendix):

$$\begin{aligned}\Delta \mathbb{f} &= \mathbb{T}^T_{(n+1)} \mathbb{k}^* \, \mathbb{T}_{(n+1)} \Delta \mathbb{u} \\ &\quad + \mathbb{T}^T_{(n+1)} \mathbb{k}^* [\mathbb{T}_{(n+1)} \Delta \mathbb{z} + \Delta \mathbb{T} \\ &\quad \cdot (\mathbb{u}_{(n)} + \mathbb{z}_{(n)})] + \Delta \mathbb{T}^T \cdot \mathbb{f}^*_{(n)} \\ &= \mathbb{k}_{(n+1)} \Delta \mathbb{u} + \mathbb{h}_{(n+1)},\end{aligned} \tag{1}$$

where

$\mathbb{f}, \mathbb{u}$ = the element nodal force and displacement vector, respectively, in the global coordinate system (GCS)

$\mathbb{f}^*, \mathbb{u}^*$ = vectors in the element coordinate system (ECS) corresponding to $\mathbb{f}$ and $\mathbb{u}$

$\mathbb{T}$ = the coordinate transformation matrix from the GCS to the ECS

$\mathbb{z}^T = \langle \mathbb{x}_0^{\mathcal{T}} - \mathbb{r}^T \rangle$

$\mathbb{x}$ = the element nodal coordinate vector in the GCS

$\mathbb{x}_0$ = the element nodal coordinate vector in the GCS at the initial state

$\mathbb{r}$ = the element rigid body rotation vector in the GCS

$\mathbb{k}^*$ = the element stiffness matrix in the ECS; for space frames dealt here, the conventional linear one of a bar element with a size of 12×12

T = a superscript denoting transposition of matrices

(n) = a subscript denoting nth equilibrium state

Δ = a prefix denoting the increments from the nth to the $(n+1)$th equilibrium state.

It is assumed that elements are straight at the initial free state and have constant bi-symmetric cross sections.

The special characteristics of the YMM method lie in the fact that the linear stiffness matrix derived under the assumption of small strain is used as the basic stiffness matrix as shown above, and the geometric nonlinearity, which is usually expressed through the nonlinear relation between strains and displacements within a body, is fully taken into account by faithfully evaluating the transforming relation of the nodal displacement vector from the GCS to the ECS with the use of the coordinates and rigid body rotation. Moreover, the coordinate transformation matrix after the deformation is kept exactly in the final equilibrium equation without any approximation. Finite rotations in the three-dimensional space are treated such that the effect of a series of rotations is considered to be the same as that of a certain single rotation around some axis. Hence, nothing is neglected in eqn (1) compared with the beam theory provided that small strain assumption is valid and that there is no load action within the element.

To extend the static equilibrium equation to the dynamic one, terms such as momentum change rate and damping must be included. In this paper, systems without damping are considered for simplicity. The incremental equation of motion of an element is then obtained as:

$$\begin{aligned}\Delta\mathbb{f} = {} & \mathbb{T}^T_{(n+1)}\,\mathbb{M}^*\,\mathbb{T}_{(n+1)}\,\ddot{\mathbb{u}}_{(n+1)} \\ & - \mathbb{T}^T_{(n)}\,\mathbb{M}^*\,\mathbb{T}_{(n)}\,\ddot{\mathbb{u}}_{(n)} \\ & + \mathrm{d}\,(\mathbb{T}^T_{(n+1)}\,\mathbb{M}^*\,\mathbb{T}_{(n+1)})/\mathrm{d}t\;\dot{\mathbb{u}}_{(n+1)} \\ & - \mathrm{d}\,(\mathbb{T}^T_{(n)}\,\mathbb{M}^*\,\mathbb{T}_{(n)})/\mathrm{d}t\;\dot{\mathbb{u}}_{(n)} \\ & + \mathbb{k}_{(n+1)}\,\Delta\mathbb{u} + \mathbb{h}_{(n+1)} \end{aligned} \tag{2}$$

where

$\mathbb{M}^*$ = the element mass matrix in the ECS

$\cdot$ = a symbol denoting differentiation with respect to time

(n) = a subscript denoting time $n\Delta t$ with Δt a constant.

In eqn (2), the first two lines and the next two lines express the increments of the product terms in the GCS, the mass matrix and the acceleration vector, and the mass matrix time differentiation and the velocity vector, respectively. The fifth line is the spring force term given in the right hand side of eqn (1). The coordinate transformation matrix $\mathbb{T}$ is a function of displacements and therefore of time. However, if concentrated mass matrices are considered as an extreme example, $\mathbb{M}^*$ is equivalent to a unit (identity) matrix multiplied by some constant as far as the part corresponding to translational displacement components is concerned. This implies that $\mathbb{T}_n^T\mathbb{M}^*\mathbb{T}_n$ is also equal to a constant time unit matrix and has no time dependency, since coordinate transformation matrices are orthogonal. Consequently, the second two lines vanish. If this conclusion is also valid, at least approximately, for the part of $\mathbb{M}^*$ corresponding to rotational components, and further for the case of consistent matrices, the second two lines can be neglected compared with the first ones. In this case the following simplified incremental equation of motion is obtained.

$$\begin{aligned}\Delta\mathbb{f} = {} & \mathbb{T}^T_{(n+1)}\,\mathbb{M}^*\,\mathbb{T}_{(n+1)}\,\Delta\ddot{\mathbb{u}} \\ & + [\mathbb{T}^T_{(n+1)}\,\mathbb{M}^*\,\mathbb{T}_{(n+1)} - \mathbb{T}^T_{(n)}\,\mathbb{M}^*\,\mathbb{T}_{(n)}]\,\ddot{\mathbb{u}}_{(n)} \\ & + \mathbb{k}_{(n+1)}\,\Delta\mathbb{u} + \mathbb{h}_{(n+1)} \end{aligned} \tag{3.1}$$

$$\begin{aligned} = {} & \mathbb{M}_{(n+1)}\,\Delta\ddot{\mathbb{u}} + (\Delta\mathbb{T}^T\,\mathbb{M}^*\,\mathbb{T}_{(n+1)} \\ & + \mathbb{T}^T_{(n+1)}\,\mathbb{M}^*\Delta\mathbb{T} + \Delta\mathbb{T}^T\,\mathbb{M}^*\,\Delta\mathbb{T}]\,\ddot{\mathbb{u}}_{(n)} \\ & + \mathbb{k}_{(n+1)}\,\Delta\mathbb{u} + \mathbb{h}_{(n+1)} \end{aligned} \tag{3.2}$$

where the coordinate transformation matrix at the end of the incremental step, $\mathbb{T}_{(n+1)}$, changes in the incremental step, $\Delta\mathbb{T}$ and $\Delta\mathbb{z}$, which is in $\mathbb{h}_{(n+1)}$, and the acceleration increment, $\Delta\ddot{\mathbb{u}}$, are functions of the displacement increment $\Delta\mathbb{u}$. Others are known at the beginning of the incremental step.

3. SOLUTION PROCEDURE FOR THE EQUATION OF MOTION

In [1, 2], a predictive–corrective solution procedure is presented which is composed of a two-step linearizing approximation and successive iterative correction. The procedure is somewhat akin to Runge–Kutta type solution processes but contains more physical aspects rather than pure mathematical formulations, and was proved effective to the system described by eqn (1). A similar procedure can be adopted to the dynamic system expressed by eqn (3), but the procedure needs a time integration scheme. Here, Newmark's β ($\beta = 1/4$) method, which is one of the simplest time integration schemes now available, is combined with the above mentioned YMM solution procedure to produce a solution procedure for the dynamic system given in eqn (3) as in the following.

In the Newmark's β ($\beta = 1/4$) method, the relation of the velocity and displacement components at the beginning and the end of a time step is assumed as:

$$\Delta\dot{\mathbb{u}} = \dot{\mathbb{u}}_{(n+1)} - \dot{\mathbb{u}}_{(n)} = \ddot{\mathbb{u}}_{(n)}\,\Delta t + \Delta\ddot{\mathbb{u}}\,\Delta t/2 \tag{4}$$

$$\Delta\mathbb{u} = \mathbb{u}_{(n+1)} - \mathbb{u}_{(n)} = \dot{\mathbb{u}}_{(n)}\Delta t + \ddot{\mathbb{u}}_{(n)}\,\Delta t^2/2 + \Delta\ddot{\mathbb{u}}\,\beta\Delta t^2. \tag{5}$$

3.1. *First approximation*

Incremental terms in eqn (3.2) are linearized at the nth equilibrium state with respect to the displacement increment $\Delta\mathbb{u}$, and the coordinate transformation matrix is evaluated at the nth equilibrium state. Then the following equation of motion to give the first approximation of $\Delta\mathbb{u}$ ($= \Delta\mathbb{u}_I$) is obtained.

$$\begin{aligned}\Delta\mathbb{f} &= (\mathbb{T}^T\,\mathbb{M}^*\,\mathbb{T})_{(n)}\,\Delta\ddot{\mathbb{u}}_I \\ &\quad + [\{(\partial\mathbb{T}^T/\partial\mathbb{u}^T)\,\mathbb{M}^*\,\mathbb{T}\ddot{\mathbb{u}} + \mathbb{T}^T\,\mathbb{M}^* \\ &\quad \cdot(\partial\mathbb{T}/\partial\mathbb{u}^T)\,\ddot{\mathbb{u}}\} + \{\mathbb{T}^T\,\mathbb{k}^*\,\mathbb{T}\} \\ &\quad + \{\mathbb{T}^T\,\mathbb{k}^*\,[\mathbb{T}\,(\partial\mathbb{z}/\partial\mathbb{u}^T) \\ &\quad + (\partial\mathbb{T}/\partial\mathbb{u}^T)(\mathbb{u}+\mathbb{z})]\} \\ &\quad + \{(\partial\mathbb{T}^T/\partial\mathbb{u}^T)\,\mathbb{f}^*\}]_{(n)}\Delta\mathbb{u}_I \\ &= \mathbb{M}_{(n)}\,\Delta\ddot{\mathbb{u}}_I + [\mathbb{M}' + \mathbb{k} + \mathbb{k}']_{(n)}\Delta\mathbb{u}_I.\end{aligned} \tag{6}$$

Here, eqn (5) is substituted for $\Delta\mathbb{u}_I$ in the above equation to eliminate the displacement increment, which leads to the following equation with the acceleration increment as the only unknown.

$$\begin{aligned}\Delta\mathbb{f} &= \{\mathbb{M} + (\mathbb{M}' + \mathbb{k} + \mathbb{k}')\}_{(n)}\Delta\ddot{\mathbb{u}}_I \\ &\quad + (\mathbb{M}' + \mathbb{k} + \mathbb{k}')_{(n)} \\ &\quad \cdot(\dot{\mathbb{u}}\,\Delta t + \ddot{\mathbb{u}}\,\Delta t^2/2)_{(n)}.\end{aligned} \tag{7}$$

The element equation thus obtained is assembled into an overall equation of motion in the usual way, and the resulting simultaneous equations are solved with respect to acceleration increment $\Delta\ddot{\mathbb{u}}$, which in turn is substituted into eqns (4) and (5), and the first approximate solutions of displacement and velocity increments $\Delta\mathbb{u}_I$ and $\Delta\dot{\mathbb{u}}_I$, respectively, are obtained.

The corresponding solution point to the first approximate solution is symbolically denoted as $(n+1)'$, and the midpoint between the points (n) and $(n+1)'$ is denoted as $(n+1/2)'$.

3.2. *Second approximation*

Second approximate solution triad $\Delta\ddot{\mathbb{u}}_{II}$, $\Delta\dot{\mathbb{u}}_{II}$ and $\Delta\mathbb{u}_{II}$ is given by a similar process to the previous one but based on the first approximation triad, namely, linearization of the incremental terms are conducted at the point $(n+1/2)'$ instead of $(n+1)'$. Thus the equation corresponding to the eqn (6) is

$$\Delta\mathbb{f} = \mathbb{M}_I\,\Delta\ddot{\mathbb{u}}_{II} + (\mathbb{M}'' + \mathbb{k} + \mathbb{k}'')_I\,\Delta\mathbb{u}_{II} \tag{8}$$

where

$$\begin{aligned}\mathbb{M}'' &= (\partial\mathbb{T}^T/\partial\mathbb{u}^T)_{(n+1/2)'}\,\mathbb{M}^*\,\mathbb{T}_I\,\ddot{\mathbb{u}}_{(n)} + \mathbb{T}_I^T\,\mathbb{M}^* \\ &\quad \cdot(\partial\mathbb{T}/\partial\mathbb{u}^T)_{(n+1/2)'}\,\ddot{\mathbb{u}}_{(n)}\end{aligned} \tag{9}$$

$$\begin{aligned}\mathbb{k}'' &= \mathbb{T}_I^T\,\mathbb{k}^*\,\{\mathbb{T}_I\,(\partial\mathbb{z}/\partial\mathbb{u}^T)_{(n+1/2)'} \\ &\quad + (\partial\mathbb{T}/\partial\mathbb{u}^T)_{(n+1/2)'}\,(\mathbb{u}_{(n)} + \mathbb{z}_{(n)})\} \\ &\quad + (\partial\mathbb{T}^T/\partial\mathbb{u}^T)_{(n+1/2)'}\,\mathbb{f}^*_{(n)}.\end{aligned} \tag{10}$$

Here, subscript I denotes that the variables with this subscript are evaluated at the point $(n+1)'$. The corresponding point to the second approximate solution is denoted as $(n+1)''$.

3.3. *Iterative-correction* [(k + 1)*th approximation* (k > 2)]

For the $(k+1)$th approximation, where k is greater than or equal to 2, linearization of incremental terms is not performed. Instead, incremental terms are also approximated by the values estimated at the previous kth approximate point, as the coordinate transformation matrix is. Thus

$$\Delta\mathbb{T}_k = \mathbb{T}_{(n+1)k} - \mathbb{T}_{(n)} = \mathbb{T}_k - \mathbb{T}_{(n)} \tag{11}$$

$$\Delta\mathbb{z}_k = \mathbb{z}_{(n+1)k} - \mathbb{z}_{(n)} = \mathbb{z}_K - \mathbb{z}_{(n)}, \tag{12}$$

where k is a subscript denoting kth approximation.

Substituting eqns (11) and (12) into eqn (3.1), we obtain the equation of motion for the $(k+1)$th approximation as

$$\Delta\mathbb{f} = \mathbb{M}_k\,\Delta\ddot{\mathbb{u}}_{k+1} + \mathbb{k}_k\,\Delta\mathbb{u}_{k+1} + \mathbb{q}_k + \mathbb{h}_k, \tag{13}$$

where $\mathbb{q}_k$ and $\mathbb{h}_k$ are known and are expressed as

$$\mathbb{q}_k = \{(\mathbb{T}^T\,\mathbb{M}^*\,\mathbb{T})_k - (\mathbb{T}^T\,\mathbb{M}^*\mathbb{T})_{(n)}\}\,\ddot{\mathbb{u}}_{(n)} \tag{14}$$

$$\begin{aligned}\mathbb{h}_k &= \mathbb{T}_k^T\,\mathbb{k}^*\,[\mathbb{T}_k\,\Delta\mathbb{z}_k + \Delta\mathbb{T}_k \\ &\quad \cdot(\mathbb{u}_{(n)} + \mathbb{z}_{(n)})] + \Delta\mathbb{T}_k^T\,\mathbb{f}^*_{(n)}.\end{aligned} \tag{15}$$

The remainder of the process is similar to that described in (3.1. First approximation). The iteration shown in eqn (13) is continued until the approximate solution satisfies the specified convergent criteria.

When a convergent solution is obtained the next increment is applied, if necessary.

4. VERIFICATION OF THE APPLICABILITY OF THE YMM METHOD FOR PROBLEMS WITH FINITE ROTATIONS IN THREE-DIMENSIONAL SPACE

There are few problems with three-dimensional finite rotations for which a theoretical solution is obtained, and also few comparable examples of numerical analyses. Here, a problem is picked up for which theoretical solution is found and the calculated result is compared with the theoretical solution. Then a curved cantilever beam with square cross section subjected to a vertical load at the free end is analyzed, and the calculated results are compared with those given by Bathe and Bolourchi [3]. Through these comparisons the validity of the YMM method can be confirmed.

4.1. *Analysis of a cantilever beam with square cross section subjected to a series of lateral loadings—different orders of loadings applied*

The principal axes of a square cross section of the cantilever beam are taken as the X and Y axes, and the longitudinal axis is defined as the Z axis. Loads are applied at the free end. Three cases of loading orders are considered, namely: (1) a load of magnitude P is first applied in the X direction, then an additional load of the same magnitude in the Y direction is applied; (2) a load P is applied in the Y direction and followed by an additional one in the X direction; and (3) loads of the same magnitude P are applied at the same time in the X and Y directions, respectively.

Deformed configurations corresponding to these three series of loadings are illustrated in Fig. 1. The final configurations for these three cases are found to be exactly the same. The validity of the YMM method for problems subjected to bendings in different planes in three-dimensional space is thus verified.

4.2. *Analysis of a curved beam subjected to an out-of-plane loading*

An out-of-plane load is applied to a curved beam with square cross section as shown in Fig. 2(a). The relations between the applied load and the displacements at the free end, where the load is applied, are shown in Fig. 2(b) in non-dimensional form. Deformed configurations at a couple of stages are illustrated in Fig. 2(c). The solid lines in Fig. 2(b) are the calculated results given by Bathe and Bolourchi [3] with eight elements and 60 steps. The results obtained by the present method with eight elements and 10 steps shown by dark circles coincide well with the solid lines. The present results are obtained by introducing modified Steffensen's iteration into the iterative–corrective procedure explained in Sec. 3.3. When the procedure shown in Sec. 3.3 itself is used, converged solutions are obtained only up to the stage indicated by the chain line in Fig. 2(b) for this problem. The modified Steffensen's iteration scheme introduced is given in the following section.

Bathe and Bolourchi checked the applicability of their method by comparing their results with the theoretical solution through the two-dimensional finite displacement analysis of a cantilever beam subjected to a concentrated moment at its free end. Their results and the results obtained here are compared in Fig. 3 with the theoretical solution. The figure shows the relations between the applied moment and the displacements at the free end. The solid lines show the theoretical solutions and the dashed lines are the ones given by Bathe and Bolourchi with 20 elements and 90 steps. The circles, triangles and rectangles are the calculated results here with eight elements and eight steps. The present results are in close agreement with the theoretical solution within the whole range. The results given by Bathe and Bolourchi also agree well with the theoretical solution up to a rotation at the free end of 90°.

4.3. *Modified Steffensen's iteration scheme*

When eqn (13) is solved with respect to $\Delta \mathbb{w}_{k+1}$, the following recurrence formula is obtained.

$$\Delta \mathbb{w}_{k+1} = \mathbb{k}_k^{-1} (\Delta \mathbb{f} - \mathbb{h}_k) = \mathbb{g}(\Delta \mathbb{w}_k). \tag{16}$$

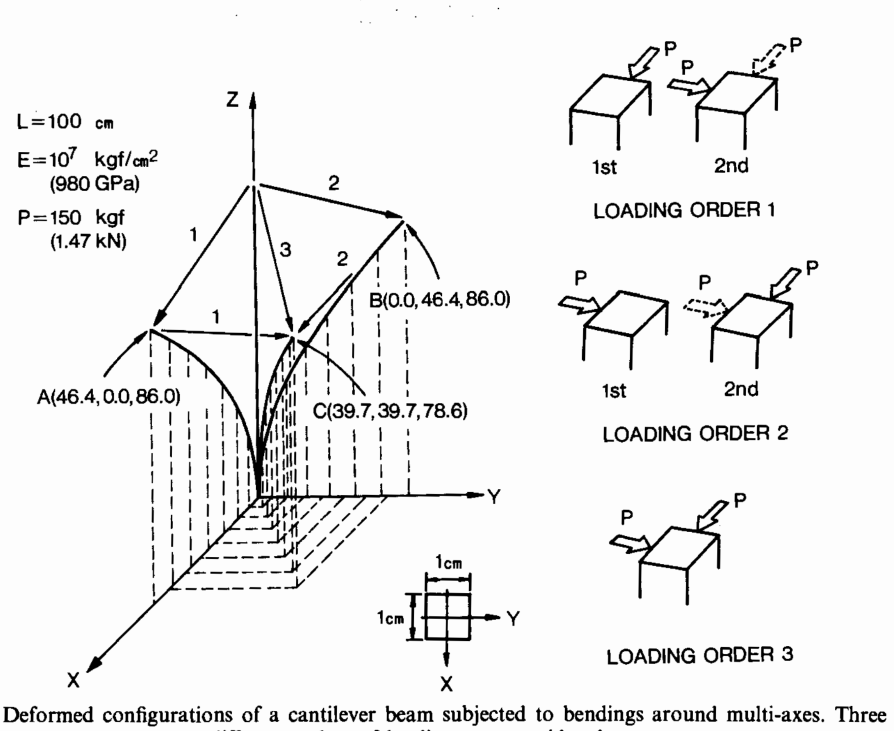


Fig. 1. Deformed configurations of a cantilever beam subjected to bendings around multi-axes. Three different orders of loadings are considered.

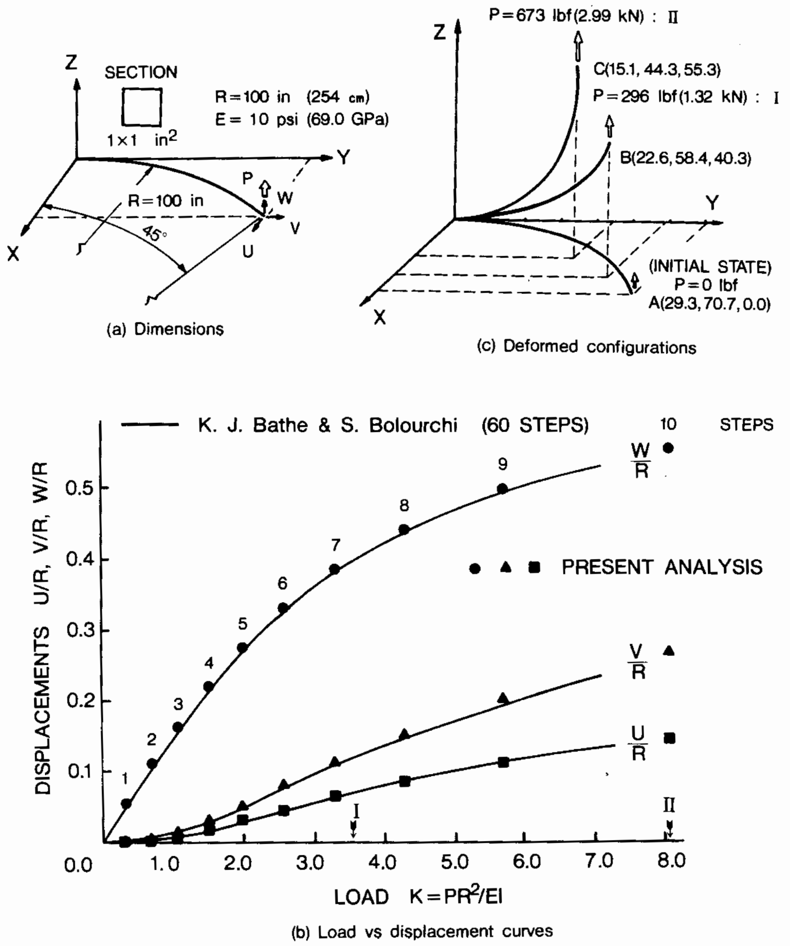


Fig. 2. Out-of-plane bending of a curved bar. Comparison with the result by Bathe and Bolourchi [3].

Here, for simplicity, static problems are to be considered.

The application of the Steffensen's iteration for this fixed point problem with $\mathfrak{g}(\Delta\mathfrak{w}_k)$ as the initial value gives the next equation.

$$\begin{aligned}\Delta\mathfrak{w}_{k+4} &= \mathfrak{g}(\Delta\mathfrak{w}_k) - [\mathfrak{g}(\mathfrak{g}(\Delta\mathfrak{w}_k)) - \mathfrak{g}(\Delta\mathfrak{w}_k)]^2 \\ &\quad /[\mathfrak{g}(\mathfrak{g}(\mathfrak{g}(\Delta\mathfrak{w}_k))) - 2\,\mathfrak{g} \\ &\quad \cdot(\mathfrak{g}(\Delta\mathfrak{w}_k)) + \mathfrak{g}(\Delta\mathfrak{w}_k)] \\ &= \mathfrak{s}(\Delta\mathfrak{w}_k). \end{aligned} \tag{17}$$

The modified Steffensen's iteration formula is constructed as the following:

(i) start with $k = 2$
(ii) calculate $\Delta\mathfrak{w}_k$, and $\Delta\mathfrak{w}_{j+1} = \mathfrak{g}(\Delta\mathfrak{w}_j)$; $j = k$, $k+1$, $k+2$
(iii) calculate $\Delta\mathfrak{w}_{k+3} = \mathfrak{s}(\Delta\mathfrak{w}_k)$
(iv) set $\Delta\mathfrak{w}_{k+3}$ as a new $\Delta\mathfrak{w}_k$
(v) go to step (ii) and repeat the steps (ii)–(iv).

The difference from the original Steffensen's iteration scheme is that the new starting point for the next iteration is set as $\mathfrak{g}(\Delta\mathfrak{w}_{k+3})$, which is given by the governing equilibrium equation using $\Delta\mathfrak{w}_{k+3}$. In the original Steffensen's iteration $\Delta\mathfrak{w}_{k+3}$ itself is used as the next starting point. The modified Steffensen's iteration scheme also has a second order convergence property just as the original Steffensen's iteration does.

5. NUMERICAL EXAMPLE—DYNAMIC INSTABILITY ANALYSIS OF A SHALLOW ARCH

The dynamic response analysis method presented in this paper is constructed so as to be able to deal with dynamic response in three-dimensional space, but, unfortunately, there exist a few problems that have analytical solutions to be compared to confirm the applicability of the presented method for such situations. Here, as the first step of examination, an in-plane dynamic instability of a shallow arch is analyzed and the result is compared with that given by Gregory and Plaut [4]. The arch has two hinges and sinusoidal initial configuration as shown in Fig. 4(a), where dimensions are also described. Forced vibration of the arch subjected to a centrally concentrated step loading (see Fig. 4(b)) is considered. Response analyses with various magnitudes

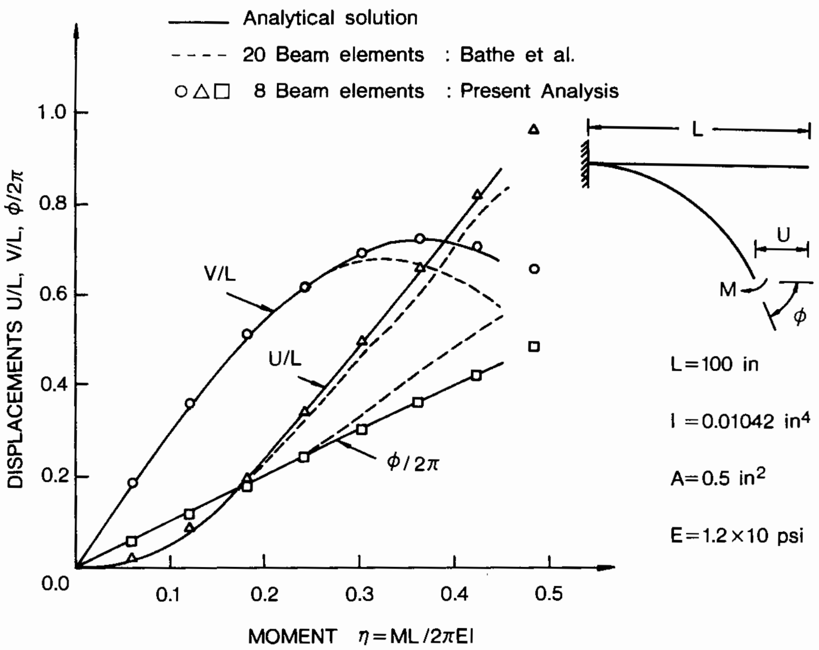
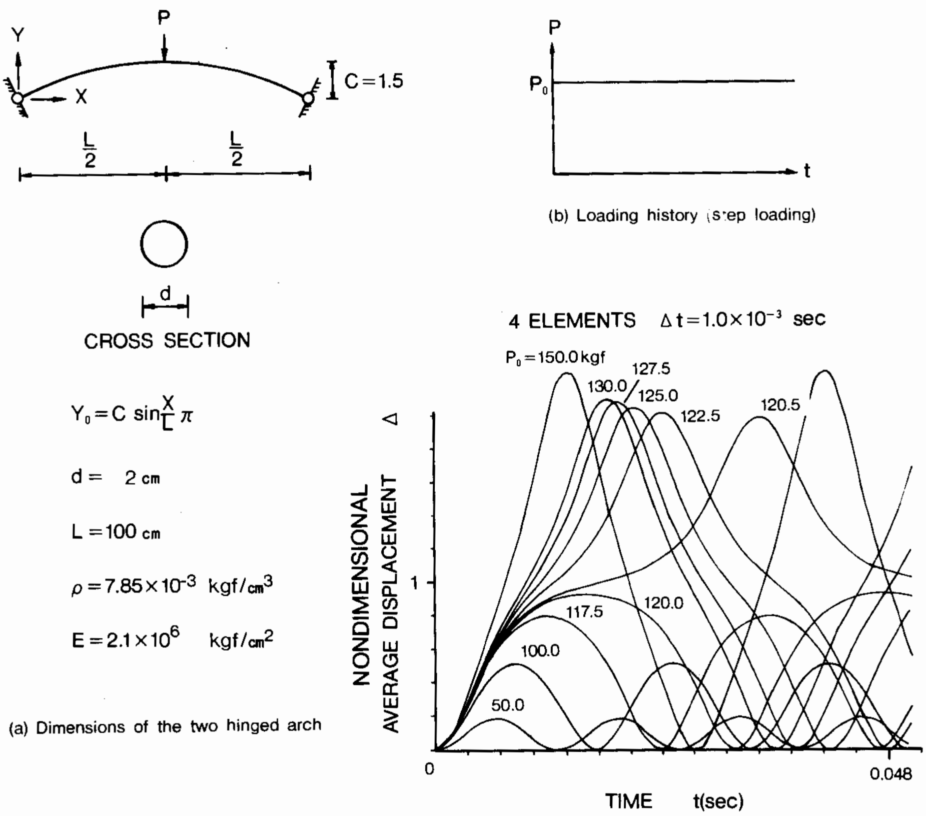


Fig. 3. Moment vs displacement curves of a cantilever beam subjected to a concentrated moment at its free end.

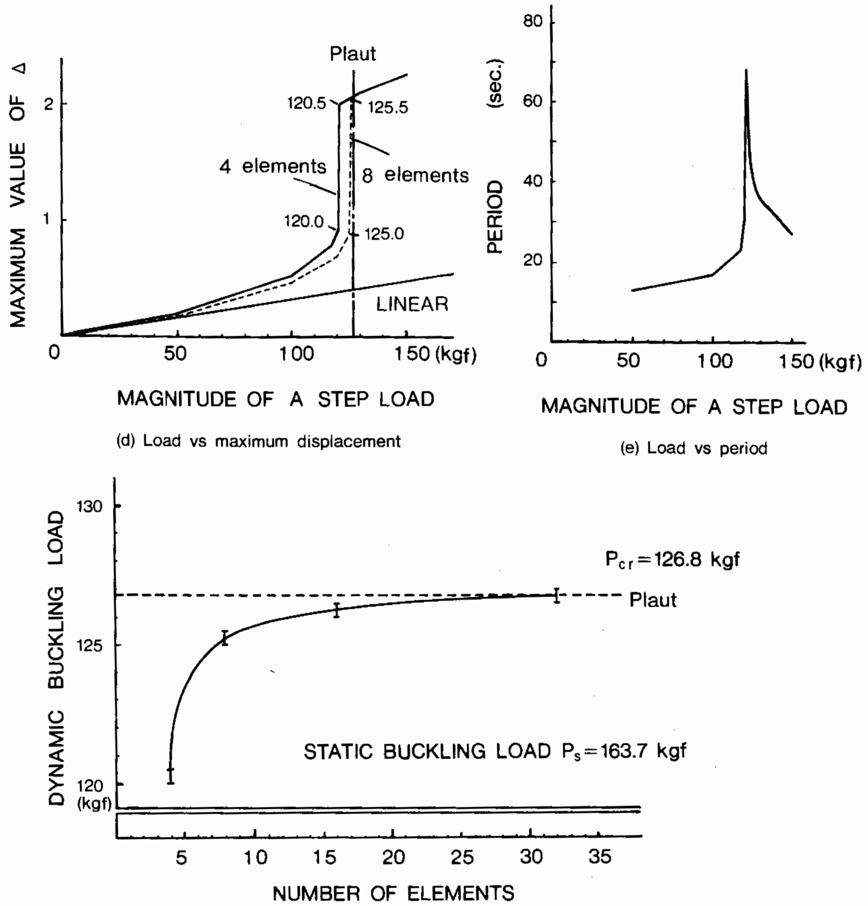


(a) Dimensions of the two hinged arch

(b) Loading history (step loading)

(c) Displacement responses to each step loading

Fig. 4. (a)–(c).



(d) Load vs maximum displacement

(e) Load vs period

(f) Convergence property with respect to number of elements

Fig. 4. Instability analysis of a shallow arch under step loadings.

of the step load are conducted and time histories of a nondimensional average displacement Δ are obtained as shown in Fig. 4(c). The displacement Δ is defined as the ratio of summation of the nodal deflection and that of the nodal height at the initial state. The figures show the results of the case where four elements with a time step of $\Delta t = 1.0 \times 10^{-3}$ sec are used.

Budiansky and Roth [5] defined dynamic instability as a state at which a small increment in loading produces sudden changes in maximum response. The dynamic buckling load obtained by Gregory and Plaut from this definition is 126.8 kgf. The static buckling load for this problem is 163.7 kgf. The maximum values of the nondimensional average displacement as a function of the magnitude of the step load are calculated by the present method as shown in Fig. 4(d). The solid line and the dotted line correspond to the cases with four and eight elements, respectively. The relation between the period of the response and the magnitude of the load is obtained as shown in Fig. 4(e). The dynamic buckling load obtained from the above results is given in Fig. 4(f) and Table 1, and it is in good agreement with that obtained by Gregory and Plaut in the limit as the number of elements is increased.

6. CONCLUSION

The problem chosen to verify the applicability of the presented method to dynamic response analysis is an in-plane vibration of a shallow arch. Therefore, in order to confirm the generality an examination needs to be made of the effects of neglecting the time differential terms of the mass matrix on the response characteristics in the presence of strong geometric

Table 1. Comparison of dynamic buckling load with the result of Gregory and Plaut

Number of elements	Dynamic buckling load calculated (kgf)	Difference with the value given by Gregory and Plaut [4] (%)
4	120.0–120.5	5.4–5.0
8	125.0–125.5	1.4–1.0
16	126.0–126.5	0.6–0.2
32	126.5–127.0	0.2–0.0

nonlinearity and three-dimensional behavior. Nevertheless, as far as the problem here is concerned, it is verified that the proposed geometrically nonlinear response analysis method which is based on the YMM method is sufficiently effective. As to the time integration, although the Newmark's β method is adopted here, other time integration schemes also can be combined.

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APPENDIX

The formulation of the equilibrium equation of an element in the YMM method is outlined as the following:

$$\{f\} = [T]^T\{f^*\}$$

(coordinates transformation of nodal forces)

$$= [T]^T[k^*]\{u^*\}$$

(nodal forces in the element coordinates system are expressed by the stiffness equation in the element coordinates system)

$$= [T]^T[k^*]([T]\{a\} - [T_0]\{a_0\})$$

(nodal displacements in the element coordinates system are expressed by generalized coordinates)

$$= [T]^T[k^*][T]\{q(\{u\},\{x_0\})\}$$

(just rewritten),

where

$\{q(\{u\},\{x_0\})\} = \{a\} - [T]^{-1}[T_0]\{a_0\}$

$\{u\}^T = \langle\{d\}^T\{\theta\}^T\rangle$ = displacements vector of the element

$\{x\}^T$ = coordinates vector of the element

$\{a\}^T = \langle\{x\}^T\{\theta\}^T - \{r\}^T\rangle$ = generalized coordinates vector of the element

$\{d\}$ = translational displacement components vector of the element in the global coordinates system

$\{\theta\}$ = rotational components vector of the element in the global system

$\{x\}$ = coordinates vector of the element in the global coordinates system

$\{x_0\}$ = initial coordinates

$\{a_0\}$ = initial generalized coordinates

$\{r\}$ = rigid body rotation vector of the element in the global system.

In the above equation, the element coordinates system is defined as an orthogonal Cartesian coordinate, and its axes are fixed on the element all the way. Therefore, the coordinate transformation matrix is a function of the generalized coordinate at the moment, even when an incremental formulation is adopted as in this paper. And the vector $\{q\}$ is a function of not only displacements $\{u\}$ but also initial coordinates $\{x_0\}$.

The incremental equation represented in this paper, namely, eqn (1), can be derived directly from the above equilibrium equation.

The solution procedure of the YMM method is included in what is explained in this paper. But it should be noted that the use of the presented two-step linearized approximation which does not have any tangent stiffness in the usual sense, has given the method such capacity as to be able to deal with bifurcation analyses without any eigenvalue calculations, although the problems chosen in this paper do not show any bifurcation phenomena.