# $\wp$-ADIC CONTINUOUS FAMILIES OF DRINFELD EIGENFORMS OF FINITE SLOPE

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Abstract. Let $p$ be a rational prime, $v_p$ the normalized $p$-adic valuation on $\mathbb{Z}$, $q > 1$ a $p$-power and $A = \mathbb{F}_q[t]$. Let $\wp \in A$ be an irreducible polynomial and $\mathfrak{n} \in A$ a non-zero element which is prime to $\wp$. Let $k \geqslant 2$ and $r \geqslant 1$ be integers. We denote by $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ the space of Drinfeld cuspforms of level $\Gamma_1(\mathfrak{n}\wp^r)$ and weight $k$ for $\mathbb{F}_q(t)$. Let $n \geqslant 1$ be an integer and $a \geqslant 0$ a rational number. Suppose that $\mathfrak{n}\wp$ has a prime factor of degree one and the generalized eigenspace in $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ of slope $a$ is one-dimensional. In this paper, under an assumption that $a$ is sufficiently small, we construct a family $\{F_{k'} \mid v_p(k'-k) \geqslant \log_p(p^n + a)\}$ of Hecke eigenforms $F_{k'} \in S_{k'}(\Gamma_1(\mathfrak{n}\wp^r))$ of slope $a$ such that, for any $Q \in A$, the Hecke eigenvalues of $F_k$ and $F_{k'}$ at $Q$ are congruent modulo $\wp^\kappa$ with some $\kappa > p^{v_p(k'-k)} - p^n - a$.

## 1. Introduction

Let $p$ be a rational prime, $q > 1$ a $p$-power and $\mathbb{F}_q$ the field of $q$ elements. Put $A = \mathbb{F}_q[t]$ and $K = \mathbb{F}_q(t)$. Let $\wp \in A$ be an irreducible polynomial of positive degree, $\mathfrak{n}$ a non-zero element of $A$ which is prime to $\wp$ and $r \geqslant 1$ an integer. Put $A_r = A/(\wp^r)$ and $\kappa(\wp) = A/(\wp)$. We denote by $K_\wp$ the $\wp$-adic completion of $K$, by $\mathbb{C}_\wp$ the $\wp$-adic completion of an algebraic closure of $K_\wp$ and by $v_\wp : \mathbb{C}_\wp \to \mathbb{Q} \cup \{+\infty\}$ the $\wp$-adic additive valuation on $\mathbb{C}_\wp$ normalized as $v_\wp(\wp) = 1$. Similarly, we denote by $K_\infty$ the $(1/t)$-adic completion of $K$ and by $\mathbb{C}_\infty$ the $(1/t)$-adic completion of an algebraic closure of $K_\infty$. Let $\bar{K}$ be the algebraic closure of $K$ inside $\mathbb{C}_\infty$ and we fix an embedding of $K$-algebras $\iota_\wp : \bar{K} \to \mathbb{C}_\wp$. For any $x \in \bar{K}$, we define its normalized $\wp$-adic valuation by $v_\wp(\iota_\wp(x))$. Let $\Omega = \mathbb{P}^1(\mathbb{C}_\infty)\backslash\mathbb{P}^1(K_\infty)$ be the Drinfeld upper half plane, which has a natural structure of a rigid analytic variety over $K_\infty$.

Let $\Gamma$ be a subgroup of $SL_2(A)$ and $k$ an integer. A Drinfeld modular form of level $\Gamma$ and weight $k$ is a rigid analytic function on $\Omega$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for any } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, z \in \Omega$$

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and a holomorphy condition at cusps. It is considered as a function field analogue of the notion of elliptic modular form.

Recently, $\wp$-adic properties of Drinfeld modular forms have attracted attention and have been studied actively (for example, [BV1, BV2, BV3, Gos, Hat1, Hat2, PZ, Vin]). However, though we have a highly developed theory of $p$-adic analytic families of elliptic eigenforms of finite slope, $\wp$-adic properties of Drinfeld modular forms are much less well-understood compared to the elliptic case. One of the difficulties in the Drinfeld case is that, since the group $\mathcal{O}_{K_\wp}^\times$ is topologically of infinitely generated, analogues of the completed group ring $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ are not Noetherian, and it seems that we have no good definition of characteristic power series applicable to non-Noetherian base rings, as mentioned in [Buz2, paragraph before Lemma 2.3].

In this paper, we will construct families of Drinfeld eigenforms in which Hecke eigenvalues vary in a $\wp$-adically continuous way. For the precise statement, we fix some notation. For any $\mathfrak{m} \in A$, we put

$$\Gamma_1(\mathfrak{m}) = \left\{ \gamma \in SL_2(A) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{m} \right\}.$$

Let $\Theta$ be any subgroup of $1 + \wp A_r \subseteq A_r^\times$. We define

$$\Gamma_0^\Theta(\wp^r) = \left\{ \gamma \in SL_2(A) \,\middle|\, \gamma \bmod \wp^r \in \begin{pmatrix} \Theta & * \\ 0 & \Theta \end{pmatrix} \right\} \subseteq \Gamma_1(\wp)$$

and $\Gamma_1^\Theta(\mathfrak{n}, \wp^r) = \Gamma_1(\mathfrak{n}) \cap \Gamma_0^\Theta(\wp^r)$, which satisfies $\Gamma_1^{\{1\}}(\mathfrak{n}, \wp^r) = \Gamma_1(\mathfrak{n}\wp^r)$.

Let $k \geqslant 2$ be an integer. For any non-zero element $Q \in A$, the Hecke operator $T_Q$ acts on the $\mathbb{C}_\infty$-vector space $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$ of Drinfeld cuspforms of level $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$ and weight $k$. The operator $T_\wp$ is also denoted by $U$. Since they stabilize an $A$-lattice $\mathcal{V}_k(A)$ (Proposition 2.2), every eigenvalue of $T_Q$ is integral over $A$. The normalized $\wp$-adic valuation of an eigenvalue of $U$ is called slope, and we denote by $d(k, a)$ the dimension of the generalized $U$-eigenspace for the eigenvalues of slope $a$. For any Hecke eigenform $F$, its $T_Q$-eigenvalue is denoted by $\lambda_Q(F)$. We denote by $v_p$ the $p$-adic valuation on $\mathbb{Z}$ satisfying $v_p(p) = 1$. Then the main theorem of this paper (Theorem 4.1) gives the following, which we will prove in §4.1.

**Theorem 1.1.** *Suppose that $\mathfrak{n}\wp$ has a prime factor $\pi$ of degree one. Let $n \geqslant 1$ and $k \geqslant 2$ be integers. Put $d = [\Gamma_1(\pi) : \Gamma_1^\Theta(\mathfrak{n}, \wp^r)]$, $\varepsilon = d(k, 0)$*

*and*

$$D_2(n,d,\varepsilon)=\frac{1}{d}\left\{\sqrt{2dp^n+(d-\varepsilon+1)(2d-\varepsilon-1)}-\frac{3}{2}d+\varepsilon\right\},$$
$$D(n,d,\varepsilon)=\min\left\{p^n\left(\frac{4+dp^n-d}{4+2dp^n-2\varepsilon}\right),D_2(n,d,\varepsilon)\right\}.$$

*Let $a$ be any non-negative rational number satisfying*

$$a<\min\{D(n,d,\varepsilon),k-1\}.$$

*Suppose $d(k,a)=1$. Then, for any integer $k'\geqslant k$ satisfying*

$$v_p(k'-k)\geqslant \log_p(p^n+a),$$

*there exists a Hecke eigenform $F_{k'}\in S_{k'}(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$ of slope $a$ such that for any $Q$ we have*

$$v_\wp(\iota_\wp(\lambda_Q(F_{k'})-\lambda_Q(F_k)))>p^{v_p(k'-k)}-p^n-a.$$

In fact, what we will prove allow nebentypus characters at $\wp$ (Remark 4.2).

For example, in the case of $\mathfrak{n}=1$, $\wp=t$ and $r=1$, we have $\Gamma_1^\Theta(\mathfrak{n},\wp^r)=\Gamma_1(t)$, $d=\varepsilon=1$ and $D(n,1,1)=\sqrt{2p^n}-\frac{1}{2}$. In this case, Theorem 1.1 implies that, for any Hecke eigenform $F_k$ of slope zero in $S_k(\Gamma_1(t))$, the $T_Q$-eigenvalue $\lambda_Q(F_k)$ is $t$-adically arbitrarily close to those coming from Hecke eigenforms with $A$-expansion [Pet], which shows $\lambda_Q(F_k)=1$ for any $Q$ (Proposition 4.3). This suggests that, though we will prove constancy results of the dimension of slope zero cuspforms with respect to $k$ and $r$ (Proposition 3.4 and Proposition 3.5), Hida theory for the level $\Gamma_0(t^r)$ should be trivial (Remark 4.5). We also note that families constructed in Theorem 1.1 contain Hecke eigenforms whose Hecke eigenvalue at $Q$ is not a power of $Q$ (§4.2), and thus they capture a more subtle $\wp$-adic structure of Hecke eigenvalues than the theory of $A$-expansions.

Let us explain the idea of the proof of Theorem 1.1. Note that a usual method to construct $p$-adic families of eigenforms of finite slope in the number field case is the use of the Riesz theory [Col, Buz2], which is not available for our case at present, due to the lack of a notion of characteristic power series over non-Noetherian Banach algebras. Instead, we follow an idea of Buzzard [Buz1] by which he constructed $p$-adically continuous families of quaternionic eigenforms over $\mathbb{Q}$.

First we will prove a variant of the Gouvêa-Mazur conjecture (Proposition 3.11), which implies $d(k,a)=d(k',a)$ if $k$ and $k'$ are highly congruent $p$-adically and $a$ is sufficiently small. With the assumption $d(k,a)=1$, it produces Hecke eigenforms $F_k$ and $F_{k'}$ of slope $a$ in weights $k$ and $k'$, respectively. For this part, we employ the same idea as in [Hat2]: a lower bound of elementary divisors of the representing matrix of $U$ with some basis and a perturbation lemma [Ked, Theorem 4.4.2] yield the equality. To obtain such a bound (Corollary 3.8), we need to define Hecke operators acting on the Steinberg complex (2.2) with respect to $\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)$, which is done in §2.3. Note that similar Hecke operators on a Steinberg complex in an adelic setting are given in [Böc, §6.4].

Then, a weight reduction map (§3.2) yields a Drinfeld cuspform $G$ of weight $k$ such that, for $m = v_p(k' - k)$, the element $G \bmod \wp^{p^m}$ is a Hecke eigenform with the same eigenvalues as those of $F_{k'} \bmod \wp^{p^m}$. Now the point is that, if two lines generated by $F_k$ and $G$ are highly congruent in some sense, then we can show that the eigenvalues of $F_k$ and $G \bmod \wp^{p^m}$ are also highly congruent, which gives Theorem 1.1; otherwise the two lines are so far apart that, again by the Gouvêa-Mazur variant mentioned above, they produce $U$-eigenvalues of slope $a$ with multiplicity more than one, which contradicts $d(k, a) = 1$ (Theorem 4.1).

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## 2. Drinfeld cuspforms via the Steinberg module

For any arithmetic subgroup $\Gamma$ of $SL_2(A)$ and any integer $k \geqslant 2$, we denote by $S_k(\Gamma)$ the space of Drinfeld cuspforms of level $\Gamma$ and weight $k$. In this section, we first recall an interpretation of $S_k(\Gamma)$ using the Steinberg module due to Teitelbaum [Tei, p. 506], following the normalization of [Böc, §5]. We also introduce Hecke operators acting on the Steinberg complex. Using them, we define an $A$-lattice of the space of Drinfeld cuspforms which is stable under the Hecke action.

2.1. **Steinberg module.** For any $A$-algebra $B$, we consider $B^2$ as the set of row vectors, and define a left action $\circ$ of $GL_2(B)$ on it by $\gamma \circ x = x\gamma^{-1}$. Let $\mathcal{T}$ be the Bruhat-Tits tree for $SL_2(K_\infty)$. We denote by $\mathcal{T}_0$ the set of vertices of $\mathcal{T}$, which is the set of $K_\infty^\times$-equivalence classes of $\mathcal{O}_{K_\infty}$-lattices in $K_\infty^2$, and by $\mathcal{T}_1$ the set of its edges. The oriented graph associated with $\mathcal{T}$ and the set of oriented edges are denoted by $\mathcal{T}^o$ and $\mathcal{T}_1^o$, respectively. For any oriented edge $e$, we denote its origin by $o(e)$, its terminus by $t(e)$ and the opposite edge by $-e$. The group $\{\pm 1\}$ acts on $\mathcal{T}_1^o$ by $(-1)e = -e$.

Let $\Gamma$ be an arithmetic subgroup of $SL_2(A)$ [Böc, §3.4], and we assume $\Gamma$ to be $p'$-torsion free (namely, every element of $\Gamma$ of finite order has $p$-power order). The group $\Gamma$ acts on $\mathcal{T}$ and $\mathcal{T}^o$ via the natural inclusion $\Gamma \to GL_2(K_\infty)$. We say a vertex or an oriented edge of $\mathcal{T}$ is $\Gamma$-stable if its stabilizer subgroup in $\Gamma$ is trivial, and $\Gamma$-unstable otherwise. We denote by $\mathcal{T}_0^{\mathrm{st}}$ and $\mathcal{T}_1^{o,\mathrm{st}}$ the subsets of $\Gamma$-stable elements. For any $\Gamma$-unstable vertex $v$, its stabilizer subgroup in $\Gamma$ is a non-trivial finite $p$-group and thus fixes a unique rational end which we denote by $b(v)$ [Ser, Ch. II, §2.9].

For any ring $R$ and any set $S$, we write $R[S]$ for the free $R$-module with basis $\{[s] \mid s \in S\}$. When $S$ admits a left action of $\Gamma$, the $R$-module $R[S]$ also admits a natural left action of the group ring $R[\Gamma]$ which we denote by $\circ$. In this case, we also define a right action of $\Gamma$ on $R[S]$ by $[s]|_\gamma = \gamma^{-1} \circ [s]$, which makes it a right $R[\Gamma]$-module.

Put

$$\mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}] = \mathbb{Z}[\mathcal{T}_1^{o,\mathrm{st}}]/\langle [e] + [-e] \mid e \in \mathcal{T}_1^{o,\mathrm{st}} \rangle.$$

We define a surjection of $\mathbb{Z}[\Gamma]$-modules $\partial_\Gamma : \mathbb{Z}[\mathcal{T}_1^{o,\mathrm{st}}] \to \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]$ by $\partial_\Gamma(e) = [t(e)] - [o(e)]$, where we put $[v] = 0$ in $\mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]$ for any $\Gamma$-unstable vertex $v$. It factors as $\partial_\Gamma : \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}] \to \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]$. Note that the both sides of this map are free left $\mathbb{Z}[\Gamma]$-modules of finite rank.

We define the Steinberg module St as the kernel of the natural augmentation map

$$\mathbb{Z}[\mathbb{P}^1(K)] \to \mathbb{Z},$$

on which the group $GL_2(K)$ acts via

$$\gamma \circ (x : y) = (x : y)\gamma^{-1}, \quad (x : y) \in \mathbb{P}^1(K).$$

We consider it as a left $\mathbb{Z}[\Gamma]$-module via the natural inclusion $\Gamma \to GL_2(K)$. Then the Steinberg module St is a finitely generated projective $\mathbb{Z}[\Gamma]$-module which sits in the split exact sequence

$$(2.1) \qquad 0 \longrightarrow \mathrm{St} \longrightarrow \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}] \xrightarrow{\partial_\Gamma} \mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}] \longrightarrow 0.$$

We consider these three left $\mathbb{Z}[\Gamma]$-modules as right $\mathbb{Z}[\Gamma]$-modules via the action $[s] \mapsto [s]|_\gamma$.

2.2. **Drinfeld cuspforms and harmonic cocycles.** For any integer $k \geqslant 2$ and any $A$-algebra $B$, we denote by $H_{k-2}(B)$ the $B$-submodule of the polynomial ring $B[X, Y]$ consisting of homogeneous polynomials of degree $k-2$. We consider the left action of the multiplicative monoid $M_2(B)$ on $H_{k-2}(B)$ defined by $(\gamma \circ X, \gamma \circ Y) = (X, Y)\gamma$. On $GL_2(B)$, it agrees with the natural left action on $\mathrm{Sym}^k(\mathrm{Hom}_B(B^2,B))$ induced by the action $\circ$ on $B^2$ after identifying $(X,Y)$ with the dual basis for the basis $((1,0),(0,1))$ of $B^2$. Put

$$V_k(B)=\mathrm{Hom}_B(H_{k-2}(B),B).$$

We denote the dual basis of the free $B$-module $V_k(B)$ with respect to the basis $\{X^iY^{k-2-i}\mid 0\leqslant i\leqslant k-2\}$ of $H_{k-2}(B)$ by

$$\{(X^iY^{k-2-i})^\vee\mid 0\leqslant i\leqslant k-2\}.$$

We also denote by $\circ$ the natural left action of $GL_2(B)$ on $V_k(B)$ induced by that on $H_{k-2}(B)$. For $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL_2(B)$, $P(X,Y)\in H_{k-2}(B)$ and $\omega\in V_k(B)$, this action is given by

$$\begin{aligned}(\gamma\circ\omega)(P(X,Y))&=\omega(\gamma^{-1}\circ P(X,Y))\\ &=\det(\gamma)^{2-k}\omega(P(dX-cY,-bX+aY))\end{aligned}$$

as in [Böc, p. 51]. The group $\Gamma$ acts on $H_{k-2}(B)$ and $V_k(B)$ via the natural map $\Gamma\to GL_2(B)$. Moreover, the monoid

$$M^{-1}=\{\xi\in GL_2(K)\mid \xi^{-1}\in M_2(A)\}$$

acts on $V_k(B)$ by

$$(\xi\circ\omega)(P(X,Y))=\omega(\xi^{-1}\circ P(X,Y)).$$

Put $\mathcal{V}_k(B)=\mathrm{St}\otimes_{\mathbb{Z}[\Gamma]}V_k(B)$ and

$$\mathcal{L}_{1,k}(B)=\mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}]\otimes_{\mathbb{Z}[\Gamma]}V_k(B),\quad \mathcal{L}_{0,k}(B)=\mathbb{Z}[\mathcal{T}_0^{\mathrm{st}}]\otimes_{\mathbb{Z}[\Gamma]}V_k(B).$$

We have the split exact sequence

$$0\longrightarrow\mathcal{V}_k(B)\longrightarrow\mathcal{L}_{1,k}(B)\xrightarrow{\partial_\Gamma\otimes 1}\mathcal{L}_{0,k}(B)\longrightarrow 0 \tag{2.2}$$

which is functorial on $B$ and compatible with any base change of $B$. Let $B'$ be any $A$-subalgebra of $B$. Since the $\mathbb{Z}[\Gamma]$-module St is projective, the natural maps $\mathcal{V}_k(B')\to\mathcal{V}_k(B)$, $\mathcal{L}_{1,k}(B')\to\mathcal{L}_{1,k}(B)$ and $\mathcal{L}_{0,k}(B')\to\mathcal{L}_{0,k}(B)$ are injective.

Let $\Lambda_1\subseteq\mathcal{T}_1^{o,\mathrm{st}}$ be a complete set of representatives of $\Gamma\backslash\mathcal{T}_1^{o,\mathrm{st}}/\{\pm1\}$. By [Ser, Ch. II, §1.2, Corollary], for any element $e\in\mathcal{T}_1^{o,\mathrm{st}}$ we can write uniquely

$$r(e)=\varepsilon_e\gamma_e e\quad(\varepsilon_e\in\{\pm1\},\gamma_e\in\Gamma,r(e)\in\Lambda_1). \tag{2.3}$$

Note that $r(e)$, $\varepsilon_e$ and $\gamma_e$ depend on the choice of $\Lambda_1$. The right $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}]$ is free with basis $\{[e]\mid e\in\Lambda_1\}$ and thus, for any

$A$-algebra $B$, any element $x$ of $\mathcal{L}_{1,k}(B)$ can be written uniquely as

$$x = \sum_{e \in \Lambda_1} [e] \otimes \omega_e, \quad \omega_e \in V_k(B).$$

**Definition 2.1.** Let $M$ be a module. A map $c : \mathcal{T}_1^o \to M$ is said to be a harmonic cocycle if the following conditions are satisfied:

(1) For any $v \in \mathcal{T}_0$, we have

$$\sum_{e \in \mathcal{T}_1^o,\ t(e)=v} c(e) = 0.$$

(2) For any $e \in \mathcal{T}_1^o$, we have $c(-e) = -c(e)$.

Any harmonic cocycle $c$ is determined by its values at $\Gamma$-stable edges, as follows. For any $e \in \mathcal{T}_1^o$, an edge $e' \in \mathcal{T}_1^{o,\mathrm{st}}$ is said to be a source of $e$ if the following conditions hold:

- When $e$ is $\Gamma$-stable, we require $e' = e$.
- When $e$ is $\Gamma$-unstable, we require that a vertex $v$ of $e'$ is $\Gamma$-unstable, $e$ lies on the unique half line from $v$ to $b(v)$ and $e$ has the same orientation as $e'$ with respect to this half line.

We denote by $\mathrm{src}(e)$ the set of sources of $e$. Then Definition 2.1 (1) gives

$$c(e) = \sum_{e' \in \mathrm{src}(e)} c(e'). \tag{2.4}$$

Moreover, for any $\gamma \in \Gamma$, we have

$$\mathrm{src}(\gamma(e)) = \gamma(\mathrm{src}(e)), \quad \mathrm{src}(-e) = -\mathrm{src}(e). \tag{2.5}$$

For any $A$-algebra $B$, we denote by $C_k^{\mathrm{har}}(\Gamma, B)$ the set of harmonic cocycles $c : \mathcal{T}_1^o \to V_k(B)$ which is $\Gamma$-equivariant (namely, $c(\gamma(e)) = \gamma \circ c(e)$ for any $\gamma \in \Gamma$ and $e \in \mathcal{T}_1^o$). For any rigid analytic function $f$ on $\Omega$ and $e \in \mathcal{T}_1^o$, we can define an element $\mathrm{Res}(f)(e) \in V_k(\mathbb{C}_\infty)$, which gives an isomorphism of $\mathbb{C}_\infty$-vector spaces

$$\mathrm{Res}_\Gamma : S_k(\Gamma) \to C_k^{\mathrm{har}}(\Gamma, \mathbb{C}_\infty), \quad f \mapsto (e \mapsto \mathrm{Res}(f)(e))$$

([Tei, Theorem 16], see also [Böc, Theorem 5.10]). By [Böc, (17)], the slash operator defined by

$$(f|_k\gamma)(z) = \det(\gamma)^{k-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$$

satisfies $\mathrm{Res}(f|_k\gamma)(e) = \gamma^{-1} \circ \mathrm{Res}(f)(\gamma(e))$.

On the other hand, the argument in [Tei, p. 506] shows that for any $A$-algebra $B$, we have a $B$-linear isomorphism

$$\Phi_\Gamma : C_k^{\mathrm{har}}(\Gamma, B) \to \mathcal{V}_k(B), \quad \Phi_\Gamma(c) = \sum_{e \in \Lambda_1} [e] \otimes c(e),$$

which is independent of the choice of a complete set of representatives $\Lambda_1$. This implies that, for any morphism $B \to B'$ of $A$-algebras, the natural map

$$C_k^{\mathrm{har}}(\Gamma, B) \otimes_B B' \to C_k^{\mathrm{har}}(\Gamma, B')$$

is an isomorphism. Moreover, we obtain an isomorphism

$$\Phi_\Gamma \circ \mathrm{Res}_\Gamma : S_k(\Gamma) \to \mathcal{V}_k(\mathbb{C}_\infty).$$

In particular, for any $A$-subalgebra $B$ of $\mathbb{C}_\infty$, we have an injection

$$\mathcal{V}_k(B) \to \mathcal{V}_k(\mathbb{C}_\infty) \simeq S_k(\Gamma).$$

2.3. **Hecke operators.** For any non-zero element $Q \in A$, we have a Hecke operator $T_Q$ acting on $S_k(\Gamma)$ defined as follows. Write

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma = \coprod_{i \in I(\Gamma, Q)} \Gamma \xi_i,$$

where $\{\xi_i \mid i \in I(\Gamma, Q)\}$ is a complete set of representatives of the right coset space $\Gamma \backslash \Gamma \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma$. For any $f \in S_k(\Gamma)$, we put

$$T_Q f = \sum_{i \in I(\Gamma, Q)} f|_k \xi_i.$$

For any $A$-algebra $B$, we define a Hecke operator $T_Q^{\mathrm{har}}$ on $C_k^{\mathrm{har}}(\Gamma, B)$ as follows. Note that $\xi_i^{-1}$ is an element of the monoid $M^{-1}$. For any $c \in C_k^{\mathrm{har}}(\Gamma, B)$ and $e \in \mathcal{T}_1^o$, we put

$$T_Q^{\mathrm{har}}(c)(e) = \sum_{i \in I(\Gamma, Q)} \xi_i^{-1} \circ c(\xi_i(e)).$$

Since $c$ is $\Gamma$-equivariant, we see that $T_Q^{\mathrm{har}}(c)$ is a harmonic cocycle which is independent of the choice of a complete set of representatives $\{\xi_i \mid i \in I(\Gamma, Q)\}$. For any $\delta \in \Gamma$, the set $\{\xi_i \delta \mid i \in I(\Gamma, Q)\}$ is also a complete set of representatives of the same right coset space. This yields $T_Q^{\mathrm{har}}(c) \in C_k^{\mathrm{har}}(\Gamma, B)$. By [Böc, (17)], for any $A$-subalgebra $B$ of $\mathbb{C}_\infty$, the endomorphism $T_Q^{\mathrm{har}}$ is identified with the restriction on $C_k^{\mathrm{har}}(\Gamma, B) \subseteq C_k^{\mathrm{har}}(\Gamma, \mathbb{C}_\infty)$ of the Hecke operator $T_Q$ on $S_k(\Gamma)$ via the isomorphism $\mathrm{Res}_\Gamma : S_k(\Gamma) \to C_k^{\mathrm{har}}(\Gamma, \mathbb{C}_\infty)$.

We also introduce a Hecke operator $T_{1,Q}$ on $\mathcal{L}_{1,k}(B)$ as follows. We denote by $C^{\pm}_{1,k}(\Gamma, B)$ the set of $\Gamma$-equivariant maps $c : \mathcal{T}_1^{o,\mathrm{st}} \to V_k(B)$ satisfying $c(-e) = -c(e)$ for any $e \in \mathcal{T}_1^{o,\mathrm{st}}$. Then the map

$$\Phi_{1,\Gamma} : C^{\pm}_{1,k}(\Gamma, B) \to \mathcal{L}_{1,k}(B), \quad \Phi_{1,\Gamma}(c) = \sum_{e\in\Lambda_1} [e] \otimes c(e)$$

is independent of the choice of $\Lambda_1$. By the uniqueness of the expression (2.3), we see that it is an isomorphism. For any $c \in C^{\pm}_{1,k}(\Gamma, B)$ and $e \in \mathcal{T}_1^{o,\mathrm{st}}$, we put

$$T^{\pm}_{1,Q}(c)(e) = \sum_{i\in I(\Gamma,Q)} \sum_{e'\in \mathrm{src}(\xi_i(e))} \xi_i^{-1} \circ c(e').$$

By (2.5), it is independent of the choice of $\{\xi_i\}$, and the same argument as in the case of $T_Q^{\mathrm{har}}$ shows that it defines an endomorphism $T^{\pm}_{1,Q}$ on $C^{\pm}_{1,k}(\Gamma, B)$. Now we put

$$T_{1,Q} = \Phi_{1,\Gamma} \circ T^{\pm}_{1,Q} \circ \Phi^{-1}_{1,\Gamma}.$$

From the construction, we see that $T_{1,Q}$ is independent of the choices of $\Lambda_1$ and $\{\xi_i\}$.

For an explicit description of $T_{1,Q}$, fix a complete set of representatives $\Lambda_1$ and take any element $x = \sum_{e\in\Lambda_1}[e] \otimes \omega_e$ of $\mathcal{L}_{1,k}(B)$. For any $e' \in \mathcal{T}_1^{o,\mathrm{st}}$, we have

$$\Phi^{-1}_{1,\Gamma}(x)(e') = \varepsilon_{e'}\gamma_{e'}^{-1} \circ \omega_{r(e')},$$

where $\varepsilon_{e'}$, $\gamma_{e'}$ and $r(e')$ are defined as (2.3) using $\Lambda_1$. Hence we obtain

$$T_{1,Q}(x) = \sum_{e\in\Lambda_1} [e] \otimes \sum_{i\in I(\Gamma,Q)} \sum_{e'\in \mathrm{src}(\xi_i(e))} \varepsilon_{e'} (\xi_i^{-1}\gamma_{e'}^{-1}) \circ \omega_{r(e')}. \tag{2.6}$$

**Proposition 2.2.** *The restriction of $T_{1,Q}$ on the submodule $\mathcal{V}_k(B) \subseteq \mathcal{L}_{1,k}(B)$ agrees with $T_Q^{\mathrm{har}}$ via the isomorphism $\Phi_\Gamma : C_k^{\mathrm{har}}(\Gamma, B) \to \mathcal{V}_k(B)$. In particular, $\mathcal{V}_k(B)$ is stable under $T_{1,Q}$, and if $B$ is an $A$-subalgebra of $\mathbb{C}_\infty$, then $\mathcal{V}_k(B)$ defines a $B$-lattice of $S_k(\Gamma)$ which is stable under Hecke operators.*

*Proof.* Take any $c \in C_k^{\mathrm{har}}(\Gamma, B)$. Since $c(r(e')) = \varepsilon_{e'}\gamma_{e'}c(e')$, (2.4) yields

$$\begin{aligned} T_{1,Q}(\Phi_\Gamma(c)) &= \sum_{e\in\Lambda_1} [e] \otimes \sum_{i\in I(\Gamma,Q)} \sum_{e'\in \mathrm{src}(\xi_i(e))} \xi_i^{-1} \circ c(e') \\ &= \sum_{e\in\Lambda_1} [e] \otimes \sum_{i\in I(\Gamma,Q)} \xi_i^{-1} \circ c(\xi_i(e)) = \sum_{e\in\Lambda_1} [e] \otimes T_Q^{\mathrm{har}}(c)(e), \end{aligned}$$

which agrees with $\Phi_\Gamma(T_Q^{\mathrm{har}}(c))$. $\square$

## 3. Variation of Gouvêa-Mazur type

Let $\mathfrak{n} \in A$ be a non-zero polynomial which is prime to $\wp$ and $r \geqslant 1$ an integer. For any $A$-algebra $B$ and any integer $m \geqslant 1$, put

$$B_m = B/\wp^m B.$$

Note that, since we have the canonical section $[-] : \kappa(\wp) \to \mathcal{O}_{K_\wp}$ of the natural surjection $\mathcal{O}_{K_\wp} \to \kappa(\wp)$, we can consider $B_m$ canonically as a $\kappa(\wp)$-algebra.

Let $\Theta$ be any subgroup of $1 + \wp A_r$. We define

$$\Gamma_0^\Theta(\wp^r) = \left\{ \gamma \in SL_2(A) \;\middle|\; \gamma \bmod \wp^r \in \begin{pmatrix} \Theta & * \\ 0 & \Theta \end{pmatrix} \right\} \subseteq \Gamma_1(\wp)$$

and $\Gamma_1^\Theta(\mathfrak{n}, \wp^r) = \Gamma_1(\mathfrak{n}) \cap \Gamma_0^\Theta(\wp^r)$. The subgroup $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$ of $SL_2(A)$ is $p'$-torsion free and contains $\Gamma_1^{\{1\}}(\mathfrak{n}, \wp^r) = \Gamma_1(\mathfrak{n}\wp^r)$. When $\Theta = 1 + \wp A_r$, we also denote $\Gamma_0^\Theta(\wp^r)$ and $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$ by $\Gamma_0^p(\wp^r)$ and $\Gamma_1^p(\mathfrak{n}, \wp^r)$, respectively. For $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$, we fix a complete set of representatives $\Lambda_1$ as in §2.2.

For Hecke operators of level $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$, we also write

$$U = T_\wp, \quad U_1 = T_{1,\wp}.$$

Let $d(k, a)$ be the dimension of the generalized $U$-eigenspace in $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$ of slope $a$. In this section, we prove $p$-adic local constancy results for $d(k, a)$ with respect to $k$, which generalize the Gouvêa-Mazur conjecture [Hat2, Theorem 1.1] for the case of level $\Gamma_1(t)$.

3.1. **Hecke operators of level $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$.** Let $Q \in A$ be any non-zero element. Write

$$\Gamma_1^\Theta(\mathfrak{n}, \wp^r) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_1^\Theta(\mathfrak{n}, \wp^r) = \coprod_{i \in I(Q)} \Gamma_1^\Theta(\mathfrak{n}, \wp^r)\xi_i.$$

For any $\gamma \in \Gamma_1^\Theta(\mathfrak{n}, \wp^r)$, $i \in I(Q)$ and $\lambda \in \kappa(\wp)^\times$, we have

$$\gamma\xi_i \equiv \begin{pmatrix} 1 & * \\ 0 & Q \end{pmatrix}, \quad \gamma \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \equiv \begin{pmatrix} \lambda^{-1} & * \\ 0 & \lambda \end{pmatrix} \bmod \wp. \tag{3.1}$$

Consider the Hecke operator $T_Q$ acting on the $\mathbb{C}_\infty$-vector space $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$, which preserves the $A$-lattice $\mathcal{V}_k(A)$ by Proposition 2.2. To describe it explicitly for the case where $Q$ is irreducible, we fix a complete set of representatives $R_Q$ of $A/(Q)$. When $Q$ divides $\mathfrak{n}\wp^r$, we have $I(Q) = R_Q$ and

$$(T_Q f)(z) = \frac{1}{Q} \sum_{\beta \in R_Q} f\left(\frac{z + \beta}{Q}\right).$$

When $Q$ does not divide $\mathfrak{n}\wp^r$, we can find $R, S \in A$ satisfying $RQ - \mathfrak{n}\wp^r S = 1$. Put

$$\eta_\diamond = \begin{pmatrix} R & S \\ \mathfrak{n}\wp^r & Q \end{pmatrix}, \quad \xi_\diamond = \begin{pmatrix} RQ & S \\ \mathfrak{n}\wp^r Q & Q \end{pmatrix} = \eta_\diamond \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have $I(Q) = \{\diamond\} \sqcup R_Q$ and

$$(T_Q f)(z) = Q^{k-1}(\langle Q\rangle_{\mathfrak{n}\wp^r} f)(Qz) + \frac{1}{Q}\sum_{\beta \in R_Q} f\left(\frac{z+\beta}{Q}\right),$$

where $\langle Q\rangle_{\mathfrak{n}\wp^r}$ is the diamond operator acting on $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$ defined by $f \mapsto f|_k \eta_\diamond$.

Note that the natural map

$$SL_2(A) \to SL_2(A/(\mathfrak{n}\wp^r)) \simeq SL_2(A/(\mathfrak{n})) \times SL_2(A_r)$$

is surjective. For any $\lambda \in \kappa(\wp)^\times$, we choose $\eta_\lambda \in SL_2(A)$ satisfying

$$\eta_\lambda \bmod \mathfrak{n} = I, \quad \eta_\lambda \bmod \wp^r = \begin{pmatrix} [\lambda]^{-1} & 0 \\ 0 & [\lambda] \end{pmatrix} \tag{3.2}$$

and put

$$\langle \lambda \rangle_{\wp^r} f = f|_k \eta_\lambda.$$

By

$$\Gamma_1(\mathfrak{n}\wp^r) \subseteq \Gamma_1^\Theta(\mathfrak{n}, \wp^r), \quad \eta_\lambda^{-1}\Gamma_1^\Theta(\mathfrak{n}, \wp^r)\eta_\lambda = \Gamma_1^\Theta(\mathfrak{n}, \wp^r), \tag{3.3}$$

this is independent of the choice of $\eta_\lambda$ and defines an action of $\kappa(\wp)^\times$ on $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$.

For any $\kappa(\wp)[\kappa(\wp)^\times]$-module $M$ and any character $\chi : \kappa(\wp)^\times \to \kappa(\wp)^\times$, we denote by $M(\chi)$ the maximal $\kappa(\wp)$-subspace of $M$ on which any $\lambda \in \kappa(\wp)^\times$ acts via $\chi(\lambda)$. Since the order of the group $\kappa(\wp)^\times$ is prime to $p$, we have the projector

$$\varepsilon_\chi : M \to M(\chi), \quad \varepsilon_\chi(m) = -\sum_{\lambda \in \kappa(\wp)^\times} \chi(\lambda)^{-1}(\lambda \cdot m)$$

and the decomposition into $\chi$-parts

$$M = \bigoplus_\chi M(\chi),$$

where the sum runs over the set of such characters $\kappa(\wp)^\times \to \kappa(\wp)^\times$.

We consider $\bar{K}$ as a $\kappa(\wp)$-algebra by the unique map $\kappa(\wp) \to \bar{K}$ which commutes the diagram

$$\begin{tikzcd} \kappa(\wp) \arrow[r] \arrow[dr, "{[-]}"'] & \bar{K} \arrow[d, "\iota_\wp"] \\ & \mathbb{C}_\wp. \end{tikzcd}$$

Then we have

$$S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r)) = \bigoplus_\chi S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))(\chi).$$

Note that, when an irreducible polynomial $Q$ does not divide $\mathfrak{n}\wp^r$, we may further assume that $\eta_\lambda$ satisfies

$$\eta_\lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \notin (Q).$$

Using this, for any irreducible polynomial $Q$ we can show

$$\Gamma_1^\Theta(\mathfrak{n}, \wp^r)\eta_\lambda^{-1} \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \eta_\lambda \Gamma_1^\Theta(\mathfrak{n}, \wp^r) = \Gamma_1^\Theta(\mathfrak{n}, \wp^r) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_1^\Theta(\mathfrak{n}, \wp^r).$$

Then (3.3) yields

$$\begin{aligned} \coprod_{i \in I(Q)} \Gamma_1^\Theta(\mathfrak{n}, \wp^r)\xi_i\eta_\lambda &= \Gamma_1^\Theta(\mathfrak{n}, \wp^r) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \eta_\lambda \Gamma_1^\Theta(\mathfrak{n}, \wp^r) \\ &= \Gamma_1^\Theta(\mathfrak{n}, \wp^r)\eta_\lambda \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_1^\Theta(\mathfrak{n}, \wp^r) = \coprod_{i \in I(Q)} \Gamma_1^\Theta(\mathfrak{n}, \wp^r)\eta_\lambda\xi_i. \end{aligned} \tag{3.4}$$

Thus $T_Q$ commutes with $\langle\lambda\rangle_{\wp^r}$ and $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))(\chi)$ is stable under Hecke operators. We denote by $d(k, \chi, a)$ be the dimension of the generalized $U$-eigenspace in $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))(\chi)$ of slope $a$. To indicate the level, we often write

$$d(k, a) = d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k, a), \quad d(k, \chi, a) = d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k, \chi, a).$$

For any $A$-algebra $B$, we also have the diamond operator $\langle\lambda\rangle_{\wp^r}$

$$\langle\lambda\rangle_{\wp^r} \in \mathrm{End}(C_k^{\mathrm{har}}(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), B)), \quad c \mapsto (e \mapsto \eta_\lambda^{-1} \circ c(\eta_\lambda(e))),$$

which is compatible with that on $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$ when $B = \mathbb{C}_\infty$. From (3.3) we see that $e$ is $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$-stable if and only of $\eta_\lambda(e)$ is, and thus the corresponding operators on $\mathcal{V}_k(B)$ and $\mathcal{L}_{1,k}(B)$ are given by

$$\langle\lambda\rangle_{\wp^r}\Big(\sum_{e \in \Lambda_1} [e] \otimes \omega_e\Big) = \sum_{e \in \Lambda_1} [e] \otimes \varepsilon_{\eta_\lambda(e)}(\eta_\lambda^{-1}\gamma_{\eta_\lambda(e)}^{-1}) \circ \omega_{r(\eta_\lambda(e))}. \tag{3.5}$$

When $B$ is also a $\kappa(\wp)$-algebra, we have the decomposition

$$C_k^{\mathrm{har}}(\Gamma_1^{\Theta}(\mathfrak{n},\wp^r),B)=\bigoplus_{\chi} C_k^{\mathrm{har}}(\Gamma_1^{\Theta}(\mathfrak{n},\wp^r),B)(\chi)$$

and similarly for $\mathcal{L}_{1,k}(B)$ and $\mathcal{V}_k(B)$. These summands are stable under Hecke operators by (3.4).

3.2. **Weight reduction.** Let $N \geqslant 1$ be any integer. For any $A$-algebra $B$, the $B$-linear map

$$\mu_{k,N}: H_{k-2}(B) \to H_{k-2+N}(B), \quad X^iY^{k-2-i} \mapsto X^{i+N}Y^{k-2-i}$$

induces the dual map

$$\rho_{k,N}: V_{k+N}(B) \to V_k(B), \quad (X^iY^{k+N-2-i})^\vee \mapsto \begin{cases} (X^{i-N}Y^{k+N-2-i})^\vee & (i \geqslant N) \\ 0 & (i < N) \end{cases}.$$

It is a surjection whose kernel is

$$V_{k+N}^{<N}(B)=\bigoplus_{i<N} B(X^iY^{k+N-2-i})^\vee.$$

**Lemma 3.1.** *Let $n \geqslant 0$ be any non-negative integer, $\bar{B}$ any $A_{p^n}$-algebra and $\lambda \in \kappa(\wp)^\times$. Let $\xi \in M_2(A)$ be any element satisfying*

$$\xi=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \bmod \wp=\lambda, \quad c \equiv 0 \bmod \wp.$$

*Let $m$ be the order of $\lambda$ in $\kappa(\wp)^\times$. Then, for any element $\omega \in V_{k+p^nm}(\bar{B})$, we have*

$$\xi^{-1}\circ\rho_{k,p^nm}(\omega)=\rho_{k,p^nm}(\xi^{-1}\circ\omega).$$

*In particular, for any integer $m' \geqslant 1$, the map $\rho_{k,p^nm'}: V_{k+p^nm'}(\bar{B}) \to V_k(\bar{B})$ is $\Gamma_1^{\Theta}(\mathfrak{n},\wp^r)$-equivariant and its kernel $V_{k+p^nm'}^{<p^nm'}(\bar{B})$ is $\Gamma_1^{\Theta}(\mathfrak{n},\wp^r)$-stable.*

*Proof.* In the ring $A_{p^n}$, we can write $a=[\lambda]+\wp a'$ with some $a' \in A_{p^n}$. For any integer $i \in [0,k-2]$, the assumption $\wp^{p^n}\bar{B}=0$ implies

$$\begin{aligned}
\xi\circ\mu_{k,p^nm}(X^iY^{k-2-i}) &= (aX+cY)^{p^nm+i}(bX+dY)^{k-2-i} \\
&= (a^{p^n}X^{p^n}+c^{p^n}Y^{p^n})^m(aX+cY)^i(bX+dY)^{k-2-i} \\
&= ([\lambda]^{p^n}X^{p^n})^m(aX+cY)^i(bX+dY)^{k-2-i} \\
&= X^{p^nm}(aX+cY)^i(bX+dY)^{k-2-i} \\
&= \mu_{k,p^nm}(\xi\circ(X^iY^{k-2-i})).
\end{aligned}$$

Taking the dual yields the lemma. □

By Lemma 3.1, for any $A_{p^n}$-algebra $\bar{B}$ and any integer $m' \geqslant 1$, we obtain the surjection

$$1 \otimes \rho_{k,p^n m'} : \mathcal{V}_{k+p^n m'}(\bar{B}) \to \mathcal{V}_k(\bar{B})$$

and similarly for $\mathcal{L}_{1,k}(\bar{B})$.

**Lemma 3.2.** *For any $A_{p^n}$-algebra $\bar{B}$, the maps*

$$1 \otimes \rho_{k,p^n} : \mathcal{V}_{k+p^n}(\bar{B}) \to \mathcal{V}_k(\bar{B}), \quad \mathcal{L}_{1,k+p^n}(\bar{B}) \to \mathcal{L}_{1,k}(\bar{B})$$

*commute with Hecke operators. Moreover, the maps*

$$1 \otimes \rho_{k,p^n(q^d-1)} : \mathcal{V}_{k+p^n(q^d-1)}(\bar{B}) \to \mathcal{V}_k(\bar{B}), \quad \mathcal{L}_{1,k+p^n(q^d-1)}(\bar{B}) \to \mathcal{L}_{1,k}(\bar{B})$$

*commute with $\langle \lambda \rangle_{\wp^r}$ for any $\lambda \in \kappa(\wp)^\times$. In particular, the $\bar{B}$-submodules*

$$\mathcal{V}^{<p^n}_{k+p^n}(\bar{B}), \quad \mathcal{V}^{<p^n(q^d-1)}_{k+p^n(q^d-1)}(\bar{B})$$

*are stable under Hecke operators.*

*Proof.* It is enough to show the assertions on $\mathcal{L}_{1,k}(\bar{B})$. By (2.6) and (3.5), we reduce ourselves to showing that, for any $\gamma \in \Gamma_1^\Theta(\mathfrak{n}, \wp^r)$, $i \in I(Q)$, $\lambda \in \kappa(\wp)^\times$, $\omega \in V_{k+p^n}(\bar{B})$ and $\omega' \in V_{k+p^n(q^d-1)}(\bar{B})$, we have

$$(\gamma\xi_i)^{-1} \circ \rho_{k,p^n}(\omega) = \rho_{k,p^n}((\gamma\xi_i)^{-1} \circ \omega),$$
$$(\gamma\eta_\lambda)^{-1} \circ \rho_{k,p^n(q^d-1)}(\omega') = \rho_{k,p^n(q^d-1)}((\gamma\eta_\lambda)^{-1} \circ \omega').$$

By (3.1), this follows from Lemma 3.1. □

3.3. **Dimension of slope zero cuspforms.** Using harmonic cocycles, the proofs of [Hid1, Corollary 8.2 and Proposition 8.3] can be adapted to obtain constancy results for the dimension of slope zero cuspforms with respect to the weight and the level at $\wp$. First we prove the following key lemma.

**Lemma 3.3.** *Let $B$ be any flat $A$-algebra. For any $s \in \mathrm{St}$ and any integer $j \in [0, k-2]$, the element $s \otimes (X^j Y^{k-2-j})^\vee \in \mathcal{V}_k(B)$ satisfies*

$$U(s \otimes (X^j Y^{k-2-j})^\vee) \in \wp^{k-2-j} \mathcal{V}_k(B).$$

*Proof.* For any non-negative integer $m$, we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}_k(B) \arrow[r] \arrow[d] & \mathcal{L}_{1,k}(B) \arrow[r, "\partial_\Gamma \otimes 1"] \arrow[d] & \mathcal{L}_{0,k}(B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{V}_k(B_m) \arrow[r] & \mathcal{L}_{1,k}(B_m) \arrow[r, "\partial_\Gamma \otimes 1"] & \mathcal{L}_{0,k}(B_m) \arrow[r] & 0.
\end{tikzcd}$$

Since the structure map $A \to B$ is flat, we see that $\wp^m \mathcal{V}_k(B)$ and $\wp^m \mathcal{L}_{1,k}(B)$ are the kernels of the left two vertical maps. Thus it suffices to show $U_1(s \otimes (X^j Y^{k-2-j})^\vee) \in \wp^{k-2-j} \mathcal{L}_{1,k}(B)$.

Any element of St is a $\mathbb{Z}$-linear combination of elements of $\mathbb{Z}[\bar{\mathcal{T}}_1^{o,\mathrm{st}}]$ of the form $[e]|_\alpha$ with $e \in \Lambda_1$ and $\alpha \in \Gamma_1^\Theta(\mathfrak{n}, \wp^r)$. Moreover, for any $\omega \in V_k(B)$, we have $[e]|_\alpha \otimes \omega = [e] \otimes \alpha \circ \omega$. By (2.6), it is enough to show that, for any $i \in I(\wp)$, $\gamma \in \Gamma_1^\Theta(\mathfrak{n}, \wp^r)$ and integers $j, l \in [0, k-2]$, we have

$$((\gamma \xi_i)^{-1} \circ (X^j Y^{k-2-j})^\vee)(X^l Y^{k-2-l}) \in \wp^{k-2-j} B.$$

Write $\gamma \xi_i = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the above evaluation is equal to

$$(X^j Y^{k-2-j})^\vee ((aX + cY)^l (bX + dY)^{k-2-l}).$$

By (3.1) we have $c, d \equiv 0 \bmod \wp$ and the coefficient of $X^j Y^{k-2-j}$ in the product $(aX + cY)^l (bX + dY)^{k-2-l}$ is divisible by $\wp^{k-2-j}$. This concludes the proof. □

**Proposition 3.4.** *(1) $d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k, 0)$ is independent of $k$.*

*(2) For any character $\chi : \kappa(\wp)^\times \to \kappa(\wp)^\times$, we have*

$$k_1 \equiv k_2 \bmod q^d - 1 \Rightarrow d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k_1, \chi, 0) = d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k_2, \chi, 0).$$

*Proof.* Note that $d(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k, 0)$ is equal to the degree of the polynomial

$$\det(I - UX; \mathcal{V}_k(\kappa(\wp))).$$

By Lemma 3.2 for $n = 0$, we have the exact sequence

$$0 \longrightarrow \mathcal{V}_{k+1}^{<1}(\kappa(\wp)) \longrightarrow \mathcal{V}_{k+1}(\kappa(\wp)) \longrightarrow \mathcal{V}_k(\kappa(\wp)) \longrightarrow 0$$

whose maps are compatible with Hecke operators. Since $(k+1) - 2 > 0$, Lemma 3.3 implies $U = 0$ on $\mathcal{V}_{k+1}^{<1}(\kappa(\wp))$ and thus we have

$$\det(I - UX; \mathcal{V}_{k+1}^{<1}(\kappa(\wp))) = 1,$$

which yields the assertion (1). Since Lemma 3.2 also gives the exact sequence

$$0 \longrightarrow \mathcal{V}_{k+p^d-1}^{<p^d-1}(\kappa(\wp))(\chi) \longrightarrow \mathcal{V}_{k+p^d-1}(\kappa(\wp))(\chi) \longrightarrow \mathcal{V}_k(\kappa(\wp))(\chi) \longrightarrow 0,$$

the assertion (2) follows similarly. □

**Proposition 3.5.** *$d(\Gamma_1^p(\mathfrak{n}, \wp^r), k, 0)$ and $d(\Gamma_1^p(\mathfrak{n}, \wp^r), k, \chi, 0)$ are independent of $r \geqslant 1$.*

*Proof.* Put $\Gamma_r = \Gamma_1^p(\mathfrak{n}, \wp^r)$. Let $\bar{\kappa}$ be an algebraic closure of $\kappa(\wp)$. We reduce ourselves to showing that the multiplicities of non-zero eigenvalues of $U$ acting on $C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa})$ and $C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa})(\chi)$ are independent of $r$. These are the same as the dimensions of the generalized eigenspaces

$$C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa})^{\mathrm{ord}}, \quad C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa})(\chi)^{\mathrm{ord}}$$

of non-zero eigenvalues, respectively.

Since any $c \in C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa})$ is also $\Gamma_{r+1}$-equivariant, we have the natural inclusion

$$\iota : C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa}) \to C_k^{\mathrm{har}}(\Gamma_{r+1}, \bar{\kappa}).$$

Since we have

$$\Gamma_{r+1} \begin{pmatrix} 1 & 0 \\ 0 & \wp \end{pmatrix} \Gamma_r = \coprod_{\beta \in R_\wp} \Gamma_{r+1} \xi_\beta, \quad \xi_\beta = \begin{pmatrix} 1 & \beta \\ 0 & \wp \end{pmatrix},$$

we obtain a map $s : C_k^{\mathrm{har}}(\Gamma_{r+1}, \bar{\kappa}) \to C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa})$ by

$$s(c)(e) = \sum_{\beta \in R_\wp} \xi_\beta^{-1} \circ c(\xi_\beta(e)),$$

which makes the following diagram commutative.

$$\begin{tikzcd} C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa}) \arrow[r, "\iota"] \arrow[d, "U"'] & C_k^{\mathrm{har}}(\Gamma_{r+1}, \bar{\kappa}) \arrow[d, "U"] \arrow[dl, "s"'] \\ C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa}) \arrow[r, "\iota"'] & C_k^{\mathrm{har}}(\Gamma_{r+1}, \bar{\kappa}) \end{tikzcd}$$

From this we see that $\iota$ and $s$ commute with $U$ and, since $U$ is isomorphic on $C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa})^{\mathrm{ord}}$, the map $\iota$ gives an isomorphism

$$\iota^{\mathrm{ord}} : C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa})^{\mathrm{ord}} \to C_k^{\mathrm{har}}(\Gamma_{r+1}, \bar{\kappa})^{\mathrm{ord}}.$$

This settles the assertion on $d(\Gamma_1^p(\mathfrak{n}, \wp^r), k, 0)$. Moreover, since the diamond operator $\langle \lambda \rangle_{\wp^r}$ is independent of the choice of $\eta_\lambda$ satisfying (3.2), we also have

$$\langle \lambda \rangle_{\wp^{r+1}} \circ \iota = \iota \circ \langle \lambda \rangle_{\wp^r}.$$

Since $U$ commutes with diamond operators, the map $\iota^{\mathrm{ord}}$ also induces an isomorphism

$$C_k^{\mathrm{har}}(\Gamma_r, \bar{\kappa})(\chi)^{\mathrm{ord}} \to C_k^{\mathrm{har}}(\Gamma_{r+1}, \bar{\kappa})(\chi)^{\mathrm{ord}},$$

from which the assertion on $d(\Gamma_1^p(\mathfrak{n}, \wp^r), k, \chi, 0)$ follows. $\square$

3.4. **Representing matrix of $U$.** Let $E/K_\wp$ be a finite extension of complete valuation fields. We extend the normalized $\wp$-adic valuation $v_\wp$ naturally to $E$. We denote by $\mathcal{O}_E$ the integer ring of $E$.

**Lemma 3.6.** *Suppose that $\mathfrak{n}\wp$ has a prime factor $\pi$ of degree one. Then the right $\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$-module* St *is free of rank $[\Gamma_1(\pi):\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$, where the rank is independent of the choice of such $\pi$.*

*Proof.* Note that, from $\Gamma_1^\Theta(\mathfrak{n},\wp^r)\subseteq\Gamma_1(\mathfrak{n}\wp)$, we see that the former is a subgroup of $\Gamma_1(\pi)$. We can show that a fundamental domain of $\Gamma_1(\pi)\backslash\mathcal{T}$ is the same as the picture of [LM, §7], and that it has no $\Gamma_1(\pi)$-stable vertex and only one $\Gamma_1(\pi)$-stable (unoriented) edge. By (2.1), the right $\mathbb{Z}[\Gamma_1(\pi)]$-module St is free of rank one. Thus the right $\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$-module St is free of rank $[\Gamma_1(\pi):\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$. Since we have

$$[\Gamma_1(\pi):\Gamma_1^\Theta(\mathfrak{n},\wp^r)]=[SL_2(A):\Gamma_1^\Theta(\mathfrak{n},\wp^r)]\left[SL_2(\mathbb{F}_q):\left\{\begin{pmatrix}1&*\\0&1\end{pmatrix}\right\}\right]^{-1},$$

the rank is independent of $\pi$. □

In the sequel, we assume that $\mathfrak{n}\wp$ has a prime factor $\pi$ of degree one. Under this assumption, Lemma 3.6 implies that the right $\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$-module St is free of rank $d$, where we put

$$d=[\Gamma_1(\pi):\Gamma_1^\Theta(\mathfrak{n},\wp^r)].$$

Hence, for any $A$-algebra $B$, the $B$-module $\mathcal{V}_k(B)$ is free of rank $d(k-1)$. We fix an ordered basis $\mathfrak{B}_k$ of the free $A$-module $\mathcal{V}_k(A)$, as follows. Take an ordered basis $(s_1,\ldots,s_d)$ of the right $\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n},\wp^r)]$-module St. The set

$$\mathfrak{B}_k=\{v_{i,j}=s_i\otimes(X^jY^{k-2-j})^\vee\mid 1\leqslant i\leqslant d,\ 0\leqslant j\leqslant k-2\}$$

forms a basis of the $A$-module $\mathcal{V}_k(A)$, and we order it as

$$v_{1,0},v_{2,0},\ldots,v_{d,0},v_{1,1},v_{2,1},\ldots,v_{d,1},v_{1,2},\ldots.$$

For any $A$-algebra $B$, the ordered basis of the $B$-module $\mathcal{V}_k(B)$ induced by $\mathfrak{B}_k$ is also denoted abusively by $\mathfrak{B}_k$. We denote by $U^{(k)}$ the representing matrix of $U$ acting on the $\mathcal{O}_E$-module $\mathcal{V}_k(\mathcal{O}_E)$ with respect to the ordered basis $\mathfrak{B}_k$. Then Lemma 3.3 gives

$$U(v_{i,j})\in\wp^{k-2-j}\mathcal{V}_k(\mathcal{O}_E). \tag{3.6}$$

In order to study perturbation of $U^{(k)}$, we use the following lemma of [Ked]. Note that the assumption $B\in GL_n(F)$ there is superfluous.

**Lemma 3.7** ([Ked], Proposition 4.4)**.** *Let $L$ be any positive integer and $A, B \in M_L(\mathcal{O}_E)$. Let $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_L$ be the elementary divisors of $A$. Namely, they are the normalized $\wp$-adic valuations of diagonal entries of the Smith normal form of $A$. Let $s'_1 \leqslant s'_2 \leqslant \cdots \leqslant s'_L$ be the elementary divisors of $AB$. Then we have*

$$s'_i \geqslant s_i \quad \text{for any } i.$$

*The same inequality also holds for the elementary divisors of $BA$.*

**Corollary 3.8.** *Suppose that $\mathfrak{n}\wp$ has a prime factor $\pi$ of degree one. Put $d = [\Gamma_1(\pi) : \Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)]$. Let $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_{d(k-1)}$ be the elementary divisors of $U^{(k)}$. Then we have*

$$s_i \geqslant \left\lfloor \frac{i-1}{d} \right\rfloor.$$

*Proof.* By (3.6), the matrix $U^{(k)}$ can be written as

$$U^{(k)} = B\mathrm{diag}(\wp^{k-2}, \ldots, \wp^{k-2}, \ldots, \wp, \ldots, \wp, 1, \ldots, 1),$$

where $B \in M_{d(k-1)}(\mathcal{O}_E)$ and the diagonal entries of the last matrix are $\{\wp^j \mid 0 \leqslant j \leqslant k-2\}$, each with multiplicity $d$. Then the corollary follows from Lemma 3.7. $\square$

**Corollary 3.9.** *Let $n \geqslant 0$ be any non-negative integer. Then, for some matrices $B_1, B_2, B_3, B_4$ with entries in $\mathcal{O}_E$, we have*

$$U^{(k+p^n)} = \left( \begin{array}{c|c} \wp^{k-1} B_1 & B_2 \\ \hline \wp^{p^n} B_3 & U^{(k)} + \wp^{p^n} B_4 \end{array} \right).$$

*Proof.* By Lemma 3.2, the lower right block is congruent to $U^{(k)}$ and the lower left block is zero modulo $\wp^{p^n}$. By (3.6), the entries on the upper left block are divisible by $\wp^{k-1}$. This concludes the proof. $\square$

For the $U$-operator acting on $\mathcal{V}_k(\mathcal{O}_E)(\chi)$, we have a similar description of its representing matrix $U_\chi^{(k)}$ as follows.

**Proposition 3.10.** *Suppose that $\mathfrak{n}\wp$ has a prime factor $\pi$ of degree one. Put $d = [\Gamma_1(\pi) : \Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)]$.*

*(1) For any integer $i \geqslant 0$, the $i$-th smallest elementary divisor $s_{\chi,i}$ of $U_\chi^{(k)}$ satisfies*

$$s_{\chi,i} \geqslant \left\lfloor \frac{i-1}{d} \right\rfloor.$$

*(2) Let $n \geqslant 0$ be any non-negative integer. Then, with some bases of $\mathcal{V}_k(\mathcal{O}_E)(\chi)$ and $\mathcal{V}_{k+p^n(q^d-1)}(\mathcal{O}_E)(\chi)$, the representing matrices $U_\chi^{(k)}$ and $U_\chi^{(k+p^n(q^d-1))}$ of $U$ acting on them satisfies*

$$U_\chi^{(k+p^n(q^d-1))} = \left(\begin{array}{c|c} \wp^{k-1}B_1 & B_2 \\ \hline \wp^{p^n}B_3 & U_\chi^{(k)} + \wp^{p^n}B_4 \end{array}\right)$$

*for some matrices $B_1, B_2, B_3, B_4$ with entries in $\mathcal{O}_E$.*

*Proof.* We have the decomposition

$$\mathcal{V}_k(\mathcal{O}_E) = \bigoplus_\chi \mathcal{V}_k(\mathcal{O}_E)(\chi),$$

where each summand is stable under Hecke operators. Thus any elementary divisor of $U_\chi^{(k)}$ is also an elementary divisor of $U^{(k)}$, and $s_{\chi,i}$ equals the $i'$-th smallest elementary divisor $s_{i'}$ of $U^{(k)}$ with some $i' \geqslant i$. Hence the assertion (1) follows from Corollary 3.8.

For (2), put $m = q^d - 1$, $k' = k + p^n m$ and consider the weight reduction map

$$\rho = 1 \otimes \rho_{k,p^n m} : \mathcal{V}_{k'}(\mathcal{O}_{E,p^n}) \to \mathcal{V}_k(\mathcal{O}_{E,p^n}).$$

By Lemma 3.1, we can define the tensor product over $\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n}, \wp^r)]$

$$\mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n}) = \mathrm{St} \otimes_{\mathbb{Z}[\Gamma_1^\Theta(\mathfrak{n},\wp^r)]} V_{k'}^{<p^n m}(\mathcal{O}_{E,p^n}),$$

which sits in the split exact sequence of $\mathcal{O}_{E,p^n}$-modules

$$0 \longrightarrow \mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n}) \longrightarrow \mathcal{V}_{k'}(\mathcal{O}_{E,p^n}) \xrightarrow{\rho} \mathcal{V}_k(\mathcal{O}_{E,p^n}) \longrightarrow 0.$$

By Lemma 3.2, the map $\rho$ is compatible with Hecke operators and $\langle \lambda \rangle_{\wp^r}$ for any $\lambda \in \kappa(\wp)^\times$. Thus the map $\rho$ also induces the split exact sequence

$$0 \longrightarrow \mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n})(\chi) \longrightarrow \mathcal{V}_{k'}(\mathcal{O}_{E,p^n})(\chi) \xrightarrow{\rho} \mathcal{V}_k(\mathcal{O}_{E,p^n})(\chi) \longrightarrow 0.$$

Let $\varepsilon_\chi : \mathcal{V}_{k'}(\mathcal{O}_E) \to \mathcal{V}_{k'}(\mathcal{O}_E)(\chi)$ be the projector to the $\chi$-part. Let $\kappa_E$ be the residue field of $E$. Consider the basis $v_{i,j} = s_i \otimes (X^j Y^{k'-2-j})^\vee$ of $\mathcal{V}_{k'}(\mathcal{O}_E)$ as before and its image $\bar{v}_{i,j}$ in $\mathcal{V}_{k'}(\kappa_E)$. Note that, for any $j < p^n m$, the image of $\varepsilon_\chi(v_{i,j})$ in $\mathcal{V}_{k'}(\mathcal{O}_{E,p^n})(\chi)$ lies in $\mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n})(\chi)$. Since the set

$$\{\varepsilon_\chi(\bar{v}_{i,j}) \mid 1 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant p^n m - 1\}$$

spans the $\kappa_E$-vector space $\mathcal{V}_{k'}^{<p^n m}(\kappa_E)(\chi)$, there exists a subset $\Sigma \subseteq [1, d] \times [0, p^n m - 1]$ such that the elements $\varepsilon_\chi(\bar{v}_{i,j})$ for $(i, j) \in \Sigma$ form its basis.

Now take a lift $\mathfrak{B}_{k',\chi,k}$ of a basis of $\mathcal{V}_k(\mathcal{O}_{E,p^n})(\chi)$ by the composite

$$\mathcal{V}_{k'}(\mathcal{O}_E)(\chi) \to \mathcal{V}_{k'}(\mathcal{O}_{E,p^n})(\chi) \xrightarrow{\rho} \mathcal{V}_k(\mathcal{O}_{E,p^n})(\chi).$$

Since the image of the set

$$\mathfrak{B}_{k',\chi} = \{\varepsilon_\chi(v_{i,j}) \mid (i,j) \in \Sigma\} \sqcup \mathfrak{B}_{k',\chi,k}$$

in $\mathcal{V}_{k'}(\kappa_E)(\chi)$ forms its basis, we see that $\mathfrak{B}_{k',\chi}$ itself forms a basis of $\mathcal{V}_{k'}(\mathcal{O}_E)(\chi)$. Moreover, by Nakayama's lemma, the images of $\varepsilon_\chi(v_{i,j})$ in $\mathcal{V}_{k'}(\mathcal{O}_{E,p^n})$ for $(i,j) \in \Sigma$ form a basis of $\mathcal{V}_{k'}^{<p^n m}(\mathcal{O}_{E,p^n})(\chi)$.

Representing $U$ by the basis $\mathfrak{B}_{k',\chi}$, we see that the lower blocks of the resulting matrix are as stated in (2). Moreover, since $U$ and $\langle\lambda\rangle_{\wp^r}$ commute with each other, (3.6) yields

$$U(\varepsilon_\chi(v_{i,j})) = \varepsilon_\chi(U(v_{i,j})) \in \wp^{k'-2-j}\mathcal{V}_{k'}(\mathcal{O}_E)(\chi)$$

for any $j < p^n m$, and thus the upper left block is divisible by $\wp^{k-1}$. This concludes the proof. $\square$

3.5. **Perturbation.** Let $E/K_\wp$ be a finite extension inside $\mathbb{C}_\wp$. Let $V$ be an $E$-vector space of finite dimension and $T : V \to V$ an $E$-linear endomorphism. For an eigenvector of $T$ with eigenvalue $\lambda \in \mathbb{C}_\wp$, we refer to $v_\wp(\lambda)$ as its slope. For any rational number $a$, we denote by $d(T,a)$ the multiplicity of $T$-eigenvalues of slope $a$. If $B$ is the representing matrix of $T$ with some basis of $V$, we also denote it by $d(B,a)$.

**Proposition 3.11.** *Let $d_0$, $n$ and $L$ be positive integers. Let $B \in M_L(\mathcal{O}_E)$ be a matrix such that its $i$-th smallest elementary divisor $s_i$ satisfies $s_i \geqslant \lfloor \frac{i-1}{d_0} \rfloor$ for any $i$. Put $\varepsilon_0 = d(B,0)$ and*

$$C_1(n,d_0,\varepsilon_0) = p^n\left(\frac{4+d_0p^n-d_0}{4+2d_0p^n-2\varepsilon_0}\right) \in (0,p^n).$$

*Moreover, we put $q_1 = r_1 = 0$ and for any $l \geqslant 2$, we write $q_l = \lfloor \frac{l-2}{d_0} \rfloor$ and $r_l = l-2-d_0q_l$. We define $C_2(n,d_0,\varepsilon_0)$ as*

$$\min\left\{ \frac{2p^n + d_0q_l(q_l-1) + 2q_l(r_l+1)}{2(l-\varepsilon_0)} \;\middle|\; \varepsilon_0 < l \leqslant 1 + d_0p^n \right\}$$

*and put*

$$C(n,d_0,\varepsilon_0) = \min\{C_1(n,d_0,\varepsilon_0), C_2(n,d_0,\varepsilon_0)\} \in (0,p^n).$$

*Let $B' \in M_L(\mathcal{O}_E)$ be any matrix satisfying $B' - B \in \wp^{p^n}M_L(\mathcal{O}_E)$. Let $a$ be any non-negative rational number satisfying*

$$a < C(n,d_0,\varepsilon_0).$$

*Then we have*

$$d(B, a) = d(B', a).$$

*Proof.* We put

$$P_B(X) = \det(I - BX) = \sum b_l X^l, \quad P_{B'}(X) = \det(I - B'X) = \sum b'_l X^l.$$

Then $b_l$ is, up to a sign, the sum of principal $l \times l$ minors of $B$. Since $P_B \equiv P_{B'} \bmod \wp$, we have $d(B', 0) = d(B, 0) = \varepsilon_0$. From the assumption on elementary divisors, we see that if $i > d_0$, then any $i \times i$ minor of $B$ is divisible by $\wp$. This yields $\varepsilon_0 \leqslant d_0$.

By [Ked, Theorem 4.4.2], for any $l \geqslant 0$ we have

$$v_\wp(b_l - b'_l) \geqslant p^n + \sum_{j=1}^{l-1} \min \left\{ \left\lfloor \frac{j-1}{d_0} \right\rfloor, p^n \right\}.$$

Here we mean that the second term of the right-hand side is zero for $l \leqslant 1$. Let $R$ be the right-hand side of the inequality. We claim that for any $l > \varepsilon_0$, we have

$$a < C(n, d_0, \varepsilon_0) \Rightarrow R > a(l - \varepsilon_0).$$

Indeed, when $l > 1 + d_0 p^n$, we have

$$R = p^n + \sum_{j=1}^{d_0 p^n} \left\lfloor \frac{j-1}{d_0} \right\rfloor + \sum_{j=1+d_0 p^n}^{l-1} p^n = p^n(l - d_0 p^n) + \frac{1}{2} d_0 p^n (p^n - 1)$$
$$= \frac{1}{2} p^n (2l - d_0 - d_0 p^n).$$

Then $R > a(l - \varepsilon_0)$ if and only if

$$(p^n - a)l - \frac{1}{2} p^n d_0 (1 + p^n) + a\varepsilon_0 > 0. \tag{3.7}$$

Since the condition $a < C(n, d_0, \varepsilon_0)$ yields $p^n > a$, the left-hand side of (3.7) is increasing with respect to $l$. Thus (3.7) holds for any $l > 1 + d_0 p^n$ if and only if it holds for $l = 2 + d_0 p^n$, which is equivalent to $a < C_1(n, d_0, \varepsilon_0)$.

On the other hand, when $l \leqslant 1 + d_0 p^n$, we have

$$R = p^n + \frac{1}{2} d_0 q_l (q_l - 1) + q_l (r_l + 1), \tag{3.8}$$

from which the claim follows.

Let $N_B$ and $N_{B'}$ be the Newton polygons of $P_B$ and $P_{B'}$, respectively. It suffices to show that the segments of $N_B$ and $N_{B'}$ with slope less than $C(n, d_0, \varepsilon_0)$ agree with each other. Suppose the contrary and take the smallest slope $a < C(n, d_0, \varepsilon_0)$ satisfying $d(B, a) \neq d(B', a)$.

Let $(l, y)$ be the right endpoint of the segment of slope $a$ in either of $N_B$ or $N_{B'}$. Since $d(B,0) = d(B',0)$, we have $a > 0$ and $l > \varepsilon_0$. Then the above claim yields

$$y \leqslant a(l - \varepsilon_0) < v_\wp(b_l - b'_l).$$

Since $y \in \{v_\wp(b_l), v_\wp(b'_l)\}$, we have $v_\wp(b_l) = v_\wp(b'_l)$. Since $a$ is minimal, this implies that slope $a$ appears in both of $N_B$ and $N_{B'}$. Applying the same argument to the right endpoint of the segment of slope $a$ in the other Newton polygon, we obtain $d(B,a) = d(B',a)$. This is the contradiction. $\square$

By a similar argument, we can show a slightly different perturbation result as follows.

**Proposition 3.12.** *With the notation in Proposition 3.11, we suppose that the following conditions hold.*

*(1) If $p = 2$, then $n \geqslant 3$ or $d_0 - \varepsilon_0 \leqslant 1$.*
*(2) $2p^n > n(d_0 n + 2 + d_0 - 2\varepsilon_0)$.*

*Then, for any non-negative rational number $a \leqslant n$, we have*

$$d(B,a) = d(B',a).$$

*Proof.* Let $R$ be as in the proof of Proposition 3.11. We claim $R > n(l - \varepsilon_0)$ for any $l > \varepsilon_0$ under the assumptions (1) and (2).

Indeed, when $l > 1 + d_0 p^n$, we have $R > n(l - \varepsilon_0)$ for any such $l$ if and only if $n < C_1(n, d_0, \varepsilon_0)$, namely

$$d_0 p^n \left(\frac{1}{2}p^n - n\right) + 2(p^n - n) + n\varepsilon_0 > \frac{1}{2} d_0 p^n.$$

If $p \geqslant 3$ or $n \geqslant 3$, then we have $\frac{1}{2}p^n - n \geqslant \frac{1}{2}$ and the above inequality holds. If $p = 2$ and $n < 3$, it is equivalent to $d_0 - \varepsilon_0 \leqslant 1$. Thus, under the condition (1), we have $R > n(l - \varepsilon_0)$ in this case.

Let us consider the case of $l \leqslant 1 + d_0 p^n$. Note that $l = 1$ is allowed only if $\varepsilon_0 = 0$, in which case the claim holds by $R = p^n > n$. For $l \geqslant 2$, by (3.8) we have $R > n(l - \varepsilon_0)$ if and only if

$$2p^n + d_0 \left(q_l - n + \frac{r_l + 1}{d_0} - \frac{1}{2}\right)^2 - d_0 \left(-n + \frac{r_l + 1}{d_0} - \frac{1}{2}\right)^2 > 2n(r_l + 2 - \varepsilon_0).$$

Note $\frac{r_l+1}{d_0} - \frac{1}{2} \in [-\frac{1}{2}, \frac{1}{2}]$. Since $q_l$ and $n$ are integers, we have

$$d_0 \left(q_l - n + \frac{r_l + 1}{d_0} - \frac{1}{2}\right)^2 \geqslant d_0 \left(\frac{r_l + 1}{d_0} - \frac{1}{2}\right)^2.$$

Thus the above inequality holds if

$$2p^n + d_0\left(\frac{r_l+1}{d_0}-\frac{1}{2}\right)^2 - d_0\left(-n+\frac{r_l+1}{d_0}-\frac{1}{2}\right)^2 > 2n(r_l+2-\varepsilon_0),$$

which is equivalent to the condition (2) and the claim follows. Now the same reasoning as in the proof of Proposition 3.11 shows $d(B,a) = d(B',a)$. □

3.6. **Dimension variation.** For the $U$-operators acting on $\mathcal{V}_k(K_\wp)$ and $\mathcal{V}(K_\wp)(\chi)$, we denote $d(U,a)$ also by

$$d(k,a) = d(\Gamma_1^\Theta(\mathfrak{n},\wp^r),k,a), \quad d(k,\chi,a) = d(\Gamma_1^\Theta(\mathfrak{n},\wp^r),k,\chi,a),$$

respectively. Note that they agree with the previously defined ones for $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$ and $S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))(\chi)$.

Now the following theorems give generalizations of [Hat2, Theorem 1.1].

**Theorem 3.13.** *Suppose that $\mathfrak{n}\wp$ has a prime factor $\pi$ of degree one. Let $n \geqslant 1$ and $k \geqslant 2$ be any integers. Put $d = [\Gamma_1(\pi) : \Gamma_1^\Theta(\mathfrak{n},\wp^r)]$ and $\varepsilon = d(k,0)$. Let $a$ be any non-negative rational number satisfying*

$$a < \min\{C(n,d,\varepsilon), k-1\}.$$

*Then, for any integer $k' \geqslant k$, we have*

$$k' \equiv k \bmod p^n \Rightarrow d(k',a) = d(k,a).$$

*Proof.* By Proposition 3.4 (1), we may assume $k' = k+p^n$. By Corollary 3.9, we can write $U^{(k+p^n)} + \wp^{p^n}W = V$ with $W \in M_{d(k+p^n-1)}(\mathcal{O}_{K_\wp})$ and

$$V = \left(\begin{array}{c|c} \wp^{k-1}B_1 & B_2 \\ \hline O & U^{(k)} \end{array}\right), \quad B_1 \in M_{dp^n}(\mathcal{O}_{K_\wp}),\ B_2 \in M_{dp^n,d(k-1)}(\mathcal{O}_{K_\wp}).$$

Corollary 3.8 and Proposition 3.4 (1) show that $U^{(k+p^n)}$ satisfies the assumptions of Proposition 3.11. Hence we obtain $d(k+p^n,a) = d(V,a)$. By [Hat2, Lemma 2.3 (2)], the matrix $\wp^{k-1}B_1$ has no eigenvalue of slope less than $k-1$. Since $a < k-1$, we also have $d(V,a) = d(k,a)$. This concludes the proof. □

**Theorem 3.14.** *Suppose that $\mathfrak{n}\wp$ has a prime factor $\pi$ of degree one. Let $n \geqslant 1$ and $k \geqslant 2$ be any integers. Let $\chi : \kappa(\wp)^\times \to \kappa(\wp)^\times$ be any character. Put $d = [\Gamma_1(\pi) : \Gamma_1^\Theta(\mathfrak{n},\wp^r)]$ and $\varepsilon_\chi = d(k,\chi,0)$. Let $a$ be any non-negative rational number satisfying*

$$a < \min\{C(n,d,\varepsilon_\chi), k-1\}.$$

*Then, for any integer $k' \geqslant k$, we have*

$$k' \equiv k \bmod p^n(q^d-1) \Rightarrow d(k',\chi,a) = d(k,\chi,a).$$

*Proof.* This follows in the same way as Theorem 3.13, using Proposition 3.10 and Proposition 3.4 (2). □

**Theorem 3.15.** *Suppose that $\mathfrak{n}\wp$ has a prime factor $\pi$ of degree one. Let $n \geqslant 1$ and $k \geqslant 2$ be any integers and $a \leqslant n$ any non-negative rational number. Put $d = [\Gamma_1(\pi) : \Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)]$ and $\varepsilon = d(k, 0)$. Suppose that the following conditions hold.*

(1) *If $p = 2$, then $n \geqslant 3$ or $d - \varepsilon \leqslant 1$.*
(2) *$2p^n > n(dn + 2 + d - 2\varepsilon)$.*

*Then, for any integer $k' \geqslant k$, we have*

$$a < k - 1,\ k' \equiv k \bmod p^n \Rightarrow d(k', a) = d(k, a).$$

*Proof.* This follows in the same way as Theorem 3.13, using Proposition 3.12 instead of Proposition 3.11. □

It will be necessary to use an increasing function no more than $C(n, d, \varepsilon)$ instead of itself. Here we give an example.

**Lemma 3.16.** *Let $n, d \geqslant 1$ and $\varepsilon \geqslant 0$ be any integers satisfying $\varepsilon \leqslant d$. Put*

$$\begin{aligned} D_2(n, d, \varepsilon) &= \frac{1}{d}\left\{\sqrt{2dp^n + (d - \varepsilon + 1)(2d - \varepsilon - 1)} - \frac{3}{2}d + \varepsilon\right\}, \\ D(n, d, \varepsilon) &= \min\{C_1(n, d, \varepsilon), D_2(n, d, \varepsilon)\}. \end{aligned}$$

*Then $D(n, d, \varepsilon)$ is an increasing function of $n$ satisfying $D(n, d, \varepsilon) \leqslant C(n, d, \varepsilon)$.*

*Proof.* Since $C_1(n, d, \varepsilon)$ is increasing for $n \geqslant 1$, it suffices to show $D_2(n, d, \varepsilon) \leqslant C_2(n, d, \varepsilon)$. Put $m = d - \varepsilon + 1$ and $x = dq_l + m \geqslant 1$. Since $r_l \in [0, d-1]$, for any $l > \varepsilon$ we have

$$\frac{2p^n + dq_l(q_l - 1) + 2q_l(r_l + 1)}{2(l - \varepsilon)} \geqslant \frac{2p^n + dq_l(q_l - 1) + 2q_l}{2x}.$$

The right-hand side equals

$$\begin{aligned} &\frac{1}{2x}\left\{2p^n + d\left(\frac{x - m}{d}\right)\left(\frac{x - m}{d} - 1\right) + 2\left(\frac{x - m}{d}\right)\right\} \\ &= \frac{x}{2d} + \frac{1}{2dx}\left(2dp^n + m(m + d - 2)\right) - \frac{m}{d} - \frac{1}{2} + \frac{1}{d}. \end{aligned}$$

By the inequality of arithmetic and geometric means, it is no less than $D_2(n, d, \varepsilon)$ and the lemma follows. □

When $\mathfrak{n} = 1$, $\wp = t$ and $r = 1$, we have $\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r) = \Gamma_1(t)$, $d = 1$ and $\varepsilon = 1$ by [Hat2, Lemma 2.4], which yields

$$C_1(n, 1, 1) = p^n \left( \frac{p^n + 3}{2p^n + 2} \right) \geqslant D_2(n, 1, 1) = \sqrt{2p^n} - \frac{1}{2}.$$

Thus we obtain

$$D(n, 1, 1) = \sqrt{2p^n} - \frac{1}{2} > 0 \tag{3.9}$$

and Theorem 3.13 gives the following improvement of [Hat2, Theorem 1.1].

**Corollary 3.17.** *Suppose $\mathfrak{n} = 1$, $\wp = t$ and $r = 1$. Let $k \geqslant 2$ be any integer and $a$ any non-negative rational number. Let $n \geqslant 1$ be any integer satisfying*

$$\frac{1}{2} \left( a + \frac{1}{2} \right)^2 < p^n.$$

*Then, for any integer $k' \geqslant k$, we have*

$$a < k - 1,\ k' \equiv k \bmod p^n \Rightarrow d(\Gamma_1(t), k', a) = d(\Gamma_1(t), k, a).$$

## 4. ℘-adic continuous family

We say $F \in \mathcal{V}_k(\mathbb{C}_\wp)$ is a Hecke eigenform if it is a non-zero eigenvector of $T_Q$ for any $Q \in A$. We denote by $\lambda_Q(F)$ the $T_Q$-eigenvalue of $F$. Since Hecke operators commute with each other, if $d(k, a) = 1$ (*resp.* $d(k, \chi, a) = 1$) then any non-zero $U$-eigenform in $\mathcal{V}_k(\mathbb{C}_\wp)$ (*resp.* $\mathcal{V}_k(\mathbb{C}_\wp)(\chi)$) of slope $a$ is a Hecke eigenform.

4.1. **Construction of the family.** Now we prove the following main theorem of this paper.

**Theorem 4.1.** *Suppose that $\mathfrak{n}\wp$ has a prime factor $\pi$ of degree one. Let $n \geqslant 1$ and $k_1 \geqslant 2$ be any integers. Put $d = [\Gamma_1(\pi) : \Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)]$ and $\varepsilon = d(k_1, 0)$. Let $a$ be any non-negative rational number satisfying*

$$a < \min\{C(n, d, \varepsilon), k_1 - 1\}.$$

*Let $n' \geqslant 1$ be any integer satisfying*

$$p^n - p^{n'} - a \geqslant 0, \quad a < C(n', d, \varepsilon).$$

*Suppose $d(k_1, a) = 1$. Let $F_1 \in \mathcal{V}_{k_1}(\mathbb{C}_\wp)$ be a Hecke eigenform of slope $a$. Then, for any integer $k_2 \geqslant k_1$ satisfying*

$$k_2 \equiv k_1 \bmod p^n,$$

*we have* $d(k_2, a) = 1$ *and thus there exists a Hecke eigenform* $F_2 \in \mathcal{V}_{k_2}(\mathbb{C}_\wp)$ *of slope* $a$ *which is unique up to a scalar multiple. Moreover, for any* $Q$ *we have*

$$v_\wp(\lambda_Q(F_1) - \lambda_Q(F_2)) > p^n - p^{n'} - a. \tag{4.1}$$

*Proof.* By Proposition 3.4 (1), we may assume $(k_1, k_2) = (k, k + p^n)$ for some integer $k \geqslant 2$. Theorem 3.13 yields $d(k + p^n, a) = 1$ and any non-zero $U$-eigenform $F_2 \in \mathcal{V}_{k_2}(\mathbb{C}_\wp)$ of slope $a$ is a Hecke eigenform. Take a finite extension $E/K_\wp$ inside $\mathbb{C}_\wp$ containing $\lambda_Q(F_i)$ and $\lambda_\wp(F_i)$ for $i = 1, 2$. We may assume $F_i \in \mathcal{V}_{k_i}(\mathcal{O}_E)$. We identify $\mathcal{V}_{k_i}(\mathcal{O}_E)$ with $\mathcal{O}_E^{d(k_i-1)}$ via the ordered basis $\mathfrak{B}_{k_i}$. Then we can write

$$F_2 = \begin{pmatrix} x \\ y \end{pmatrix}, \quad x \in \mathcal{O}_E^{dp^n},\ y \in \mathcal{O}_E^{d(k-1)},$$

where each entry of $x$ is the coefficient of $v_{s,t} \in \mathfrak{B}_{k_2}$ in $F_2$ with $t < p^n$. For any integer $N$ and $z = {}^t(z_1, \ldots, z_N) \in \mathcal{O}_E^N$, we put

$$v_\wp(z) = \min\{v_\wp(z_i) \mid i = 1, \ldots, N\}.$$

Replacing $F_i$ by its scalar multiple, we may assume $v_\wp(F_i) = 0$.

For any $H \in \mathcal{V}_{k_i}(\mathcal{O}_E)$, we denote by $\bar{H}$ its image by the natural map $\mathcal{V}_{k_i}(\mathcal{O}_E) \to \mathcal{V}_{k_i}(\mathcal{O}_{E,p^n})$. Consider the weight reduction map

$$1 \otimes \rho_{k,p^n} : \mathcal{V}_{k+p^n}(\mathcal{O}_{E,p^n}) \to \mathcal{V}_k(\mathcal{O}_{E,p^n})$$

as in §3.2, which we denote by $\rho$. Then $\rho(\bar{F}_2) = y \bmod \wp^{p^n}$.

We claim $v_\wp(y) \leqslant a$. Indeed, if $v_\wp(x) \geqslant v_\wp(y)$, then the assumption $v_\wp(F_2) = 0$ yields $v_\wp(y) = 0$. If $v_\wp(x) < v_\wp(y)$, then $v_\wp(x) = 0$ and Corollary 3.9 gives

$$\lambda_\wp(F_2)x = \wp^{k-1}B_1 x + B_2 y.$$

Since $v_\wp(\lambda_\wp(F_2)) = a < k - 1$, this forces $v_\wp(y) \leqslant a$ and the claim follows.

Take $G_1 \in \mathcal{V}_k(\mathcal{O}_E)$ satisfying $\bar{G}_1 = \rho(\bar{F}_2)$. By Lemma 3.2, we have

$$T_Q(G_1) \equiv \lambda_Q(F_2)G_1, \quad U(G_1) \equiv \lambda_\wp(F_2)G_1 \bmod \wp^{p^n}\mathcal{V}_k(\mathcal{O}_E). \tag{4.2}$$

Since we have $a < C(n, d, \varepsilon) < p^n$, the above claim yields $v_\wp(G_1) \leqslant a$. If $G_1 \in \mathcal{O}_E F_1$, then $G_1$ is a Hecke eigenform with the same eigenvalues as those of $F_1$. Thus we have

$$\lambda_Q(F_1)\bar{G}_1 = T_Q(\bar{G}_1) = \lambda_Q(F_2)\bar{G}_1,$$

which gives

$$v_\wp(\lambda_Q(F_1) - \lambda_Q(F_2)) \geqslant p^n - a. \tag{4.3}$$

Suppose $G_1 \notin \mathcal{O}_E F_1$, and take $H_1 \in \mathcal{V}_k(\mathcal{O}_E)$ such that $F_1$ and $H_1$ form a basis of a direct summand of $\mathcal{V}_k(\mathcal{O}_E)$ containing $G_1$. Write

$$G_1 = \alpha F_1 + \beta H_1, \quad \alpha, \beta \in \mathcal{O}_E. \tag{4.4}$$

Then $\beta \neq 0$. By (4.2), for any $R \in \{\wp, Q\}$ we have

$$\lambda_R(F_2) G_1 \equiv T_R(G_1) = \alpha \lambda_R(F_1) F_1 + \beta T_R(H_1) \bmod \wp^{p^n} \mathcal{V}_k(\mathcal{O}_E).$$

Combined with (4.4), this implies

$$\beta T_R(H_1) \equiv \alpha(\lambda_R(F_2) - \lambda_R(F_1)) F_1 + \beta \lambda_R(F_2) H_1 \bmod \wp^{p^n} \mathcal{V}_k(\mathcal{O}_E) \tag{4.5}$$

and thus we obtain

$$\alpha(\lambda_R(F_1) - \lambda_R(F_2)) \equiv 0 \bmod (\beta, \wp^{p^n}). \tag{4.6}$$

Put $b = v_\wp(\beta)$. Suppose $b > p^n - p^{n'}$. Since $v_\wp(F_1) = 0$ and

$$v_\wp(G_1) \leqslant a \leqslant p^n - p^{n'} < b,$$

(4.4) gives $v_\wp(\alpha) \leqslant a$ and (4.6) yields

$$v_\wp(\lambda_Q(F_1) - \lambda_Q(F_2)) > p^n - p^{n'} - a. \tag{4.7}$$

Suppose $b \leqslant p^n - p^{n'}$. In this case we have $\beta^{-1}\wp^{p^n} \in \mathcal{O}_E$ and by (4.6) we can write

$$\alpha(\lambda_\wp(F_2) - \lambda_\wp(F_1)) = \beta\nu$$

with some $\nu \in \mathcal{O}_E$. Then (4.5) shows

$$U(H_1) \equiv \nu F_1 + \lambda_\wp(F_2) H_1 \bmod \beta^{-1}\wp^{p^n} \mathcal{V}_k(\mathcal{O}_E). \tag{4.8}$$

Take an ordered basis $(F_1, H_1, \tilde{v}_3, \ldots, \tilde{v}_{d(k-1)})$ of the $\mathcal{O}_E$-module $\mathcal{V}_k(\mathcal{O}_E)$, and we denote by $\tilde{U}^{(k)}$ the representing matrix of $U$ with respect to it. By (4.8), we can write

$$\tilde{U}^{(k)} = \left(\begin{array}{cc|ccc} \lambda_\wp(F_1) & \nu + \beta^{-1}\wp^{p^n} c_1 & * & \cdots & * \\ 0 & \lambda_\wp(F_2) + \beta^{-1}\wp^{p^n} c_2 & * & \cdots & * \\ 0 & \beta^{-1}\wp^{p^n} c_3 & * & \cdots & * \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \beta^{-1}\wp^{p^n} c_{d(k-1)} & * & \cdots & * \end{array}\right), \quad c_1, \ldots, c_{d(k-1)} \in \mathcal{O}_E.$$

Note that the elementary divisors of $\tilde{U}^{(k)}$ and $U^{(k)}$ agree with each other. Let $V$ be the element of $M_{d(k-1)}(\mathcal{O}_E)$ with the same columns as those of $\tilde{U}^{(k)}$ except the second column which we require to be

$$\begin{pmatrix} \nu \\ \lambda_\wp(F_2) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then we have $d(V,a) \geqslant 2$. On the other hand, since $p^n - b \geqslant p^{n'}$, the assumption $a < C(n',d,\varepsilon)$ and Proposition 3.11 yield $d(V,a) = d(k,a) = 1$, which is the contradiction. Thus the case $b \leqslant p^n - p^{n'}$ never occurs. Now the theorem follows from (4.3) and (4.7). $\square$

**Remark 4.2.** Putting $\varepsilon = d(k_1,\chi,0)$ and assuming $d(k_1,\chi,a) = 1$, the same proof using Proposition 3.10 and Theorem 3.14 shows that we can construct, from a Hecke eigenform $F_1 \in \mathcal{V}_{k_1}(\mathbb{C}_\wp)(\chi)$ of slope $a$, a Hecke eigenform $F_2 \in \mathcal{V}_{k_2}(\mathbb{C}_\wp)(\chi)$ of slope $a$ satisfying (4.1) for any integer $k_2 \geqslant k_1$ with

$$k_2 \equiv k_1 \bmod p^n(q^d - 1).$$

*Proof of Theorem 1.1.* Suppose that $n$, $k$ and $a$ satisfy the assumptions of Theorem 1.1. Take any $k' \geqslant k$ satisfying

$$m = v_p(k' - k) \geqslant \log_p(p^n + a).$$

Since $n \leqslant m$ and $D(n,d,\varepsilon)$ is an increasing function of $n$ satisfying $D(n,d,\varepsilon) \leqslant C(n,d,\varepsilon)$, we have

$$a < \min\{C(m,d,\varepsilon), k-1\}, \quad p^m - p^n - a \geqslant 0, \quad a < C(n,d,\varepsilon).$$

Note that, if $d(k,a) = 1$, then any $U$-eigenform of slope $a$ in $\mathcal{V}_k(\mathbb{C}_\wp)$ is identified with a scalar multiple of that in $\mathcal{V}_k(\bar{K}) \subseteq S_k(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$ via the fixed embedding $\iota_\wp$. Thus Theorem 4.1 produces a Hecke eigenform $F_{k'} \in S_{k'}(\Gamma_1^\Theta(\mathfrak{n},\wp^r))$ such that for any $Q$ we have

$$v_\wp(\iota_\wp(\lambda_Q(F_{k'}) - \lambda_Q(F_k))) > p^m - p^n - a.$$

This concludes the proof of Theorem 1.1. $\square$

4.2. **Examples.** We assume $\mathfrak{n} = 1$, $\wp = t$, $r = 1$ and $\Gamma_1^\Theta(\mathfrak{n},\wp^r) = \Gamma_1(t)$. In this case we have $d = 1$ and $d(k,0) = 1$ for any $k \geqslant 2$. In the following, we give examples of congruences between Hecke eigenvalues obtained by Theorem 1.1 for this case, using results of [BV2, LM, Pet]. Note that the Hecke operator at $Q$ considered in [BV2, Pet] is $QT_Q$ with our normalization.

4.2.1. *Slope zero forms.* By $d(k,0) = 1$, any $U$-eigenform of slope zero in $S_k(\Gamma_1(t))$ is a member of a $t$-adic continuous family obtained by Theorem 1.1. Some of such eigenforms can be given by the theory of $A$-expansions [Pet].

For any integer $k \geqslant 3$ satisfying $k \equiv 2 \bmod q-1$, Petrov constructed an element $f_{k,1} \in S_k(SL_2(A))$ with $A$-expansion [Pet, Theorem 1.3]. We know that $f_{k,1}$ is a Hecke eigenform whose Hecke eigenvalue at $Q$ is one for any $Q$; this follows from a formula for the Hecke action [Pet, p. 2252] and $c_a = a^{k-n}$.

For such $k$, let $f_{k,1}^{(t)} \in S_k(\Gamma_1(t))$ be the $t$-stabilization of $f_{k,1}$ of finite slope, namely

$$f_{k,1}^{(t)}(z) = f_{k,1}(z) - t^{k-1} f_{k,1}(tz).$$

It is non-zero by [Pet, Theorem 2.2]. Moreover, we can show that $f_{k,1}^{(t)}$ is a Hecke eigenform which also satisfies $\lambda_Q(f_{k,1}^{(t)}) = 1$ for any $Q$.

**Proposition 4.3.** *Let $k \geqslant 2$ be any integer and $F_k$ any non-zero element of $S_k(\Gamma_1(t))$ of slope zero. Then we have $\lambda_Q(F_k) = 1$ for any $Q$.*

*Proof.* Let $r \in \{0, 1, \dots, q-2\}$ be an integer satisfying $k \equiv r \bmod q-1$. For $a = 0$, we see from (3.9) that the assumptions of Theorem 1.1 are satisfied by $n = 1$. Then, for any integer $s \geqslant 1$, we obtain a Hecke eigenform of slope zero

$$F_{k'} \in S_{k'}(\Gamma_1(t)), \quad k' = k + (q+1-r)q^s$$

such that, with the fixed embedding $\iota_t : \bar{K} \to \mathbb{C}_t$, we have

$$\iota_t(\lambda_Q(F_{k'})) \equiv \iota_t(\lambda_Q(F_k)) \bmod t^{q^s - p} \quad \text{for any } Q.$$

Since $k' \geqslant 3$, $k' \equiv 2 \bmod q-1$ and $d(k', 0) = 1$, we see that $F_{k'}$ is a scalar multiple of $f_{k',1}^{(t)}$ and thus $\lambda_Q(F_{k'}) = 1$. Since $s$ is arbitrary, this implies $\lambda_Q(F_k) = 1$. □

**Corollary 4.4.** *Let $k \geqslant 2$ and $r \geqslant 1$ be any integers. Then there exists a unique character $\chi : \kappa(\wp)^\times \to \kappa(\wp)^\times$ satisfying $d(\Gamma_0^p(t^r), k, \chi, 0) \neq 0$. For such $\chi$, we have $d(\Gamma_0^p(t^r), k, \chi, 0) = 1$ and any Hecke eigenform $F$ of slope zero in $S_k(\Gamma_0^p(t^r))(\chi)$ satisfies $\lambda_Q(F) = 1$ for any $Q$.*

*Proof.* Since $\Gamma_0^p(t) = \Gamma_1(t)$, Proposition 3.5 implies $d(\Gamma_0^p(t^r), k, 0) = 1$. Since we have

$$d(\Gamma_0^p(t^r), k, 0) = \sum_{\chi} d(\Gamma_0^p(t^r), k, \chi, 0),$$

the uniqueness of $\chi$ and the assertion on the dimension follow. Let $F_k$ be any Hecke eigenform of slope zero in $S_k(\Gamma_1(t))$. Since the natural inclusion $S_k(\Gamma_1(t)) \to S_k(\Gamma_0^p(t^r))$ is compatible with Hecke operators, $F$ is a scalar multiple of the image of $F_k$. Hence the last assertion follows from Proposition 4.3. □

**Remark 4.5.** Note that, since the only $p$-power root of unity in $\mathbb{C}_\wp$ is one, there exists no non-trivial finite order character $1 + \wp\mathcal{O}_{K_\wp} \to \mathbb{C}_\wp^\times$. Thus it seems to the author that, if we try to generalize Hida theory including [Hid2, §7.3, Theorem 3] to Drinfeld cuspforms of level $\Gamma_1(t^r)$, then it would be natural to restrict ourselves to those of level $\Gamma_0^p(t^r)$. However, Corollary 4.4 shows that such a generalization is trivial.

4.2.2. *Slope one forms.* Let us consider the case $p = q = 3$ and $a = 1$. Since $D(1,1,1) = \sqrt{6} - \frac{1}{2} = 1.949\ldots$, the assumptions of Theorem 1.1 are satisfied by $k \geqslant 3$ and $n = 1$. Then a computation using [BV2, (17)] shows $d(10,1) = 1$. Let $G_{10}$ and $G_{19}$ be any non-zero Drinfeld cuspforms of level $\Gamma_1(t)$ and slope one in weights 10 and 19, respectively. Then Theorem 1.1 gives

$$v_t(\iota_t(\lambda_Q(G_{10}) - \lambda_Q(G_{19}))) > 5 \tag{4.9}$$

for any $Q$.

For $Q = t$, using [BV2, (17)] we can show that $\lambda_t(G_{10}) = -t - t^3$, and $\lambda_t(G_{19})$ is a root of the polynomial

$$\begin{aligned} X^4 &+ (t + t^3)X^3 + (-t^8 + t^{10} + t^{12} + t^{14} + t^{16})X^2 \\ &+ (-t^9 - t^{11} + t^{13} + t^{15} + t^{17} + t^{19})X + (-t^{18} - t^{20} + t^{24} + t^{26} + t^{28}) \end{aligned}$$

(see also [Val]). Put $\iota_t(\lambda_t(G_{19})) = ty$ with $v_t(y) = 0$. Then we obtain $y^3(y + 1 + t^2) \equiv 0 \bmod t^6$ and $\iota_t(\lambda_t(G_{10})) \equiv \iota_t(\lambda_t(G_{19})) \bmod t^7$, which satisfies (4.9). In fact, plugging in $X = -t - t^3 + Z$ to the polynomial above yields $v_t(\iota_t(\lambda_t(G_{10}) - \lambda_t(G_{19}))) = 9$.

We identify $S_k(\Gamma_1(t))$ with $\mathbb{C}_\infty^{k-1}$ via the ordered basis

$$\{\mathbf{c}_j(\gamma_0) = \mathbf{c}_j(\bar{e}) \mid 0 \leqslant j \leqslant k-2\}$$

defined in [LM, BV2]. Then $G_{10}$ is identified with the vector

$${}^t\big(0, 1 + t^2, 0, -(1 + t^2), 0, -t^2, 0, 1, 0\big).$$

Thus $\lambda_{1+t}(G_{10})$ agrees with the evaluation $T_{1+t}(G_{10})(\gamma_0)(X^7Y)$ after identifying $G_{10}$ with a harmonic cocycle. By [LM, (7.1)], we have $\lambda_{1+t}(G_{10}) = 1 - t - t^3$. On the other hand, by computing the characteristic polynomial of $T_{1+t}$ acting on $S_{19}(\Gamma_1(t))$ using [LM, (7.1)] and plugging in $X = 1 - t - t^3 + Z$ into it, (4.9) implies $v_t(\iota_t(\lambda_{1+t}(G_{10}) - \lambda_{1+t}(G_{19}))) = 9$.

Note that, since these eigenvalues are not powers of $t$ or $1 + t$, the Hecke eigenforms $G_{10}$ and $G_{19}$ are not the $t$-stabilizations of Hecke eigenforms with $A$-expansion.

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