$\wp$-ADIC CONTINUOUS FAMILIES OF DRINFELD EIGENFORMS OF FINITE SLOPE

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Abstract. Let p be a rational prime, v_p the normalized p-adic valuation on $\mathbb{Z}$, $q > 1$ a power of p and $A = \mathbb{F}_q[t]$. Let $\wp \in A$ be an irreducible polynomial and $\mathfrak{n} \in A$ a non-zero element which is prime to $\wp$. Let $k \geqslant 2$ and $r \geqslant 1$ be integers. We denote by $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ the space of Drinfeld cuspforms of level $\Gamma_1(\mathfrak{n}\wp^r)$ and weight k for $\mathbb{F}_q(t)$. Let $n \geqslant 1$ be an integer and $a \geqslant 0$ a rational number. Suppose that $\mathfrak{n}\wp$ has an irreducible factor of degree one and the generalized eigenspace in $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ of slope a is one-dimensional. In this paper, under an assumption that a is sufficiently small, we construct a family $\{F_{k'} \mid v_p(k'-k) \geqslant \log_p(p^n + a)\}$ of Hecke eigenforms $F_{k'} \in S_{k'}(\Gamma_1(\mathfrak{n}\wp^r))$ of slope a such that, for any $Q \in A$, the Hecke eigenvalues of F_k and $F_{k'}$ at Q are congruent modulo $\wp^\kappa$ with some $\kappa > p^{v_p(k'-k)} - p^n - a$.

1. Introduction

Let p be a rational prime, $q > 1$ a power of p and $\mathbb{F}_q$ the field of q elements. Put $A = \mathbb{F}_q[t]$ and $K = \mathbb{F}_q(t)$. Let $\wp \in A$ be an irreducible polynomial of degree $d > 0$, $\mathfrak{n}$ a non-zero element of A which is prime to $\wp$ and $r \geqslant 1$ an integer. Put $A_r = A/(\wp^r)$ and $\kappa(\wp) = A/(\wp)$. We denote by $K_\wp$ the $\wp$-adic completion of K, by $\mathbb{C}_\wp$ the $\wp$-adic completion of an algebraic closure of $K_\wp$ and by $v_\wp : \mathbb{C}_\wp \to \mathbb{Q} \cup \{+\infty\}$ the $\wp$-adic additive valuation on $\mathbb{C}_\wp$ normalized by $v_\wp(\wp) = 1$. Similarly, we denote by K_∞ the $(1/t)$-adic completion of K and by $\mathbb{C}_\infty$ the $(1/t)$-adic completion of an algebraic closure of K_∞. Let $\bar{K}$ be the algebraic closure of K inside $\mathbb{C}_\infty$ and let us fix an embedding of K-algebras $\iota_\wp : \bar{K} \to \mathbb{C}_\wp$. For any $x \in \bar{K}$, we define its normalized $\wp$-adic valuation by $v_\wp(\iota_\wp(x))$. Let $\Omega = \mathbb{P}^1(\mathbb{C}_\infty)\backslash\mathbb{P}^1(K_\infty)$ be the Drinfeld upper half plane, which has a natural structure of a rigid analytic variety over K_∞.

Let Γ be an arithmetic subgroup of $SL_2(A)$ and k an integer. A Drinfeld modular form of level Γ and weight k is a rigid analytic function

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on Ω satisfying

$$f\left(\frac{az+b}{cz+d}\right)=(cz+d)^k f(z)\quad \text{for any } \gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma, z\in\Omega$$

and a holomorphy condition at cusps. It is considered as a function field analogue of the notion of elliptic modular form.

Recently, $\wp$-adic properties of Drinfeld modular forms have attracted attention and have been studied actively (for example, [BV1, BV2, BV3, Gos, Hat1, Hat2, PZ, Vin]). Especially, in [Hat2] the author proved an analogue of the Gouvêa-Mazur conjecture for Drinfeld cuspforms of level $\Gamma_1(t)$ and observed that the t-adic slopes seem to form periodic sequences.

However, though we have a highly developed theory of p-adic analytic families of elliptic eigenforms of finite slope, $\wp$-adic properties of Drinfeld modular forms are much less well-understood compared to the elliptic case. One of the difficulties in the Drinfeld case is that, since the group $\mathcal{O}_{K_\wp}^\times$ is topologically infinitely generated, analogues of the completed group ring $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ are not Noetherian, and it seems that we have no good definition of characteristic power series applicable to non-Noetherian base rings, as mentioned in [Buz2, paragraph before Lemma 2.3].

For elliptic modular forms, the structure of slopes is often explained by (or related to) the existence of p-adic analytic families of elliptic eigenforms and their geometry. Contrarily, for Drinfeld modular forms we do not know at all where such structure of slopes comes from, due to the lack of a theory of $\wp$-adic analytic families as in [Col, Buz2].

In this paper, in the hope of compensating the lack, we will construct families of Drinfeld eigenforms in which Hecke eigenvalues vary in a $\wp$-adically continuous way. For the precise statement, we fix some notation. For any $\mathfrak{m}\in A$, we put

$$\Gamma_1(\mathfrak{m})=\left\{\gamma\in SL_2(A)\ \middle|\ \gamma\equiv\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix}\bmod \mathfrak{m}\right\}.$$

Let Θ be any subgroup of $1+\wp A_r\subseteq A_r^\times$. We define

$$\Gamma_0^\Theta(\wp^r)=\left\{\gamma\in SL_2(A)\ \middle|\ \gamma \bmod \wp^r\in\begin{pmatrix}\Theta & *\\ 0 & \Theta\end{pmatrix}\right\}\subseteq\Gamma_1(\wp)$$

and $\Gamma_1^\Theta(\mathfrak{n},\wp^r)=\Gamma_1(\mathfrak{n})\cap\Gamma_0^\Theta(\wp^r)$, which satisfies $\Gamma_1^{\{1\}}(\mathfrak{n},\wp^r)=\Gamma_1(\mathfrak{n}\wp^r)$.

Let $k\geqslant 2$ be an integer. For any non-zero element $Q\in A$, the Hecke operator T_Q acts on the $\mathbb{C}_\infty$-vector space $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$ of Drinfeld cuspforms of level $\Gamma_1^\Theta(\mathfrak{n},\wp^r)$ and weight k. The operator $T_\wp$ is also denoted by U. Since the Hecke operators stabilize an A-lattice $\mathcal{V}_k(A)$

(Proposition 2.2), every eigenvalue of T_Q is integral over A. The normalized $\wp$-adic valuation of an eigenvalue of U is called slope, and we denote by $d(k,a)$ the dimension of the generalized U-eigenspace for the eigenvalues of slope a. Namely, it is defined as

$$d(k,a) = \sum_{\lambda} m(\lambda),$$

where λ runs over the set of U-eigenvalues of slope a and $m(\lambda)$ denotes the multiplicity of λ.

For any Hecke eigenform F, its T_Q-eigenvalue is denoted by $\lambda_Q(F)$. We denote by v_p the p-adic valuation on $\mathbb{Z}$ satisfying $v_p(p) = 1$. Then the main theorem of this paper (Theorem 4.1) gives the following, which we will prove in §4.1.

Theorem 1.1. *Suppose that $\mathfrak{n}\wp$ has an irreducible factor π of degree one. Let $n \geqslant 1$ and $k \geqslant 2$ be integers. Put $\delta = [\Gamma_1(\pi) : \Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)]$, $\varepsilon = d(k,0)$ and*

$$D_2(n,\delta,\varepsilon) = \frac{1}{\delta}\left\{\sqrt{2\delta p^n + (\delta-\varepsilon+1)(2\delta-\varepsilon-1)} - \frac{3}{2}\delta + \varepsilon\right\},$$

$$D(n,\delta,\varepsilon) = \min\left\{p^n\left(\frac{4+\delta p^n - \delta}{4+2\delta p^n - 2\varepsilon}\right), D_2(n,\delta,\varepsilon)\right\}.$$

Let a be any non-negative rational number satisfying

$$a < \min\{D(n,\delta,\varepsilon), k-1\}.$$

Suppose $d(k,a) = 1$. Then, for any integer $k' \geqslant k$ satisfying

$$v_p(k'-k) \geqslant \log_p(p^n + a),$$

there exists a Hecke eigenform $F_{k'} \in S_{k'}(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r))$ of slope a such that for any Q we have

$$v_\wp(\iota_\wp(\lambda_Q(F_{k'}) - \lambda_Q(F_k))) > p^{v_p(k'-k)} - p^n - a.$$

In fact, what we will prove allow nebentypus characters at $\wp$ (Remark 4.2).

For example, in the case of $\mathfrak{n} = 1$, $\wp = t$ and $r = 1$, we have $\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r) = \Gamma_1(t)$, $\delta = \varepsilon = 1$ and $D(n,1,1) = \sqrt{2p^n} - \frac{1}{2}$. In this case, Theorem 1.1 implies that, for any Hecke eigenform F_k of slope zero in $S_k(\Gamma_1(t))$, the T_Q-eigenvalue $\lambda_Q(F_k)$ is t-adically arbitrarily close to those coming from Hecke eigenforms with A-expansion [Pet], which shows $\lambda_Q(F_k) = 1$ for any Q (Proposition 4.3). This suggests that, though we will prove constancy results of the dimension of slope zero cuspforms with respect to k and r (Proposition 3.4 and Proposition 3.5), Hida theory for the level $\Gamma_0(t^r)$ should be trivial (Remark 4.5).

We also note that families constructed in Theorem 1.1 contain Hecke eigenforms whose Hecke eigenvalue at Q is not a power of Q (§4.2), and thus they capture a more subtle $\wp$-adic structure of Hecke eigenvalues than the theory of A-expansions.

Let us explain the idea of the proof of Theorem 1.1. Note that a usual method to construct p-adic families of eigenforms of finite slope in the number field case is the use of the Riesz theory [Col, Buz2], which is not available for our case at present, due to the lack of a notion of characteristic power series over non-Noetherian Banach algebras. Instead, we follow an idea of Buzzard [Buz1] by which he constructed p-adically continuous families of quaternionic eigenforms over $\mathbb{Q}$.

First we will prove a variant of the Gouvêa-Mazur conjecture (Proposition 3.11), which implies $d(k,a) = d(k',a)$ if k and k' are highly congruent p-adically and a is sufficiently small. With the assumption $d(k,a) = 1$, it produces Hecke eigenforms F_k and $F_{k'}$ of slope a in weights k and k', respectively. For this part, we employ the same idea as in [Hat2]: a lower bound of elementary divisors of the representing matrix of U with some basis and a perturbation lemma [Ked, Theorem 4.4.2] yield the equality. To obtain such a bound (Corollary 3.8), we need to define Hecke operators acting on the Steinberg complex (2.2) with respect to $\Gamma_1^{\Theta}(\mathfrak{n},\wp^r)$, which is done in §2.3. Note that similar Hecke operators on a Steinberg complex in an adelic setting are given in [Böc, §6.4].

Then, a weight reduction map (§3.2) yields a Drinfeld cuspform G of weight k such that, for $m = v_p(k'-k)$, the element $G \bmod \wp^{p^m}$ is a Hecke eigenform with the same eigenvalues as those of $F_{k'} \bmod \wp^{p^m}$. Now the point is that, if two lines generated by F_k and G are highly congruent in some sense, then we can show that the eigenvalues of F_k and $G \bmod \wp^{p^m}$ are also highly congruent, which gives Theorem 1.1; otherwise the two lines are so far apart that, again by the Gouvêa-Mazur variant mentioned above, they produce U-eigenvalues of slope a with multiplicity more than one, which contradicts $d(k,a) = 1$ (Theorem 4.1).

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2. Drinfeld cuspforms via the Steinberg module

For any arithmetic subgroup Γ of $SL_2(A)$ and any integer $k \geqslant 2$, we denote by $S_k(\Gamma)$ the space of Drinfeld cuspforms of level Γ and weight k. In this section, we first recall an interpretation of $S_k(\Gamma)$ using the Steinberg module due to Teitelbaum [Tei, p. 506], following the normalization of [Böc, §5]. We also introduce Hecke operators acting on the Steinberg complex. Using them, we define an A-lattice of the space of Drinfeld cuspforms which is stable under the Hecke action.

2.1. **Steinberg module.** For any A-algebra B, we consider B^2 as the set of row vectors, and define a left action $\circ$ of $GL_2(B)$ on it by $\gamma \circ x = x\gamma^{-1}$. Let $\mathcal{T}$ be the Bruhat-Tits tree for $SL_2(K_\infty)$. We denote by $\mathcal{T}_0$ the set of vertices of $\mathcal{T}$, which is the set of $K_\infty^\times$-equivalence classes of $\mathcal{O}_{K_\infty}$-lattices in K_∞^2, and by $\mathcal{T}_1$ the set of its edges. The oriented graph associated with $\mathcal{T}$ and the set of oriented edges are denoted by $\mathcal{T}^o$ and $\mathcal{T}_1^o$, respectively. For any oriented edge e, we denote its origin by $o(e)$, its terminus by $t(e)$ and the opposite edge by $-e$. The group $\{\pm 1\}$ acts on $\mathcal{T}_1^o$ by $(-1)e = -e$.

Let Γ be an arithmetic subgroup of $SL_2(A)$ [Böc, §3.4], and we assume Γ to be p'-torsion free (namely, every element of Γ of finite order has p-power order). The group Γ acts on $\mathcal{T}$ and $\mathcal{T}^o$ via the natural inclusion $\Gamma \to GL_2(K_\infty)$. We say a vertex or an oriented edge of $\mathcal{T}$ is Γ-stable if its stabilizer subgroup in Γ is trivial, and Γ-unstable otherwise. We denote by $\mathcal{T}_0^{\mathrm{st}}$ and $\mathcal{T}_1^{o,\mathrm{st}}$ the subsets of Γ-stable elements. For any Γ-unstable vertex v, its stabilizer subgroup in Γ is a non-trivial finite p-group and thus fixes a unique rational end which we denote by $b(v)$ [Ser, Ch. II, §2.9].

For any ring R and any set S, we write $R[S]$ for the free R-module with basis $\{[s] \mid s \in S\}$. When S admits a left action of Γ, the R-module $R[S]$ also admits a natural left action of the group ring $R[\Gamma]$ which we denote by $\circ$. In this case, we also define a right action of Γ on $R[S]$ by $[s]|_\gamma = \gamma^{-1} \circ [s]$, which makes it a right $R[\Gamma]$-module.

Put

$$\mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}] = \mathbb{Z}[\mathcal{T}_1^{o,\mathrm{st}}]/\langle [e] + [-e] \mid e \in \mathcal{T}_1^{o,\mathrm{st}} \rangle.$$

We define a surjection of $\mathbb{Z}[\Gamma]$-modules $\partial_\Gamma : \mathbb{Z}[\mathcal{T}_1^{o,\mathrm{st}}] \to \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]$ by $\partial_\Gamma(e) = [t(e)] - [o(e)]$, where we understand $[v] = 0$ in $\mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]$ for any Γ-unstable vertex v. It factors as $\partial_\Gamma : \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}] \to \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]$. Note that the both sides of this map are free left $\mathbb{Z}[\Gamma]$-modules of finite rank.

We define the Steinberg module St as the kernel of the natural augmentation map

$$\mathbb{Z}[\mathbb{P}^1(K)] \to \mathbb{Z},$$

on which the group $GL_2(K)$ acts via

$$\gamma \circ (x:y) = (x:y)\gamma^{-1}, \quad (x:y) \in \mathbb{P}^1(K).$$

We consider it as a left $\mathbb{Z}[\Gamma]$-module via the natural inclusion $\Gamma \to GL_2(K)$. Then the Steinberg module St is a finitely generated projective $\mathbb{Z}[\Gamma]$-module which sits in the split exact sequence of $\mathbb{Z}[\Gamma]$-modules

$$0 \longrightarrow \mathrm{St} \longrightarrow \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}] \xrightarrow{\partial_\Gamma} \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}] \longrightarrow 0 \tag{2.1}$$

[Böc, §5.3]. We consider these three left $\mathbb{Z}[\Gamma]$-modules as right $\mathbb{Z}[\Gamma]$-modules via the action $[s] \mapsto [s]|_\gamma$.

2.2. **Drinfeld cuspforms and harmonic cocycles.** For any integer $k \geqslant 2$ and any A-algebra B, we denote by $H_{k-2}(B)$ the B-submodule of the polynomial ring $B[X,Y]$ consisting of homogeneous polynomials of degree $k-2$. We consider the left action of the multiplicative monoid $M_2(B)$ on $H_{k-2}(B)$ defined by $(\gamma \circ X, \gamma \circ Y) = (X,Y)\gamma$. On $GL_2(B)$, it agrees with the natural left action on $\mathrm{Sym}^k(\mathrm{Hom}_B(B^2,B))$ induced by the action $\circ$ on B^2 after identifying (X,Y) with the dual basis for the basis $((1,0),(0,1))$ of B^2. Put

$$V_k(B) = \mathrm{Hom}_B(H_{k-2}(B), B).$$

We denote the dual basis of the free B-module $V_k(B)$ with respect to the basis $\{X^iY^{k-2-i} \mid 0 \leqslant i \leqslant k-2\}$ of $H_{k-2}(B)$ by

$$\{(X^iY^{k-2-i})^\vee \mid 0 \leqslant i \leqslant k-2\}.$$

We also denote by $\circ$ the natural left action of $GL_2(B)$ on $V_k(B)$ induced by that on $H_{k-2}(B)$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(B)$, $P(X,Y) \in H_{k-2}(B)$ and $\omega \in V_k(B)$, this action is given by

$$\begin{aligned}(\gamma \circ \omega)(P(X,Y)) &= \omega(\gamma^{-1} \circ P(X,Y)) \\ &= \det(\gamma)^{2-k}\omega(P(dX - cY, -bX + aY))\end{aligned}$$

as in [Böc, p. 51]. The group Γ acts on $H_{k-2}(B)$ and $V_k(B)$ via the natural map $\Gamma \to GL_2(B)$. Moreover, the monoid

$$M^{-1} = \{\xi \in GL_2(K) \mid \xi^{-1} \in M_2(A)\}$$

acts on $V_k(B)$ by

$$(\xi \circ \omega)(P(X,Y)) = \omega(\xi^{-1} \circ P(X,Y)).$$

Put $\mathcal{V}_k(B) = \mathrm{St} \otimes_{\mathbb{Z}[\Gamma]} V_k(B)$ and

$$\mathcal{L}_{1,k}(B) = \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}] \otimes_{\mathbb{Z}[\Gamma]} V_k(B), \quad \mathcal{L}_{0,k}(B) = \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}] \otimes_{\mathbb{Z}[\Gamma]} V_k(B).$$

We have the split exact sequence of B-modules

$$0 \longrightarrow \mathcal{V}_k(B) \longrightarrow \mathcal{L}_{1,k}(B) \xrightarrow{\partial_\Gamma \otimes 1} \mathcal{L}_{0,k}(B) \longrightarrow 0 \tag{2.2}$$

which is functorial on B and compatible with any base change of B. Let B' be any A-subalgebra of B. Since the $\mathbb{Z}[\Gamma]$-module St is projective, the natural maps $\mathcal{V}_k(B') \to \mathcal{V}_k(B)$, $\mathcal{L}_{1,k}(B') \to \mathcal{L}_{1,k}(B)$ and $\mathcal{L}_{0,k}(B') \to \mathcal{L}_{0,k}(B)$ are injective.

Let $\Lambda_1 \subseteq \mathcal{T}_1^{o,\mathrm{st}}$ be a complete set of representatives of $\Gamma\backslash\mathcal{T}_1^{o,\mathrm{st}}/\{\pm 1\}$. By [Ser, Ch. II, §1.2, Corollary], for any element $e \in \mathcal{T}_1^{o,\mathrm{st}}$ there exist unique elements $\varepsilon_e \in \{\pm 1\}$, $\gamma_e \in \Gamma$, $r(e) \in \Lambda_1$ satisfying

$$r(e) = \varepsilon_e \gamma_e e. \tag{2.3}$$

Note that $r(e)$, ε_e and γ_e depend on the choice of Λ_1. The right $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}]$ is free with basis $\{[e] \mid e \in \Lambda_1\}$ and thus, for any A-algebra B, any element x of $\mathcal{L}_{1,k}(B)$ can be written uniquely as

$$x = \sum_{e \in \Lambda_1} [e] \otimes \omega_e, \quad \omega_e \in V_k(B).$$

Definition 2.1. Let M be a $\mathbb{Z}$-module. A map $c : \mathcal{T}_1^o \to M$ is said to be a harmonic cocycle if the following conditions are satisfied:

(1) For any $v \in \mathcal{T}_0$, we have
$$\sum_{e \in \mathcal{T}_1^o,\ t(e)=v} c(e) = 0.$$

(2) For any $e \in \mathcal{T}_1^o$, we have $c(-e) = -c(e)$.

Any harmonic cocycle c is determined by its values at Γ-stable edges, as follows. For any $e \in \mathcal{T}_1^o$, an edge $e' \in \mathcal{T}_1^{o,\mathrm{st}}$ is said to be a source of e if the following conditions hold:

- When e is Γ-stable, we require $e' = e$.
- When e is Γ-unstable, we require that a vertex v of e' is Γ-unstable, e lies on the unique half-line from v to the rational end $b(v)$ and e has the same orientation as e' with respect to this half line.

We denote by $\mathrm{src}(e)$ the set of sources of e. Then Definition 2.1 (1) gives

$$c(e) = \sum_{e' \in \mathrm{src}(e)} c(e'). \tag{2.4}$$

Moreover, for any $\gamma \in \Gamma$, we have

$$\mathrm{src}(\gamma(e)) = \gamma(\mathrm{src}(e)), \quad \mathrm{src}(-e) = -\mathrm{src}(e). \tag{2.5}$$

For any A-algebra B, we denote by $C_k^{\mathrm{har}}(\Gamma, B)$ the set of harmonic cocycles $c : \mathcal{T}_1^o \to V_k(B)$ which is Γ-equivariant (namely, $c(\gamma(e)) = \gamma \circ c(e)$ for any $\gamma \in \Gamma$ and $e \in \mathcal{T}_1^o$). For any rigid analytic function f on Ω and $e \in \mathcal{T}_1^o$, we can define an element $\mathrm{Res}(f)(e) \in V_k(\mathbb{C}_\infty)$, which gives an isomorphism of $\mathbb{C}_\infty$-vector spaces

$$\mathrm{Res}_\Gamma : S_k(\Gamma) \to C_k^{\mathrm{har}}(\Gamma, \mathbb{C}_\infty), \quad f \mapsto (e \mapsto \mathrm{Res}(f)(e))$$

([Tei, Theorem 16], see also [Böc, Theorem 5.10]). By [Böc, (17)], the slash operator defined by

$$(f|_k\gamma)(z) = \det(\gamma)^{k-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$$

satisfies $\mathrm{Res}(f|_k\gamma)(e) = \gamma^{-1} \circ \mathrm{Res}(f)(\gamma(e))$.

On the other hand, the argument in [Tei, p. 506] shows that for any A-algebra B, we have a B-linear isomorphism

$$\Phi_\Gamma : C_k^{\mathrm{har}}(\Gamma, B) \to \mathcal{V}_k(B), \quad \Phi_\Gamma(c) = \sum_{e \in \Lambda_1} [e] \otimes c(e),$$

which is independent of the choice of a complete set of representatives Λ_1. This implies that, for any morphism $B \to B'$ of A-algebras, the natural map

$$C_k^{\mathrm{har}}(\Gamma, B) \otimes_B B' \to C_k^{\mathrm{har}}(\Gamma, B')$$

is an isomorphism. Moreover, we obtain an isomorphism

$$\Phi_\Gamma \circ \mathrm{Res}_\Gamma : S_k(\Gamma) \to \mathcal{V}_k(\mathbb{C}_\infty).$$

In particular, for any A-subalgebra B of $\mathbb{C}_\infty$, we have an injection

$$\mathcal{V}_k(B) \to \mathcal{V}_k(\mathbb{C}_\infty) \simeq S_k(\Gamma).$$

2.3. **Hecke operators.** For any non-zero element $Q \in A$, we have a Hecke operator T_Q acting on $S_k(\Gamma)$ defined as follows. Write

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma = \coprod_{i \in I(\Gamma, Q)} \Gamma \xi_i,$$

where $I(\Gamma, Q)$ is a set of indices and $\{\xi_i \mid i \in I(\Gamma, Q)\}$ is a complete set of representatives of the right coset space $\Gamma \backslash \Gamma \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma$. For any $f \in S_k(\Gamma)$, we put

$$T_Q f = \sum_{i \in I(\Gamma, Q)} f|_k \xi_i.$$

For any A-algebra B, we define a Hecke operator T_Q^{har} on $C_k^{\mathrm{har}}(\Gamma, B)$ as follows. Note that ξ_i^{-1} is an element of the monoid M^{-1}. For any $c \in C_k^{\mathrm{har}}(\Gamma, B)$ and $e \in \mathcal{T}_1^o$, we put

$$T_Q^{\mathrm{har}}(c)(e) = \sum_{i \in I(\Gamma, Q)} \xi_i^{-1} \circ c(\xi_i(e)).$$

Since c is Γ-equivariant, we see that $T_Q^{\mathrm{har}}(c)$ is a harmonic cocycle which is independent of the choice of a complete set of representatives $\{\xi_i \mid i \in I(\Gamma, Q)\}$. For any $\gamma \in \Gamma$, the set $\{\xi_i \gamma \mid i \in I(\Gamma, Q)\}$ is also a complete set of representatives of the same right coset space. This yields $T_Q^{\mathrm{har}}(c) \in C_k^{\mathrm{har}}(\Gamma, B)$. By [Böc, (17)], for any A-subalgebra B of $\mathbb{C}_\infty$, the endomorphism T_Q^{har} is identified with the restriction on $C_k^{\mathrm{har}}(\Gamma, B) \subseteq C_k^{\mathrm{har}}(\Gamma, \mathbb{C}_\infty)$ of the Hecke operator T_Q on $S_k(\Gamma)$ via the isomorphism $\mathrm{Res}_\Gamma : S_k(\Gamma) \to C_k^{\mathrm{har}}(\Gamma, \mathbb{C}_\infty)$.

We also introduce a Hecke operator $T_{1,Q}$ on $\mathcal{L}_{1,k}(B)$ as follows. We denote by $C_{1,k}^{\pm}(\Gamma, B)$ the set of Γ-equivariant maps $c : \mathcal{T}_1^{o,\mathrm{st}} \to V_k(B)$ satisfying $c(-e) = -c(e)$ for any $e \in \mathcal{T}_1^{o,\mathrm{st}}$. Then the map

$$\Phi_{1,\Gamma} : C_{1,k}^{\pm}(\Gamma, B) \to \mathcal{L}_{1,k}(B), \quad \Phi_{1,\Gamma}(c) = \sum_{e \in \Lambda_1} [e] \otimes c(e)$$

is independent of the choice of Λ_1. By the uniqueness of the expression (2.3), we see that it is an isomorphism. For any $c \in C_{1,k}^{\pm}(\Gamma, B)$ and $e \in \mathcal{T}_1^{o,\mathrm{st}}$, we put

$$T_{1,Q}^{\pm}(c)(e) = \sum_{i \in I(\Gamma, Q)} \sum_{e' \in \mathrm{src}(\xi_i(e))} \xi_i^{-1} \circ c(e').$$

By (2.5), it is independent of the choice of $\{\xi_i\}$, and the same argument as in the case of T_Q^{har} shows that it defines an endomorphism $T_{1,Q}^{\pm}$ on $C_{1,k}^{\pm}(\Gamma, B)$. Now we put

$$T_{1,Q} = \Phi_{1,\Gamma} \circ T_{1,Q}^{\pm} \circ \Phi_{1,\Gamma}^{-1}.$$

From the construction, we see that $T_{1,Q}$ is independent of the choices of Λ_1 and $\{\xi_i\}$.

For an explicit description of $T_{1,Q}$, fix a complete set of representatives Λ_1 and take any element $x = \sum_{e \in \Lambda_1} [e] \otimes \omega_e$ of $\mathcal{L}_{1,k}(B)$. For any $e' \in \mathcal{T}_1^{o,\mathrm{st}}$, we have

$$\Phi_{1,\Gamma}^{-1}(x)(e') = \varepsilon_{e'} \gamma_{e'}^{-1} \circ \omega_{r(e')},$$

where $\varepsilon_{e'}$, $\gamma_{e'}$ and $r(e')$ are defined as (2.3) using Λ_1. Hence we obtain

$$T_{1,Q}(x) = \sum_{e \in \Lambda_1} [e] \otimes \sum_{i \in I(\Gamma, Q)} \sum_{e' \in \mathrm{src}(\xi_i(e))} \varepsilon_{e'} (\xi_i^{-1} \gamma_{e'}^{-1}) \circ \omega_{r(e')}. \tag{2.6}$$

Proposition 2.2. *The restriction of $T_{1,Q}$ on the submodule $\mathcal{V}_k(B) \subseteq \mathcal{L}_{1,k}(B)$ agrees with T_Q^{har} via the isomorphism $\Phi_\Gamma : C_k^{\mathrm{har}}(\Gamma, B) \to \mathcal{V}_k(B)$. In particular, $\mathcal{V}_k(B)$ is stable under $T_{1,Q}$, and if B is an A-subalgebra of $\mathbb{C}_\infty$, then $\mathcal{V}_k(B)$ defines a B-lattice of $S_k(\Gamma)$ which is stable under Hecke operators.*

Proof. Take any $c \in C_k^{\mathrm{har}}(\Gamma, B)$. Since $c(r(e')) = \varepsilon_{e'}\gamma_{e'}c(e')$, (2.4) yields

$$\begin{aligned} T_{1,Q}(\Phi_\Gamma(c)) &= \sum_{e \in \Lambda_1} [e] \otimes \sum_{i \in I(\Gamma,Q)} \sum_{e' \in \mathrm{src}(\xi_i(e))} \xi_i^{-1} \circ c(e') \\ &= \sum_{e \in \Lambda_1} [e] \otimes \sum_{i \in I(\Gamma,Q)} \xi_i^{-1} \circ c(\xi_i(e)) = \sum_{e \in \Lambda_1} [e] \otimes T_Q^{\mathrm{har}}(c)(e), \end{aligned}$$

which agrees with $\Phi_\Gamma(T_Q^{\mathrm{har}}(c))$. □

3. Variation of Gouvêa-Mazur type

Let $\mathfrak{n} \in A$ be a non-zero polynomial which is prime to $\wp$. For any A-algebra B and any integer $m \geqslant 1$, put

$$B_m = B/\wp^m B.$$

Note that, since we have the canonical section $[-] : \kappa(\wp) \to \mathcal{O}_{K_\wp}$ of the natural surjection $\mathcal{O}_{K_\wp} \to \kappa(\wp)$, we can consider B_m canonically as a $\kappa(\wp)$-algebra.

Let $r \geqslant 1$ be an integer and Θ any subgroup of $1 + \wp A_r$. We define

$$\Gamma_0^\Theta(\wp^r) = \left\{ \gamma \in SL_2(A) \,\middle|\, \gamma \bmod \wp^r \in \begin{pmatrix} \Theta & * \\ 0 & \Theta \end{pmatrix} \right\} \subseteq \Gamma_1(\wp)$$

and $\Gamma_1^\Theta(\mathfrak{n}, \wp^r) = \Gamma_1(\mathfrak{n}) \cap \Gamma_0^\Theta(\wp^r)$ (The notation $\Gamma_0^\Theta(\wp^r)$ is meant to indicate that it consists of elements of $\Gamma_0(\wp^r)$ whose diagonal entries lie in Θ modulo $\wp^r$). The subgroup $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$ of $SL_2(A)$ is p'-torsion free and contains $\Gamma_1^{\{1\}}(\mathfrak{n}, \wp^r) = \Gamma_1(\mathfrak{n}\wp^r)$. When $\Theta = 1 + \wp A_r$, we also denote $\Gamma_0^\Theta(\wp^r)$ and $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$ by $\Gamma_0^p(\wp^r)$ and $\Gamma_1^p(\mathfrak{n}, \wp^r)$, respectively (The exponent p is meant to indicate that Θ is the p-Sylow subgroup of $A_r^\times$). For $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$, we fix a complete set of representatives Λ_1 as in §2.2.

For Hecke operators of level $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$, we also write

$$U = T_\wp, \quad U_1 = T_{1,\wp}.$$

Let $d(k, a)$ be the dimension of the generalized U-eigenspace in $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$ of slope a. In this section, we prove p-adic local constancy results for $d(k, a)$ with respect to k, which generalize the Gouvêa-Mazur conjecture [Hat2, Theorem 1.1] for the case of level $\Gamma_1(t)$.

3.1. **Hecke operators of level $\Gamma_1^\Theta(\mathfrak{n},\wp^r)$.** Let $Q\in A$ be any non-zero element. Write

$$\Gamma_1^\Theta(\mathfrak{n},\wp^r)\begin{pmatrix}1&0\\0&Q\end{pmatrix}\Gamma_1^\Theta(\mathfrak{n},\wp^r)=\coprod_{i\in I(Q)}\Gamma_1^\Theta(\mathfrak{n},\wp^r)\xi_i.$$

For any $\gamma\in\Gamma_1^\Theta(\mathfrak{n},\wp^r)$, $i\in I(Q)$ and $\lambda\in\kappa(\wp)^\times$, we have

$$\gamma\xi_i\equiv\begin{pmatrix}1&*\\0&Q\end{pmatrix},\quad \gamma\begin{pmatrix}\lambda^{-1}&0\\0&\lambda\end{pmatrix}\equiv\begin{pmatrix}\lambda^{-1}&*\\0&\lambda\end{pmatrix}\bmod\wp. \tag{3.1}$$

Consider the Hecke operator T_Q acting on the $\mathbb{C}_\infty$-vector space $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$, which preserves the A-lattice $\mathcal{V}_k(A)$ by Proposition 2.2. To describe it explicitly for the case where Q is irreducible, we fix a complete set of representatives R_Q of $A/(Q)$. When Q divides $\mathfrak{n}\wp^r$, we have $I(Q)=R_Q$ and

$$(T_Qf)(z)=\frac{1}{Q}\sum_{\beta\in R_Q}f\left(\frac{z+\beta}{Q}\right).$$

When Q does not divide $\mathfrak{n}\wp^r$, we can find $R,S\in A$ satisfying $RQ-\mathfrak{n}\wp^rS=1$. Put

$$\eta_\diamond=\begin{pmatrix}R&S\\\mathfrak{n}\wp^r&Q\end{pmatrix},\quad \xi_\diamond=\begin{pmatrix}RQ&S\\\mathfrak{n}\wp^rQ&Q\end{pmatrix}=\eta_\diamond\begin{pmatrix}Q&0\\0&1\end{pmatrix}.$$

Then we have $I(Q)=\{\diamond\}\sqcup R_Q$ and

$$(T_Qf)(z)=Q^{k-1}(\langle Q\rangle_{\mathfrak{n}\wp^r}f)(Qz)+\frac{1}{Q}\sum_{\beta\in R_Q}f\left(\frac{z+\beta}{Q}\right),$$

where $\langle Q\rangle_{\mathfrak{n}\wp^r}$ is the diamond operator acting on $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$ defined by $f\mapsto f|_k\eta_\diamond$.

Note that the natural map

$$SL_2(A)\to SL_2(A/(\mathfrak{n}\wp^r))\simeq SL_2(A/(\mathfrak{n}))\times SL_2(A_r)$$

is surjective. For any $\lambda\in\kappa(\wp)^\times$, we choose $\eta_\lambda\in SL_2(A)$ satisfying

$$\eta_\lambda\bmod\mathfrak{n}=I,\quad \eta_\lambda\bmod\wp^r=\begin{pmatrix}[\lambda]^{-1}&0\\0&[\lambda]\end{pmatrix} \tag{3.2}$$

and put

$$\langle\lambda\rangle_{\wp^r}f=f|_k\eta_\lambda.$$

By

$$\Gamma_1(\mathfrak{n}\wp^r)\subseteq\Gamma_1^\Theta(\mathfrak{n},\wp^r),\quad \eta_\lambda^{-1}\Gamma_1^\Theta(\mathfrak{n},\wp^r)\eta_\lambda=\Gamma_1^\Theta(\mathfrak{n},\wp^r), \tag{3.3}$$

this is independent of the choice of η_λ and defines an action of $\kappa(\wp)^\times$ on $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$.

For any $\kappa(\wp)[\kappa(\wp)^\times]$-module M and any character $\chi:\kappa(\wp)^\times\to\kappa(\wp)^\times$, we denote by $M(\chi)$ the maximal $\kappa(\wp)$-subspace of M on which any $\lambda\in\kappa(\wp)^\times$ acts via $\chi(\lambda)$. Since the order of the group $\kappa(\wp)^\times$ is prime to p, we have the projector

$$\varepsilon_\chi : M\to M(\chi),\quad \varepsilon_\chi(m)=-\sum_{\lambda\in\kappa(\wp)^\times}\chi(\lambda)^{-1}(\lambda\cdot m)$$

and the decomposition into χ-parts

$$M=\bigoplus_\chi M(\chi),$$

where the sum runs over the set of such characters $\kappa(\wp)^\times\to\kappa(\wp)^\times$.

We consider $\bar{K}$ as a $\kappa(\wp)$-algebra by the unique map $\kappa(\wp)\to\bar{K}$ which commutes the diagram

$$\begin{array}{ccc}\kappa(\wp) & \longrightarrow & \bar{K}\\ & \overset{[-]}{\searrow} & \downarrow{\scriptstyle \iota_\wp}\\ & & \mathbb{C}_\wp.\end{array}$$

Then we have

$$S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))=\bigoplus_\chi S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))(\chi).$$

Note that, when an irreducible polynomial Q does not divide $\mathfrak{n}\wp^r$, we may further assume that η_λ satisfies

$$\eta_\lambda=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad a\notin(Q).$$

Using this, for any irreducible polynomial Q we can show

$$\Gamma_1^\Theta(\mathfrak{n},\wp^r)\eta_\lambda^{-1}\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\eta_\lambda\Gamma_1^\Theta(\mathfrak{n},\wp^r)=\Gamma_1^\Theta(\mathfrak{n},\wp^r)\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\Gamma_1^\Theta(\mathfrak{n},\wp^r).$$

Then (3.3) yields

$$\begin{aligned}\coprod_{i\in I(Q)}\Gamma_1^\Theta(\mathfrak{n},\wp^r)\xi_i\eta_\lambda&=\Gamma_1^\Theta(\mathfrak{n},\wp^r)\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\eta_\lambda\Gamma_1^\Theta(\mathfrak{n},\wp^r)\\ &=\Gamma_1^\Theta(\mathfrak{n},\wp^r)\eta_\lambda\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\Gamma_1^\Theta(\mathfrak{n},\wp^r)=\coprod_{i\in I(Q)}\Gamma_1^\Theta(\mathfrak{n},\wp^r)\eta_\lambda\xi_i.\end{aligned}\tag{3.4}$$

Thus T_Q commutes with $\langle\lambda\rangle_{\wp^r}$ and $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))(\chi)$ is stable under Hecke operators. We denote by $d(k,\chi,a)$ be the dimension of the generalized U-eigenspace in $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))(\chi)$ of slope a. To indicate the level, we often write

$$d(k,a) = d(\Gamma_1^\Theta(\mathfrak{n},\wp^r),k,a), \quad d(k,\chi,a) = d(\Gamma_1^\Theta(\mathfrak{n},\wp^r),k,\chi,a).$$

For any A-algebra B, we also have the diamond operator $\langle\lambda\rangle_{\wp^r}$

$$\langle\lambda\rangle_{\wp^r} \in \mathrm{End}(C_k^{\mathrm{har}}(\Gamma_1^\Theta(\mathfrak{n},\wp^r),B)), \quad c \mapsto (e \mapsto \eta_\lambda^{-1} \circ c(\eta_\lambda(e))),$$

which is compatible with that on $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$ when $B = \mathbb{C}_\infty$. From (3.3) we see that e is $\Gamma_1^\Theta(\mathfrak{n},\wp^r)$-stable if and only of $\eta_\lambda(e)$ is, and thus the corresponding operators on $\mathcal{V}_k(B)$ and $\mathcal{L}_{1,k}(B)$ are given by

$$\langle\lambda\rangle_{\wp^r}\Big(\sum_{e\in\Lambda_1}[e]\otimes\omega_e\Big) = \sum_{e\in\Lambda_1}[e]\otimes\varepsilon_{\eta_\lambda(e)}(\eta_\lambda^{-1}\gamma_{\eta_\lambda(e)}^{-1})\circ\omega_{r(\eta_\lambda(e))}. \tag{3.5}$$

When B is also a $\kappa(\wp)$-algebra, we have the decomposition

$$C_k^{\mathrm{har}}(\Gamma_1^\Theta(\mathfrak{n},\wp^r),B) = \bigoplus_\chi C_k^{\mathrm{har}}(\Gamma_1^\Theta(\mathfrak{n},\wp^r),B)(\chi)$$

and similarly for $\mathcal{L}_{1,k}(B)$ and $\mathcal{V}_k(B)$. These summands are stable under Hecke operators by (3.4).

3.2. **Weight reduction.** Let $N \geqslant 1$ be any integer. For any A-algebra B, the B-linear map

$$\mu_{k,N} : H_{k-2}(B) \to H_{k-2+N}(B), \quad X^iY^{k-2-i} \mapsto X^{i+N}Y^{k-2-i}$$

induces the dual map

$$\rho_{k,N} : V_{k+N}(B) \to V_k(B), \quad (X^iY^{k+N-2-i})^\vee \mapsto \begin{cases} (X^{i-N}Y^{k+N-2-i})^\vee & (i \geqslant N) \\ 0 & (i < N) \end{cases}.$$

It is a surjection whose kernel is

$$V_{k+N}^{<N}(B) = \bigoplus_{i<N} B(X^iY^{k+N-2-i})^\vee.$$

Lemma 3.1. *Let $n \geqslant 0$ be any non-negative integer, $\bar{B}$ any A_{p^n}-algebra and $\lambda \in \kappa(\wp)^\times$. Let $\xi \in M_2(A)$ be any element satisfying*

$$\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \bmod \wp = \lambda, \quad c \equiv 0 \bmod \wp.$$

Let m be the order of λ in $\kappa(\wp)^\times$. Then, for any element $\omega \in V_{k+p^nm}(\bar{B})$, we have

$$\xi^{-1} \circ \rho_{k,p^nm}(\omega) = \rho_{k,p^nm}(\xi^{-1}\circ\omega).$$

In particular, for any integer $m' \geqslant 1$, the map $\rho_{k,p^nm'} : V_{k+p^nm'}(\bar{B}) \to V_k(\bar{B})$ is $\Gamma_1^\Theta(\mathfrak{n},\wp^r)$-equivariant and its kernel $V_{k+p^nm'}^{<p^nm'}(\bar{B})$ is $\Gamma_1^\Theta(\mathfrak{n},\wp^r)$-stable.

Proof. In the ring A_{p^n}, we can write $a = [\lambda] + \wp a'$ with some $a' \in A_{p^n}$. For any integer $i \in [0, k-2]$, the assumption $\wp^{p^n}\bar{B} = 0$ implies

$$\begin{aligned}\xi \circ \mu_{k,p^n m}(X^i Y^{k-2-i}) &= (aX + cY)^{p^n m+i}(bX + dY)^{k-2-i} \\ &= (a^{p^n} X^{p^n} + c^{p^n} Y^{p^n})^m (aX + cY)^i (bX + dY)^{k-2-i} \\ &= ([\lambda]^{p^n} X^{p^n})^m (aX + cY)^i (bX + dY)^{k-2-i} \\ &= X^{p^n m}(aX + cY)^i (bX + dY)^{k-2-i} \\ &= \mu_{k,p^n m}(\xi \circ (X^i Y^{k-2-i})).\end{aligned}$$

Taking the dual yields the lemma. ☐

By Lemma 3.1, for any A_{p^n}-algebra $\bar{B}$ and any integer $m' \geqslant 1$, we obtain the surjection

$$1 \otimes \rho_{k,p^n m'} : \mathcal{V}_{k+p^n m'}(\bar{B}) \to \mathcal{V}_k(\bar{B})$$

and similarly for $\mathcal{L}_{1,k}(\bar{B})$.

Lemma 3.2. *For any A_{p^n}-algebra $\bar{B}$, the maps*

$$1 \otimes \rho_{k,p^n} : \mathcal{V}_{k+p^n}(\bar{B}) \to \mathcal{V}_k(\bar{B}), \quad \mathcal{L}_{1,k+p^n}(\bar{B}) \to \mathcal{L}_{1,k}(\bar{B})$$

commute with Hecke operators. Moreover, the maps

$$1 \otimes \rho_{k,p^n(q^d-1)} : \mathcal{V}_{k+p^n(q^d-1)}(\bar{B}) \to \mathcal{V}_k(\bar{B}), \quad \mathcal{L}_{1,k+p^n(q^d-1)}(\bar{B}) \to \mathcal{L}_{1,k}(\bar{B})$$

commute with $\langle \lambda \rangle_{\wp^r}$ for any $\lambda \in \kappa(\wp)^\times$. In particular, the $\bar{B}$-submodules

$$\mathcal{V}^{<p^n}_{k+p^n}(\bar{B}), \quad \mathcal{V}^{<p^n(q^d-1)}_{k+p^n(q^d-1)}(\bar{B})$$

are stable under Hecke operators.

Proof. It is enough to show the assertions on $\mathcal{L}_{1,k}(\bar{B})$. By (2.6) and (3.5), we reduce ourselves to showing that, for any $\gamma \in \Gamma_1^\Theta(\mathfrak{n}, \wp^r)$, $i \in I(Q)$, $\lambda \in \kappa(\wp)^\times$, $\omega \in V_{k+p^n}(\bar{B})$ and $\omega' \in V_{k+p^n(q^d-1)}(\bar{B})$, we have

$$\begin{aligned}(\gamma\xi_i)^{-1} \circ \rho_{k,p^n}(\omega) &= \rho_{k,p^n}((\gamma\xi_i)^{-1} \circ \omega), \\ (\gamma\eta_\lambda)^{-1} \circ \rho_{k,p^n(q^d-1)}(\omega') &= \rho_{k,p^n(q^d-1)}((\gamma\eta_\lambda)^{-1} \circ \omega').\end{aligned}$$

By (3.1), this follows from Lemma 3.1. ☐

3.3. **Dimension of slope zero cuspforms.** Using harmonic cocycles, the proofs of [Hid1, Corollary 8.2 and Proposition 8.3] can be adapted to obtain constancy results for the dimension of slope zero cuspforms with respect to the weight and the level at $\wp$. First we prove the following key lemma.

Lemma 3.3. *Let B be any flat A-algebra. For any $s \in \mathrm{St}$ and any integer $j \in [0, k-2]$, the element $s \otimes (X^j Y^{k-2-j})^\vee \in \mathcal{V}_k(B)$ satisfies*

$$U(s \otimes (X^j Y^{k-2-j})^\vee) \in \wp^{k-2-j}\mathcal{V}_k(B).$$

Proof. For any non-negative integer m, we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V}_k(B) & \longrightarrow & \mathcal{L}_{1,k}(B) & \xrightarrow{\partial_\Gamma \otimes 1} & \mathcal{L}_{0,k}(B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{V}_k(B_m) & \longrightarrow & \mathcal{L}_{1,k}(B_m) & \xrightarrow{\partial_\Gamma \otimes 1} & \mathcal{L}_{0,k}(B_m) & \longrightarrow & 0. \end{array}$$

Since the structure map $A \to B$ is flat, we see that $\wp^m\mathcal{V}_k(B)$ and $\wp^m\mathcal{L}_{1,k}(B)$ are the kernels of the left two vertical maps. Thus it suffices to show $U_1(s \otimes (X^j Y^{k-2-j})^\vee) \in \wp^{k-2-j}\mathcal{L}_{1,k}(B)$.

Any element of St is a $\mathbb{Z}$-linear combination of elements of $\mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}]$ of the form $[e]|_\alpha$ with $e \in \Lambda_1$ and $\alpha \in \Gamma_1^\Theta(\mathfrak{n}, \wp^r)$. Moreover, for any $\omega \in V_k(B)$, we have $[e]|_\alpha \otimes \omega = [e] \otimes \alpha \circ \omega$. By (2.6), it is enough to show that, for any $i \in I(\wp)$, $\gamma \in \Gamma_1^\Theta(\mathfrak{n}, \wp^r)$ and integers $j, l \in [0, k-2]$, we have

$$((\gamma\xi_i)^{-1} \circ (X^j Y^{k-2-j})^\vee)(X^l Y^{k-2-l}) \in \wp^{k-2-j}B.$$

Write $\gamma\xi_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the above evaluation is equal to

$$(X^j Y^{k-2-j})^\vee((aX + cY)^l(bX + dY)^{k-2-l}).$$

By (3.1) we have $c, d \equiv 0 \bmod \wp$ and the coefficient of $X^j Y^{k-2-j}$ in the product $(aX + cY)^l(bX + dY)^{k-2-l}$ is divisible by $\wp^{k-2-j}$. This concludes the proof. □

Proposition 3.4. *(1) $d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k, 0)$ is independent of k.*

(2) For any character $\chi : \kappa(\wp)^\times \to \kappa(\wp)^\times$, we have

$$k_1 \equiv k_2 \bmod q^d - 1 \Rightarrow d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k_1, \chi, 0) = d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k_2, \chi, 0).$$

Proof. Let us consider the operator U acting on the $k(\wp)$-module $\mathcal{V}_k(\kappa(\wp))$. Note that $d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k, 0)$ is equal to the degree in X of the polynomial

$$\det(I - UX; \mathcal{V}_k(\kappa(\wp))).$$

By Lemma 3.2 for $n = 0$, we have the exact sequence

$$0 \longrightarrow \mathcal{V}_{k+1}^{<1}(\kappa(\wp)) \longrightarrow \mathcal{V}_{k+1}(\kappa(\wp)) \longrightarrow \mathcal{V}_k(\kappa(\wp)) \longrightarrow 0$$

whose maps are compatible with Hecke operators. Since $(k+1)-2>0$, Lemma 3.3 implies $U=0$ on $\mathcal{V}^{<1}_{k+1}(\kappa(\wp))$ and thus we have

$$\det(I-UX;\mathcal{V}^{<1}_{k+1}(\kappa(\wp)))=1,$$

which yields the assertion (1). Since Lemma 3.2 also gives the exact sequence

$$0\longrightarrow \mathcal{V}^{<q^d-1}_{k+q^d-1}(\kappa(\wp))(\chi)\longrightarrow \mathcal{V}_{k+q^d-1}(\kappa(\wp))(\chi)\longrightarrow \mathcal{V}_k(\kappa(\wp))(\chi)\longrightarrow 0,$$

the assertion (2) follows similarly. $\square$

Proposition 3.5. *$d(\Gamma_1^p(\mathfrak{n},\wp^r),k,0)$ and $d(\Gamma_1^p(\mathfrak{n},\wp^r),k,\chi,0)$ are independent of $r\geqslant 1$.*

Proof. Put $\Gamma_r=\Gamma_1^p(\mathfrak{n},\wp^r)$. Let $\bar{\kappa}$ be an algebraic closure of $\kappa(\wp)$. We reduce ourselves to showing that the multiplicities of non-zero eigenvalues of U acting on $C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa})$ and $C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa})(\chi)$ are independent of r. These are the same as the dimensions of the generalized eigenspaces

$$C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa})^{\mathrm{ord}},\quad C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa})(\chi)^{\mathrm{ord}}$$

of non-zero eigenvalues, respectively.

Since any $c\in C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa})$ is also Γ_{r+1}-equivariant, we have the natural inclusion

$$\iota: C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa})\to C_k^{\mathrm{har}}(\Gamma_{r+1},\bar{\kappa}).$$

Since we have

$$\Gamma_{r+1}\begin{pmatrix}1&0\\0&\wp\end{pmatrix}\Gamma_r=\coprod_{\beta\in R_\wp}\Gamma_{r+1}\xi_\beta,\quad \xi_\beta=\begin{pmatrix}1&\beta\\0&\wp\end{pmatrix}$$

with some set of indices $R_\wp$, we obtain a map $s: C_k^{\mathrm{har}}(\Gamma_{r+1},\bar{\kappa})\to C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa})$ by

$$s(c)(e)=\sum_{\beta\in R_\wp}\xi_\beta^{-1}\circ c(\xi_\beta(e)),$$

which makes the following diagram commutative.

$$\begin{array}{ccc} C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa}) & \xrightarrow{\ \iota\ } & C_k^{\mathrm{har}}(\Gamma_{r+1},\bar{\kappa}) \\ {\scriptstyle U}\downarrow & \overset{s}{\swarrow} & \downarrow{\scriptstyle U} \\ C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa}) & \xrightarrow[\ \iota\ ]{} & C_k^{\mathrm{har}}(\Gamma_{r+1},\bar{\kappa}) \end{array}$$

From this we see that ι and s commute with U and, since U induces an isomorphism on $C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa})^{\mathrm{ord}}$, the map ι gives an isomorphism

$$\iota^{\mathrm{ord}}: C_k^{\mathrm{har}}(\Gamma_r,\bar{\kappa})^{\mathrm{ord}}\to C_k^{\mathrm{har}}(\Gamma_{r+1},\bar{\kappa})^{\mathrm{ord}}.$$

This settles the assertion on $d(\Gamma_1^p(\mathfrak{n},\wp^r),k,0)$. Moreover, since the diamond operator $\langle\lambda\rangle_{\wp^r}$ is independent of the choice of η_λ satisfying (3.2), we also have

$$\langle\lambda\rangle_{\wp^{r+1}}\circ\iota=\iota\circ\langle\lambda\rangle_{\wp^r}.$$

Since U commutes with diamond operators, the map ι^{ord} also induces an isomorphism

$$C_k^{\mathrm{har}}(\Gamma_r,\bar\kappa)(\chi)^{\mathrm{ord}}\to C_k^{\mathrm{har}}(\Gamma_{r+1},\bar\kappa)(\chi)^{\mathrm{ord}},$$

from which the assertion on $d(\Gamma_1^p(\mathfrak{n},\wp^r),k,\chi,0)$ follows. □

3.4. **Representing matrix of U.** Let $E/K_\wp$ be a finite extension of complete valuation fields. We extend the normalized $\wp$-adic valuation $v_\wp$ naturally to E. We denote by $\mathcal{O}_E$ the ring of integers of E.

Lemma 3.6. *Suppose that $\mathfrak{n}\wp$ has an irreducible factor π of degree one. Then the right $\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$-module* St *is free of rank $[\Gamma_1(\pi):\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$, where the rank is independent of the choice of such π.*

Proof. Note that, from $\Gamma_1^\Theta(\mathfrak{n},\wp^r)\subseteq\Gamma_1(\mathfrak{n}\wp)$, we see that the former is a subgroup of $\Gamma_1(\pi)$. We can show that a fundamental domain of $\Gamma_1(\pi)\backslash\mathcal{T}$ is the same as the picture of [LM, §7], and that it has no $\Gamma_1(\pi)$-stable vertex and only one $\Gamma_1(\pi)$-stable (unoriented) edge. By (2.1), the right $\mathbb{Z}[\Gamma_1(\pi)]$-module St is free of rank one. Thus the right $\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$-module St is free of rank $[\Gamma_1(\pi):\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$. Since we have

$$[\Gamma_1(\pi):\Gamma_1^\Theta(\mathfrak{n},\wp^r)]=[SL_2(A):\Gamma_1^\Theta(\mathfrak{n},\wp^r)]\left[SL_2(\mathbb{F}_q):\left\{\begin{pmatrix}1&*\\0&1\end{pmatrix}\right\}\right]^{-1},$$

the rank is independent of π. □

In the sequel, we assume that $\mathfrak{n}\wp$ has an irreducible factor π of degree one. Under this assumption, Lemma 3.6 implies that the right $\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$-module St is free of rank δ, where we put

$$\delta=[\Gamma_1(\pi):\Gamma_1^\Theta(\mathfrak{n},\wp^r)].$$

Hence, for any A-algebra B, the B-module $\mathcal{V}_k(B)$ is free of rank $\delta(k-1)$. We fix an ordered basis $\mathfrak{B}_k$ of the free A-module $\mathcal{V}_k(A)$, as follows. Take an ordered basis $(v_1,\ldots,v_\delta)$ of the right $\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$-module St. The set

$$\mathfrak{B}_k=\{v_{i,j}=v_i\otimes(X^jY^{k-2-j})^\vee\mid 1\leqslant i\leqslant\delta,\ 0\leqslant j\leqslant k-2\}$$

forms a basis of the A-module $\mathcal{V}_k(A)$, and we order it as

$$v_{1,0},v_{2,0},\ldots,v_{\delta,0},v_{1,1},v_{2,1},\ldots,v_{\delta,1},v_{1,2},\ldots.$$

For any A-algebra B, the ordered basis of the B-module $\mathcal{V}_k(B)$ induced by $\mathfrak{B}_k$ is also denoted abusively by $\mathfrak{B}_k$. We denote by $U^{(k)}$ the representing matrix of U acting on the $\mathcal{O}_E$-module $\mathcal{V}_k(\mathcal{O}_E)$ with respect to the ordered basis $\mathfrak{B}_k$. Then Lemma 3.3 gives

$$U(v_{i,j}) \in \wp^{k-2-j}\mathcal{V}_k(\mathcal{O}_E). \tag{3.6}$$

In order to study perturbation of $U^{(k)}$, we use the following lemma of [Ked]. Note that the assumption "$B \in GL_n(F)$" imposed in [Ked] is superfluous.

Lemma 3.7 ([Ked], Proposition 4.4.4)**.** *Let L be any positive integer and $A, B \in M_L(\mathcal{O}_E)$. Let $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_L$ be the elementary divisors of A. Namely, they are the normalized $\wp$-adic valuations of diagonal entries of the Smith normal form of A. Let $s'_1 \leqslant s'_2 \leqslant \cdots \leqslant s'_L$ be the elementary divisors of AB. Then we have*

$$s'_i \geqslant s_i \quad \text{for any } i.$$

The same inequality also holds for the elementary divisors of BA.

Corollary 3.8. *Suppose that $\mathfrak{n}\wp$ has an irreducible factor π of degree one. Put $\delta = [\Gamma_1(\pi) : \Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)]$. Let $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_{\delta(k-1)}$ be the elementary divisors of $U^{(k)}$. Then we have*

$$s_i \geqslant \left\lfloor \frac{i-1}{\delta} \right\rfloor.$$

Proof. By (3.6), the matrix $U^{(k)}$ can be written as

$$U^{(k)} = B\mathrm{diag}(\wp^{k-2}, \ldots, \wp^{k-2}, \ldots, \wp, \ldots, \wp, 1, \ldots, 1),$$

where $B \in M_{\delta(k-1)}(\mathcal{O}_E)$ and the diagonal entries of the last matrix are $\{\wp^j \mid 0 \leqslant j \leqslant k-2\}$, each with multiplicity δ. Then the corollary follows from Lemma 3.7. □

Corollary 3.9. *Let $n \geqslant 0$ be any non-negative integer. Then, for some matrices B_1, B_2, B_3, B_4 with entries in $\mathcal{O}_E$, we have*

$$U^{(k+p^n)} = \left(\begin{array}{c|c} \wp^{k-1}B_1 & B_2 \\ \hline \wp^{p^n}B_3 & U^{(k)} + \wp^{p^n}B_4 \end{array}\right).$$

Proof. By Lemma 3.2, the lower right block is congruent to $U^{(k)}$ and the lower left block is zero modulo $\wp^{p^n}$. By (3.6), the entries on the upper left block are divisible by $\wp^{k-1}$. This concludes the proof. □

For the U-operator acting on $\mathcal{V}_k(\mathcal{O}_E)(\chi)$, we have a similar description of its representing matrix $U_\chi^{(k)}$ as follows.

Proposition 3.10. *Suppose that* $\mathfrak{n}\wp$ *has an irreducible factor* π *of degree one. Put* $\delta = [\Gamma_1(\pi) : \Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)]$.

(1) *For any integer* $i \geqslant 0$*, the* i*-th smallest elementary divisor* $s_{\chi,i}$ *of* $U_\chi^{(k)}$ *satisfies*

$$s_{\chi,i} \geqslant \left\lfloor \frac{i-1}{\delta} \right\rfloor.$$

(2) *Let* $n \geqslant 0$ *be any non-negative integer. Then, with some bases of* $\mathcal{V}_k(\mathcal{O}_E)(\chi)$ *and* $\mathcal{V}_{k+p^n(q^d-1)}(\mathcal{O}_E)(\chi)$*, the representing matrices* $U_\chi^{(k)}$ *and* $U_\chi^{(k+p^n(q^d-1))}$ *of* U *acting on them satisfies*

$$U_\chi^{(k+p^n(q^d-1))} = \left(\begin{array}{c|c} \wp^{k-1} B_1 & B_2 \\ \hline \wp^{p^n} B_3 & U_\chi^{(k)} + \wp^{p^n} B_4 \end{array}\right)$$

for some matrices B_1, B_2, B_3, B_4 *with entries in* $\mathcal{O}_E$.

Proof. We have the decomposition

$$\mathcal{V}_k(\mathcal{O}_E) = \bigoplus_{\chi} \mathcal{V}_k(\mathcal{O}_E)(\chi),$$

where each summand is stable under Hecke operators. Thus any elementary divisor of $U_\chi^{(k)}$ is also an elementary divisor of $U^{(k)}$, and $s_{\chi,i}$ equals the i'-th smallest elementary divisor $s_{i'}$ of $U^{(k)}$ with some $i' \geqslant i$. Hence the assertion (1) follows from Corollary 3.8.

For (2), put $m = q^d - 1$, $k' = k + p^n m$ and consider the weight reduction map

$$\rho = 1 \otimes \rho_{k,p^n m} : \mathcal{V}_{k'}(\mathcal{O}_{E,p^n}) \to \mathcal{V}_k(\mathcal{O}_{E,p^n})$$

for $\mathcal{O}_{E,p^n} = \mathcal{O}_E/\wp^{p^n}\mathcal{O}_E$. By Lemma 3.1, we can define the tensor product over $\mathbb{Z}[\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)]$

$$\mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n}) = \mathrm{St} \otimes_{\mathbb{Z}[\Gamma_1^{\Theta}(\mathfrak{n},\wp^r)]} V_{k'}^{<p^n m}(\mathcal{O}_{E,p^n}),$$

which sits in the split exact sequence of $\mathcal{O}_{E,p^n}$-modules

$$0 \longrightarrow \mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n}) \longrightarrow \mathcal{V}_{k'}(\mathcal{O}_{E,p^n}) \xrightarrow{\rho} \mathcal{V}_k(\mathcal{O}_{E,p^n}) \longrightarrow 0.$$

By Lemma 3.2, the map ρ is compatible with Hecke operators and $\langle \lambda \rangle_{\wp^r}$ for any $\lambda \in \kappa(\wp)^\times$. Thus the map ρ also induces the split exact sequence

$$0 \longrightarrow \mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n})(\chi) \longrightarrow \mathcal{V}_{k'}(\mathcal{O}_{E,p^n})(\chi) \xrightarrow{\rho} \mathcal{V}_k(\mathcal{O}_{E,p^n})(\chi) \longrightarrow 0.$$

Let $\varepsilon_\chi : \mathcal{V}_{k'}(\mathcal{O}_E) \to \mathcal{V}_{k'}(\mathcal{O}_E)(\chi)$ be the projector to the χ-part. Let κ_E be the residue field of E. Consider the basis $v_{i,j} = v_i \otimes (X^j Y^{k'-2-j})^\vee$ of $\mathcal{V}_{k'}(\mathcal{O}_E)$ as before and its image $\bar{v}_{i,j}$ in $\mathcal{V}_{k'}(\kappa_E)$. Note that, for any

$j < p^n m$, the image of $\varepsilon_\chi(v_{i,j})$ in $\mathcal{V}_{k'}(\mathcal{O}_{E,p^n})(\chi)$ lies in $\mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n})(\chi)$. Since the set

$$\{\varepsilon_\chi(\bar{v}_{i,j}) \mid 1 \leqslant i \leqslant \delta,\ 0 \leqslant j \leqslant p^n m - 1\}$$

spans the κ_E-vector space $\mathcal{V}_{k'}^{<p^n m}(\kappa_E)(\chi)$, there exists a subset $\Sigma \subseteq [1, \delta] \times [0, p^n m - 1]$ such that the elements $\varepsilon_\chi(\bar{v}_{i,j})$ for $(i, j) \in \Sigma$ form its basis.

Now take a lift $\mathfrak{B}_{k',\chi,k}$ of a basis of $\mathcal{V}_k(\mathcal{O}_{E,p^n})(\chi)$ by the composite

$$\mathcal{V}_{k'}(\mathcal{O}_E)(\chi) \to \mathcal{V}_{k'}(\mathcal{O}_{E,p^n})(\chi) \xrightarrow{\rho} \mathcal{V}_k(\mathcal{O}_{E,p^n})(\chi).$$

Since the image of the set

$$\mathfrak{B}_{k',\chi} = \{\varepsilon_\chi(v_{i,j}) \mid (i, j) \in \Sigma\} \sqcup \mathfrak{B}_{k',\chi,k}$$

in $\mathcal{V}_{k'}(\kappa_E)(\chi)$ forms its basis, we see that $\mathfrak{B}_{k',\chi}$ itself forms a basis of $\mathcal{V}_{k'}(\mathcal{O}_E)(\chi)$. Moreover, by Nakayama's lemma, the images of $\varepsilon_\chi(v_{i,j})$ in $\mathcal{V}_{k'}(\mathcal{O}_{E,p^n})$ for $(i, j) \in \Sigma$ form a basis of $\mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n})(\chi)$.

Representing U by the basis $\mathfrak{B}_{k',\chi}$, we see that the lower blocks of the resulting matrix are as stated in (2). Moreover, since U and $\langle\lambda\rangle_{\wp^r}$ commute with each other, (3.6) yields

$$U(\varepsilon_\chi(v_{i,j})) = \varepsilon_\chi(U(v_{i,j})) \in \wp^{k'-2-j}\mathcal{V}_{k'}(\mathcal{O}_E)(\chi)$$

for any $j < p^n m$, and thus the upper left block is divisible by $\wp^{k-1}$. This concludes the proof. □

3.5. **Perturbation.** Let $E/K_\wp$ be a finite extension inside $\mathbb{C}_\wp$. Let V be an E-vector space of finite dimension and $T : V \to V$ an E-linear endomorphism. For an eigenvector of T with eigenvalue $\lambda \in \mathbb{C}_\wp$, we refer to $v_\wp(\lambda)$ as its slope. For any rational number a, we denote by $d(T, a)$ the multiplicity of T-eigenvalues of slope a. If B is the representing matrix of T with some basis of V, we also denote it by $d(B, a)$.

Proposition 3.11. *Let δ_0, n and L be positive integers. Let $B \in M_L(\mathcal{O}_E)$ be a matrix such that its i-th smallest elementary divisor s_i satisfies $s_i \geqslant \lfloor \frac{i-1}{\delta_0} \rfloor$ for any i. Put $\varepsilon_0 = d(B, 0)$ and*

$$C_1(n, \delta_0, \varepsilon_0) = p^n \left(\frac{4 + \delta_0 p^n - \delta_0}{4 + 2\delta_0 p^n - 2\varepsilon_0} \right) \in (0, p^n).$$

Moreover, we put $q_1 = r_1 = 0$ and for any $l \geqslant 2$, we write $q_l = \lfloor \frac{l-2}{\delta_0} \rfloor$ and $r_l = l - 2 - \delta_0 q_l$. We define $C_2(n, \delta_0, \varepsilon_0)$ as

$$\min \left\{ \frac{2p^n + \delta_0 q_l(q_l - 1) + 2q_l(r_l + 1)}{2(l - \varepsilon_0)} \;\middle|\; \varepsilon_0 < l \leqslant 1 + \delta_0 p^n \right\}$$

and put

$$C(n,\delta_0,\varepsilon_0)=\min\{C_1(n,\delta_0,\varepsilon_0),C_2(n,\delta_0,\varepsilon_0)\}\in(0,p^n).$$

Let $B'\in M_L(\mathcal{O}_E)$ *be any matrix satisfying* $B'-B\in\wp^{p^n}M_L(\mathcal{O}_E)$. *Let* a *be any non-negative rational number satisfying*

$$a<C(n,\delta_0,\varepsilon_0).$$

Then we have

$$d(B,a)=d(B',a).$$

Proof. We put

$$P_B(X)=\det(I-BX)=\sum b_l X^l,\quad P_{B'}(X)=\det(I-B'X)=\sum b'_l X^l.$$

Then b_l is, up to a sign, the sum of principal $l\times l$ minors of B. Since $P_B\equiv P_{B'}\bmod\wp$, we have $d(B',0)=d(B,0)=\varepsilon_0$. From the assumption on elementary divisors, we see that if $i>\delta_0$, then any $i\times i$ minor of B is divisible by $\wp$. This yields $\varepsilon_0\leqslant\delta_0$. From this inequality and $n\geqslant 1$, we see $C_1(n,\delta_0,\varepsilon_0)\in(0,p^n)$. Since each member of the set in the definition of $C_2(n,\delta_0,\varepsilon_0)$ is positive, we obtain $C_2(n,\delta_0,\varepsilon_0)>0$ and thus $C(n,\delta_0,\varepsilon_0)\in(0,p^n)$.

By [Ked, Theorem 4.4.2], for any $l\geqslant 0$ we have

$$v_\wp(b_l-b'_l)\geqslant p^n+\sum_{j=1}^{l-1}\min\left\{\left\lfloor\frac{j-1}{\delta_0}\right\rfloor,p^n\right\}.$$

Here we mean that the second term of the right-hand side is zero for $l\leqslant 1$. Let R be the right-hand side of the inequality. We claim that for any $l>\varepsilon_0$, we have

$$a<C(n,\delta_0,\varepsilon_0)\Rightarrow R>a(l-\varepsilon_0).$$

Indeed, when $l>1+\delta_0p^n$, we have

$$R=p^n+\sum_{j=1}^{\delta_0p^n}\left\lfloor\frac{j-1}{\delta_0}\right\rfloor+\sum_{j=1+\delta_0p^n}^{l-1}p^n=p^n(l-\delta_0p^n)+\frac{1}{2}\delta_0p^n(p^n-1)$$
$$=\frac{1}{2}p^n(2l-\delta_0-\delta_0p^n).$$

Then $R>a(l-\varepsilon_0)$ if and only if

$$(p^n-a)l-\frac{1}{2}p^n\delta_0(1+p^n)+a\varepsilon_0>0. \tag{3.7}$$

Since the condition $a<C(n,\delta_0,\varepsilon_0)$ yields $p^n>a$, the left-hand side of (3.7) is increasing with respect to l. Thus (3.7) holds for any $l>1+\delta_0p^n$ if and only if it holds for $l=2+\delta_0p^n$, which is equivalent to $a<C_1(n,\delta_0,\varepsilon_0)$.

On the other hand, when $l \leqslant 1+\delta_0 p^n$, we have

$$R = p^n + \frac{1}{2}\delta_0 q_l(q_l-1) + q_l(r_l+1), \tag{3.8}$$

from which the claim follows.

Let N_B and $N_{B'}$ be the Newton polygons of P_B and $P_{B'}$, respectively. It suffices to show that the segments of N_B and $N_{B'}$ with slope less than $C(n,\delta_0,\varepsilon_0)$ agree with each other. Suppose the contrary and take the smallest slope $a < C(n,\delta_0,\varepsilon_0)$ satisfying $d(B,a) \neq d(B',a)$.

Let (l,y) be the right endpoint of the segment of slope a in either of N_B or $N_{B'}$. Since $d(B,0)=d(B',0)$, we have $a>0$ and $l>\varepsilon_0$. Then the above claim yields

$$y \leqslant a(l-\varepsilon_0) < v_\wp(b_l - b'_l).$$

Since $y \in \{v_\wp(b_l), v_\wp(b'_l)\}$, we have $v_\wp(b_l)=v_\wp(b'_l)$. Since a is minimal, this implies that the slope a appears in both of N_B and $N_{B'}$. Applying the same argument to the right endpoint of the segment of slope a in the other Newton polygon, we obtain $d(B,a)=d(B',a)$. This is the contradiction. □

By a similar argument, we can show a slightly different perturbation result as follows.

Proposition 3.12. *With the notation in Proposition 3.11, we suppose that the following conditions hold.*

(1) If $p=2$, then $n \geqslant 3$ or $\delta_0-\varepsilon_0 \leqslant 1$.
(2) $2p^n > n(\delta_0 n + 2 + \delta_0 - 2\varepsilon_0)$.

Then, for any non-negative rational number $a \leqslant n$, we have

$$d(B,a)=d(B',a).$$

Proof. Let R be as in the proof of Proposition 3.11. We claim $R > n(l-\varepsilon_0)$ for any $l>\varepsilon_0$ under the assumptions (1) and (2).

Indeed, when $l > 1+\delta_0 p^n$, we have $R > n(l-\varepsilon_0)$ for any such l if and only if $n < C_1(n,\delta_0,\varepsilon_0)$, namely

$$\delta_0 p^n\left(\frac{1}{2}p^n - n\right) + 2(p^n-n) + n\varepsilon_0 > \frac{1}{2}\delta_0 p^n.$$

If $p \geqslant 3$ or $n \geqslant 3$, then we have $\frac{1}{2}p^n - n \geqslant \frac{1}{2}$ and the above inequality holds. If $p=2$ and $n<3$, it is equivalent to $\delta_0-\varepsilon_0 \leqslant 1$. Thus, under the condition (1), we have $R > n(l-\varepsilon_0)$ in this case.

Let us consider the case of $l \leqslant 1+\delta_0 p^n$. Note that $l=1$ is allowed only if $\varepsilon_0=0$, in which case the claim holds by $R=p^n>n$. For $l \geqslant 2$, by (3.8) we have $R > n(l-\varepsilon_0)$ if and only if

$$2p^n + \delta_0\left(q_l - n + \frac{r_l+1}{\delta_0} - \frac{1}{2}\right)^2 - \delta_0\left(-n + \frac{r_l+1}{\delta_0} - \frac{1}{2}\right)^2 > 2n(r_l + 2 - \varepsilon_0).$$

Note $\frac{r_l+1}{\delta_0} - \frac{1}{2} \in [-\frac{1}{2}, \frac{1}{2}]$. Since q_l and n are integers, we have

$$\delta_0\left(q_l - n + \frac{r_l+1}{\delta_0} - \frac{1}{2}\right)^2 \geqslant \delta_0\left(\frac{r_l+1}{\delta_0} - \frac{1}{2}\right)^2.$$

Thus the above inequality holds if

$$2p^n + \delta_0\left(\frac{r_l+1}{\delta_0} - \frac{1}{2}\right)^2 - \delta_0\left(-n + \frac{r_l+1}{\delta_0} - \frac{1}{2}\right)^2 > 2n(r_l + 2 - \varepsilon_0),$$

which is equivalent to the condition (2) and the claim follows. Now the same reasoning as in the proof of Proposition 3.11 shows $d(B,a) = d(B',a)$. □

3.6. **Dimension variation.** For the U-operators acting on $\mathcal{V}_k(K_\wp)$ and $\mathcal{V}(K_\wp)(\chi)$, we denote $d(U,a)$ also by

$$d(k,a) = d(\Gamma_1^\Theta(\mathfrak{n},\wp^r),k,a), \quad d(k,\chi,a) = d(\Gamma_1^\Theta(\mathfrak{n},\wp^r),k,\chi,a),$$

respectively. Note that they agree with the previously defined ones for $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$ and $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))(\chi)$.

Now the following theorems give generalizations of [Hat2, Theorem 1.1].

Theorem 3.13. *Suppose that $\mathfrak{n}\wp$ has an irreducible factor π of degree one. Let $n \geqslant 1$ and $k \geqslant 2$ be arbitrary integers. Put $\delta = [\Gamma_1(\pi) : \Gamma_1^\Theta(\mathfrak{n},\wp^r)]$ and $\varepsilon = d(k,0)$. Let a be any non-negative rational number satisfying*

$$a < \min\{C(n,\delta,\varepsilon), k-1\}.$$

Then, for any integer $k' \geqslant k$, we have

$$k' \equiv k \bmod p^n \Rightarrow d(k',a) = d(k,a).$$

Proof. By Proposition 3.4 (1), we may assume $k' = k+p^n$. By Corollary 3.9, we can write $U^{(k+p^n)} + \wp^{p^n}W = V$ with $W \in M_{\delta(k+p^n-1)}(\mathcal{O}_{K_\wp})$ and

$$V = \left(\begin{array}{c|c} \wp^{k-1}B_1 & B_2 \\ \hline O & U^{(k)} \end{array}\right), \quad B_1 \in M_{\delta p^n}(\mathcal{O}_{K_\wp}),\ B_2 \in M_{\delta p^n,\delta(k-1)}(\mathcal{O}_{K_\wp}).$$

Corollary 3.8 and Proposition 3.4 (1) show that $U^{(k+p^n)}$ satisfies the assumptions of Proposition 3.11. Hence we obtain $d(k+p^n,a) = d(V,a)$. By [Hat2, Lemma 2.3 (2)], the matrix $\wp^{k-1}B_1$ has no eigenvalue of slope less than $k-1$. Since $a < k-1$, we also have $d(V,a) = d(k,a)$. This concludes the proof. □

Theorem 3.14. *Suppose that* $\mathfrak{n}\wp$ *has an irreducible factor* π *of degree one. Let* $n \geqslant 1$ *and* $k \geqslant 2$ *be arbitrary integers. Let* $\chi : \kappa(\wp)^\times \to \kappa(\wp)^\times$ *be any character. Put* $\delta = [\Gamma_1(\pi) : \Gamma_1^\Theta(\mathfrak{n}, \wp^r)]$ *and* $\varepsilon_\chi = d(k, \chi, 0)$. *Let* a *be any non-negative rational number satisfying*

$$a < \min\{C(n, \delta, \varepsilon_\chi), k - 1\}.$$

Then, for any integer $k' \geqslant k$, *we have*

$$k' \equiv k \bmod p^n(q^d - 1) \Rightarrow d(k', \chi, a) = d(k, \chi, a).$$

Proof. This follows in the same way as Theorem 3.13, using Proposition 3.10 and Proposition 3.4 (2). □

Theorem 3.15. *Suppose that* $\mathfrak{n}\wp$ *has an irreducible factor* π *of degree one. Let* $n \geqslant 1$ *and* $k \geqslant 2$ *be arbitrary integers and* $a \leqslant n$ *any non-negative rational number. Put* $\delta = [\Gamma_1(\pi) : \Gamma_1^\Theta(\mathfrak{n}, \wp^r)]$ *and* $\varepsilon = d(k, 0)$. *Suppose that the following conditions hold.*

1. *If* $p = 2$, *then* $n \geqslant 3$ *or* $\delta - \varepsilon \leqslant 1$.
2. $2p^n > n(\delta n + 2 + \delta - 2\varepsilon)$.

Then, for any integer $k' \geqslant k$, *we have*

$$a < k - 1,\ k' \equiv k \bmod p^n \Rightarrow d(k', a) = d(k, a).$$

Proof. This follows in the same way as Theorem 3.13, using Proposition 3.12 instead of Proposition 3.11. □

It will be necessary to use an increasing function no more than $C(n, \delta, \varepsilon)$, instead of using $C(n, \delta, \varepsilon)$. Here we give an example.

Lemma 3.16. *Let* $n, \delta \geqslant 1$ *and* $\varepsilon \geqslant 0$ *be arbitrary integers satisfying* $\varepsilon \leqslant \delta$. *Put*

$$D_2(n, \delta, \varepsilon) = \frac{1}{\delta}\left\{\sqrt{2\delta p^n + (\delta - \varepsilon + 1)(2\delta - \varepsilon - 1)} - \frac{3}{2}\delta + \varepsilon\right\},$$
$$D(n, \delta, \varepsilon) = \min\{C_1(n, \delta, \varepsilon), D_2(n, \delta, \varepsilon)\}.$$

Then $D(n, \delta, \varepsilon)$ *is an increasing function of* n *satisfying* $D(n, \delta, \varepsilon) \leqslant C(n, \delta, \varepsilon)$.

Proof. Since $C_1(n, \delta, \varepsilon)$ is increasing for $n \geqslant 1$, it suffices to show $D_2(n, \delta, \varepsilon) \leqslant C_2(n, \delta, \varepsilon)$. Put $m = \delta - \varepsilon + 1$ and $x = \delta q_l + m \geqslant 1$. Since $r_l \in [0, \delta - 1]$, for any $l > \varepsilon$ we have

$$\frac{2p^n + \delta q_l(q_l - 1) + 2q_l(r_l + 1)}{2(l - \varepsilon)} \geqslant \frac{2p^n + \delta q_l(q_l - 1) + 2q_l}{2x}.$$

The right-hand side equals

$$\begin{aligned}&\frac{1}{2x}\left\{2p^n+\delta\left(\frac{x-m}{\delta}\right)\left(\frac{x-m}{\delta}-1\right)+2\left(\frac{x-m}{\delta}\right)\right\}\\&=\frac{x}{2\delta}+\frac{1}{2\delta x}\left(2\delta p^n+m(m+\delta-2)\right)-\frac{m}{\delta}-\frac{1}{2}+\frac{1}{\delta}.\end{aligned}$$

By the inequality of arithmetic and geometric means, it is not less than $D_2(n,\delta,\varepsilon)$ and the lemma follows. □

When $\mathfrak{n}=1$, $\wp=t$ and $r=1$, we have $\Gamma_1^{\Theta}(\mathfrak{n},\wp^r)=\Gamma_1(t)$, $\delta=1$ and $\varepsilon=1$ by [Hat2, Lemma 2.4], which yields

$$C_1(n,1,1)=p^n\left(\frac{p^n+3}{2p^n+2}\right)\geqslant D_2(n,1,1)=\sqrt{2p^n}-\frac{1}{2}.$$

Thus we obtain

$$D(n,1,1)=\sqrt{2p^n}-\frac{1}{2}>0 \tag{3.9}$$

and Theorem 3.13 gives the following improvement of [Hat2, Theorem 1.1].

Corollary 3.17. *Suppose $\mathfrak{n}=1$, $\wp=t$ and $r=1$. Let $k\geqslant 2$ be any integer and a any non-negative rational number. Let $n\geqslant 1$ be any integer satisfying*

$$\frac{1}{2}\left(a+\frac{1}{2}\right)^2<p^n.$$

Then, for any integer $k'\geqslant k$, we have

$$a<k-1,\ k'\equiv k \bmod p^n\Rightarrow d(\Gamma_1(t),k',a)=d(\Gamma_1(t),k,a).$$

4. ℘-adic continuous family

We say $F\in\mathcal{V}_k(\mathbb{C}_\wp)$ is a Hecke eigenform if it is a non-zero eigenvector of T_Q for any $Q\in A$. We denote by $\lambda_Q(F)$ the T_Q-eigenvalue of F. Since Hecke operators commute with each other, if $d(k,a)=1$ (*resp.* $d(k,\chi,a)=1$) then any non-zero U-eigenform in $\mathcal{V}_k(\mathbb{C}_\wp)$ (*resp.* $\mathcal{V}_k(\mathbb{C}_\wp)(\chi)$) of slope a is a Hecke eigenform.

4.1. Construction of the family.

Now we prove the following main theorem of this paper.

Theorem 4.1. *Suppose that $\mathfrak{n}\wp$ has an irreducible factor π of degree one. Let $n\geqslant 1$ and $k_1\geqslant 2$ be any integers. Put $\delta=[\Gamma_1(\pi):\Gamma_1^{\Theta}(\mathfrak{n},\wp^r)]$ and $\varepsilon=d(k_1,0)$. Let a be any non-negative rational number satisfying*

$$a<\min\{C(n,\delta,\varepsilon),k_1-1\}.$$

Let $n' \geqslant 1$ be any integer satisfying

$$p^n - p^{n'} - a \geqslant 0, \quad a < C(n', \delta, \varepsilon).$$

Suppose $d(k_1, a) = 1$. Let $F_1 \in \mathcal{V}_{k_1}(\mathbb{C}_\wp)$ be a Hecke eigenform of slope a. Then, for any integer $k_2 \geqslant k_1$ satisfying

$$k_2 \equiv k_1 \bmod p^n,$$

we have $d(k_2, a) = 1$ and thus there exists a Hecke eigenform $F_2 \in \mathcal{V}_{k_2}(\mathbb{C}_\wp)$ of slope a which is unique up to a scalar multiple. Moreover, for any Q we have

$$v_\wp(\lambda_Q(F_1) - \lambda_Q(F_2)) > p^n - p^{n'} - a. \tag{4.1}$$

Proof. By Proposition 3.4 (1), we may assume $(k_1, k_2) = (k, k + p^n)$ for some integer $k \geqslant 2$. Theorem 3.13 yields $d(k + p^n, a) = 1$ and any non-zero U-eigenform $F_2 \in \mathcal{V}_{k_2}(\mathbb{C}_\wp)$ of slope a is a Hecke eigenform. Take a finite extension $E/K_\wp$ inside $\mathbb{C}_\wp$ containing $\lambda_Q(F_i)$ and $\lambda_\wp(F_i)$ for $i = 1, 2$. We may assume $F_i \in \mathcal{V}_{k_i}(\mathcal{O}_E)$. We identify $\mathcal{V}_{k_i}(\mathcal{O}_E)$ with $\mathcal{O}_E^{\delta(k_i - 1)}$ via the ordered basis $\mathfrak{B}_{k_i}$. Then we can write

$$F_2 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad x \in \mathcal{O}_E^{\delta p^n},\ y \in \mathcal{O}_E^{\delta(k-1)},$$

where each entry of x is the coefficient of $v_{s,t} \in \mathfrak{B}_{k_2}$ in F_2 with $t < p^n$. For any integer N and $z = {}^t(z_1, \ldots, z_N) \in \mathcal{O}_E^N$, we put

$$v_\wp(z) = \inf\{v_\wp(z_i) \mid i = 1, \ldots, N\}.$$

Replacing F_i by its scalar multiple, we may assume $v_\wp(F_i) = 0$.

For any $H \in \mathcal{V}_{k_i}(\mathcal{O}_E)$, we denote by $\bar{H}$ its image by the natural map $\mathcal{V}_{k_i}(\mathcal{O}_E) \to \mathcal{V}_{k_i}(\mathcal{O}_{E,p^n})$. Consider the weight reduction map

$$1 \otimes \rho_{k,p^n} : \mathcal{V}_{k+p^n}(\mathcal{O}_{E,p^n}) \to \mathcal{V}_k(\mathcal{O}_{E,p^n})$$

as in §3.2, which we denote by ρ. Then $\rho(\bar{F}_2) = y \bmod \wp^{p^n}$.

We claim $v_\wp(y) \leqslant a$. Indeed, if $v_\wp(x) \geqslant v_\wp(y)$, then the assumption $v_\wp(F_2) = 0$ yields $v_\wp(y) = 0$. If $v_\wp(x) < v_\wp(y)$, then $v_\wp(x) = 0$ and Corollary 3.9 gives

$$\lambda_\wp(F_2)x = \wp^{k-1}B_1 x + B_2 y.$$

Since $v_\wp(\lambda_\wp(F_2)) = a < k - 1$, this forces $v_\wp(y) \leqslant a$ and the claim follows.

Take $G_1 \in \mathcal{V}_k(\mathcal{O}_E)$ satisfying $\bar{G}_1 = \rho(\bar{F}_2)$. By Lemma 3.2, we have

$$T_Q(G_1) \equiv \lambda_Q(F_2)G_1, \quad U(G_1) \equiv \lambda_\wp(F_2)G_1 \bmod \wp^{p^n}\mathcal{V}_k(\mathcal{O}_E). \tag{4.2}$$

Since we have $a < C(n, \delta, \varepsilon) < p^n$, the above claim yields $v_\wp(G_1) \leqslant a$. If $G_1 \in \mathcal{O}_E F_1$, then G_1 is a Hecke eigenform with the same eigenvalues as those of F_1. Thus we have

$$\lambda_Q(F_1)\bar{G}_1 = T_Q(\bar{G}_1) = \lambda_Q(F_2)\bar{G}_1,$$

which gives

$$v_\wp(\lambda_Q(F_1) - \lambda_Q(F_2)) \geqslant p^n - a. \tag{4.3}$$

Suppose $G_1 \notin \mathcal{O}_E F_1$, and take $H_1 \in \mathcal{V}_k(\mathcal{O}_E)$ such that F_1 and H_1 form a basis of a direct summand of $\mathcal{V}_k(\mathcal{O}_E)$ containing G_1. Write

$$G_1 = \alpha F_1 + \beta H_1, \quad \alpha, \beta \in \mathcal{O}_E. \tag{4.4}$$

Then $\beta \neq 0$. By (4.2), for any $R \in \{\wp, Q\}$ we have

$$\lambda_R(F_2)G_1 \equiv T_R(G_1) = \alpha\lambda_R(F_1)F_1 + \beta T_R(H_1) \bmod \wp^{p^n}\mathcal{V}_k(\mathcal{O}_E).$$

Combined with (4.4), this implies

$$\beta T_R(H_1) \equiv \alpha(\lambda_R(F_2) - \lambda_R(F_1))F_1 + \beta\lambda_R(F_2)H_1 \bmod \wp^{p^n}\mathcal{V}_k(\mathcal{O}_E) \tag{4.5}$$

and thus we obtain

$$\alpha(\lambda_R(F_1) - \lambda_R(F_2)) \equiv 0 \bmod (\beta, \wp^{p^n}). \tag{4.6}$$

Put $b = v_\wp(\beta)$. Suppose $b > p^n - p^{n'}$. Since $v_\wp(F_1) = 0$ and

$$v_\wp(G_1) \leqslant a \leqslant p^n - p^{n'} < b,$$

(4.4) gives $v_\wp(\alpha) \leqslant a$ and (4.6) yields

$$v_\wp(\lambda_Q(F_1) - \lambda_Q(F_2)) > p^n - p^{n'} - a. \tag{4.7}$$

Suppose $b \leqslant p^n - p^{n'}$. In this case we have $\beta^{-1}\wp^{p^n} \in \mathcal{O}_E$ and by (4.6) we can write

$$\alpha(\lambda_\wp(F_2) - \lambda_\wp(F_1)) = \beta\nu$$

with some $\nu \in \mathcal{O}_E$. Then (4.5) shows

$$U(H_1) \equiv \nu F_1 + \lambda_\wp(F_2)H_1 \bmod \beta^{-1}\wp^{p^n}\mathcal{V}_k(\mathcal{O}_E). \tag{4.8}$$

Take an ordered basis $(F_1, H_1, \tilde{v}_3, \ldots, \tilde{v}_{\delta(k-1)})$ of the $\mathcal{O}_E$-module $\mathcal{V}_k(\mathcal{O}_E)$, and we denote by $\tilde{U}^{(k)}$ the representing matrix of U with respect to it. By (4.8), we can write

$$\tilde{U}^{(k)} = \left(\begin{array}{cc|ccc} \lambda_\wp(F_1) & \nu + \beta^{-1}\wp^{p^n}c_1 & * & \cdots & * \\ 0 & \lambda_\wp(F_2) + \beta^{-1}\wp^{p^n}c_2 & * & \cdots & * \\ 0 & \beta^{-1}\wp^{p^n}c_3 & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \beta^{-1}\wp^{p^n}c_{\delta(k-1)} & * & \cdots & * \end{array}\right), \quad c_1, \ldots, c_{\delta(k-1)} \in \mathcal{O}_E.$$

Note that the elementary divisors of $\tilde{U}^{(k)}$ and $U^{(k)}$ agree with each other. Let V be the element of $M_{\delta(k-1)}(\mathcal{O}_E)$ with the same columns as those of $\tilde{U}^{(k)}$ except the second column which we require to be

$$\begin{pmatrix} \nu \\ \lambda_{\wp}(F_2) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then we have $d(V,a) \geqslant 2$. On the other hand, since $p^n - b \geqslant p^{n'}$, the assumption $a < C(n',\delta,\varepsilon)$ and Proposition 3.11 yield $d(V,a) = d(k,a) = 1$, which is the contradiction. Thus the case $b \leqslant p^n - p^{n'}$ never occurs. Now the theorem follows from (4.3) and (4.7). □

Remark 4.2. Putting $\varepsilon = d(k_1,\chi,0)$ and assuming $d(k_1,\chi,a) = 1$, the same proof using Proposition 3.10 and Theorem 3.14 shows that we can construct, from a Hecke eigenform $F_1 \in \mathcal{V}_{k_1}(\mathbb{C}_{\wp})(\chi)$ of slope a, a Hecke eigenform $F_2 \in \mathcal{V}_{k_2}(\mathbb{C}_{\wp})(\chi)$ of slope a satisfying (4.1) for any integer $k_2 \geqslant k_1$ with

$$k_2 \equiv k_1 \bmod p^n(q^d - 1).$$

Proof of Theorem 1.1. Suppose that n, k and a satisfy the assumptions of Theorem 1.1. Take any $k' \geqslant k$ satisfying

$$m = v_p(k' - k) \geqslant \log_p(p^n + a).$$

Since $n \leqslant m$ and $D(n,\delta,\varepsilon)$ is an increasing function of n satisfying $D(n,\delta,\varepsilon) \leqslant C(n,\delta,\varepsilon)$, we have

$$a < \min\{C(m,\delta,\varepsilon), k-1\}, \quad p^m - p^n - a \geqslant 0, \quad a < C(n,\delta,\varepsilon).$$

Note that, if $d(k,a) = 1$, then any U-eigenform of slope a in $\mathcal{V}_k(\mathbb{C}_{\wp})$ is identified with a scalar multiple of that in $\mathcal{V}_k(\bar{K}) \subseteq S_k(\Gamma_1^{\Theta}(\mathfrak{n},\wp^r))$ via the fixed embedding $\iota_{\wp}$. Thus Theorem 4.1 produces a Hecke eigenform $F_{k'} \in S_{k'}(\Gamma_1^{\Theta}(\mathfrak{n},\wp^r))$ such that for any Q we have

$$v_{\wp}(\iota_{\wp}(\lambda_Q(F_{k'}) - \lambda_Q(F_k))) > p^m - p^n - a.$$

This concludes the proof of Theorem 1.1. □

4.2. **Examples.** We assume $\mathfrak{n} = 1$, $\wp = t$, $r = 1$ and $\Gamma_1^{\Theta}(\mathfrak{n},\wp^r) = \Gamma_1(t)$. In this case we have $\delta = 1$ and $d(k,0) = 1$ for any $k \geqslant 2$. In the following, we give examples of congruences between Hecke eigenvalues obtained by Theorem 1.1 for this case, using results of [BV2, LM, Pet]. Note that the Hecke operator at Q considered in [BV2, Pet] is QT_Q with our normalization.

4.2.1. *Slope zero forms.* By $d(k,0)=1$, any U-eigenform of slope zero in $S_k(\Gamma_1(t))$ is a member of a t-adic continuous family obtained by Theorem 1.1. Some of such eigenforms can be given by the theory of A-expansions [Pet].

For any integer $k \geqslant 3$ satisfying $k \equiv 2 \bmod q-1$, Petrov constructed an element $f_{k,1} \in S_k(SL_2(A))$ with A-expansion [Pet, Theorem 1.3]. We know that $f_{k,1}$ is a Hecke eigenform whose Hecke eigenvalue at Q is one for any Q; this follows from a formula for the Hecke action [Pet, p. 2252] and $c_a = a^{k-n}$.

For such k, let $f_{k,1}^{(t)} \in S_k(\Gamma_1(t))$ be the t-stabilization of $f_{k,1}$ of finite slope, namely

$$f_{k,1}^{(t)}(z) = f_{k,1}(z) - t^{k-1} f_{k,1}(tz).$$

It is non-zero by [Pet, Theorem 2.2]. Moreover, we can show that $f_{k,1}^{(t)}$ is a Hecke eigenform which also satisfies $\lambda_Q(f_{k,1}^{(t)}) = 1$ for any Q.

Proposition 4.3. *Let $k \geqslant 2$ be any integer and F_k any non-zero element of $S_k(\Gamma_1(t))$ of slope zero. Then we have $\lambda_Q(F_k) = 1$ for any Q.*

Proof. Recall that $d(k,0)=1$ implies that F_k is a Hecke eigenform. Let $r \in \{0,1,\ldots,q-2\}$ be an integer satisfying $k \equiv r \bmod q-1$. For $a=0$, we see from (3.9) that the assumptions of Theorem 1.1 are satisfied by $n=1$. Then, for any integer $s \geqslant 1$, we obtain a Hecke eigenform of slope zero

$$F_{k'} \in S_{k'}(\Gamma_1(t)), \quad k' = k + (q+1-r)q^s$$

such that, with the fixed embedding $\iota_t : \bar{K} \to \mathbb{C}_t$, we have

$$\iota_t(\lambda_Q(F_{k'})) \equiv \iota_t(\lambda_Q(F_k)) \bmod t^{q^s-p} \quad \text{for any } Q.$$

Since $k' \geqslant 3$, $k' \equiv 2 \bmod q-1$ and $d(k',0)=1$, we see that $F_{k'}$ is a scalar multiple of $f_{k',1}^{(t)}$ and thus $\lambda_Q(F_{k'}) = 1$. Since s is arbitrary, this implies $\lambda_Q(F_k) = 1$. □

Corollary 4.4. *Let $k \geqslant 2$ and $r \geqslant 1$ be arbitrary integers. Then there exists a unique character $\chi : \kappa(\wp)^\times \to \kappa(\wp)^\times$ satisfying $d(\Gamma_0^p(t^r),k,\chi,0) \neq 0$. For such χ, we have $d(\Gamma_0^p(t^r),k,\chi,0) = 1$ and any Hecke eigenform F of slope zero in $S_k(\Gamma_0^p(t^r))(\chi)$ satisfies $\lambda_Q(F) = 1$ for any Q.*

Proof. Since $\Gamma_0^p(t) = \Gamma_1(t)$, Proposition 3.5 implies $d(\Gamma_0^p(t^r),k,0) = 1$. Since we have

$$d(\Gamma_0^p(t^r),k,0) = \sum_{\chi} d(\Gamma_0^p(t^r),k,\chi,0),$$

the uniqueness of χ and the assertion on the dimension follow. Let F_k be any Hecke eigenform of slope zero in $S_k(\Gamma_1(t))$. Since the natural inclusion $S_k(\Gamma_1(t)) \to S_k(\Gamma_0^p(t^r))$ is compatible with Hecke operators, F is a scalar multiple of the image of F_k. Hence the last assertion follows from Proposition 4.3. □

Remark 4.5. Note that, since the only p-power root of unity in $\mathbb{C}_\wp$ is one, there exists no non-trivial finite order character $1+\wp\mathcal{O}_{K_\wp} \to \mathbb{C}_\wp^\times$. Thus it seems to the author that, if we try to generalize Hida theory including [Hid2, §7.3, Theorem 3] to Drinfeld cuspforms of level $\Gamma_1(t^r)$, then it would be natural to restrict ourselves to those of level $\Gamma_0^p(t^r)$. However, Corollary 4.4 shows that such a generalization is trivial.

4.2.2. *Slope one forms.* Let us consider the case $p=q=3$ and $a=1$. Since $D(1,1,1)=\sqrt{6}-\frac{1}{2}=1.949\ldots$, the assumptions of Theorem 1.1 are satisfied by $k \geqslant 3$ and $n=1$. Then a computation using [BV2, (17)] shows $d(10,1)=1$. Let G_{10} and G_{19} be any non-zero Drinfeld cuspforms of level $\Gamma_1(t)$ and slope one in weights 10 and 19, respectively. Then Theorem 1.1 gives

$$v_t(\iota_t(\lambda_Q(G_{10})-\lambda_Q(G_{19})))>5 \tag{4.9}$$

for any Q.

For $Q=t$, using [BV2, (17)] we can show that $\lambda_t(G_{10})=-t-t^3$, and $\lambda_t(G_{19})$ is a root of the polynomial

$$\begin{aligned} &X^4+(t+t^3)X^3+(-t^8+t^{10}+t^{12}+t^{14}+t^{16})X^2 \\ &\quad +(-t^9-t^{11}+t^{13}+t^{15}+t^{17}+t^{19})X+(-t^{18}-t^{20}+t^{24}+t^{26}+t^{28}) \end{aligned}$$

(see also [Val]). Put $\iota_t(\lambda_t(G_{19}))=ty$ with $v_t(y)=0$. Then we obtain $y^3(y+1+t^2)\equiv 0 \bmod t^6$ and $\iota_t(\lambda_t(G_{10}))\equiv \iota_t(\lambda_t(G_{19})) \bmod t^7$, which satisfies (4.9). In fact, plugging in $X=-t-t^3+Z$ to the polynomial above yields $v_t(\iota_t(\lambda_t(G_{10})-\lambda_t(G_{19})))=9$.

We identify $S_k(\Gamma_1(t))$ with $\mathbb{C}_\infty^{k-1}$ via the ordered basis

$$\{\mathbf{c}_j(\gamma_0)=\mathbf{c}_j(\bar{e}) \mid 0\leqslant j\leqslant k-2\}$$

defined in [LM, BV2]. Then G_{10} is identified with the vector

$${}^t\left(0,1+t^2,0,-(1+t^2),0,-t^2,0,1,0\right).$$

Thus $\lambda_{1+t}(G_{10})$ agrees with the evaluation $T_{1+t}(G_{10})(\gamma_0)(X^7Y)$ after identifying G_{10} with a harmonic cocycle. By [LM, (7.1)], we have $\lambda_{1+t}(G_{10})=1-t-t^3$. On the other hand, by computing the characteristic polynomial of T_{1+t} acting on $S_{19}(\Gamma_1(t))$ using [LM, (7.1)] and plugging in $X=1-t-t^3+Z$ into it, (4.9) implies $v_t(\iota_t(\lambda_{1+t}(G_{10})-\lambda_{1+t}(G_{19})))=9$.

Note that, since these eigenvalues are not powers of t or $1+t$, the Hecke eigenforms G_{10} and G_{19} are not the t-stabilizations of Hecke eigenforms with A-expansion.

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