ON SOME CONSTANCY OF HECKE EIGENSYSTEMS FOR DRINFELD CUSPFORMS OF LEVEL $\Gamma_1(\mathfrak{n}\wp^r)$

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Abstract. Let p be a rational prime, let $q > 1$ be a p-power integer, let $\mathbb{F}_q$ be the field of q elements and let $A = \mathbb{F}_q[t]$ be the polynomial ring over $\mathbb{F}_q$. Let $\mathfrak{n} \in A$ be a nonzero element and let $\wp \in A$ be a monic irreducible polynomial of positive degree. Let $k \geqslant 2$ and $r \geqslant 1$ be integers. Let $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ be the space of Drinfeld cuspforms of level $\Gamma_1(\mathfrak{n}\wp^r)$ and weight k. In this paper, we show that a Hecke eigensystem of finite $\wp$-slope appears in $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ if and only if it appears in $S_k(\Gamma_1(\mathfrak{n}\wp))$.

1. Introduction

Let p be a rational prime, let $q > 1$ be a p-power integer, let $\mathbb{F}_q$ be the field of q elements, let t be an indeterminate, let $A = \mathbb{F}_q[t]$, $K = \mathbb{F}_q(t)$, $K_\infty = \mathbb{F}_q((1/t))$ and $\mathcal{O}_{K_\infty} = \mathbb{F}_q[[1/t]]$. Let $\mathbb{C}_\infty$ be the $(1/t)$-adic completion of an algebraic closure of K_∞. Let $\Omega = \mathbb{C}_\infty \backslash K_\infty$ be the Drinfeld upper half plane, which is equipped with a natural structure of a rigid analytic variety over $\mathbb{C}_\infty$.

Let Π be the set of monic irreducible polynomials of positive degree in A. Let $\mathfrak{n} \in A$ be a nonzero element and let $\wp \in \Pi$ satisfy $\wp \nmid \mathfrak{n}$. Let $k \geqslant 2$ and $r \geqslant 1$ be integers. For any nonzero $\mathfrak{m} \in A$, let

$$\Gamma_1(\mathfrak{m}) = \left\{ \gamma \in SL_2(A) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{m} \right\}.$$

A Drinfeld modular form is a rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ satisfying a transformation condition and regularity at cusps similar to those for elliptic modular forms. Though the definitions of elliptic and Drinfeld modular forms are parallel, arithmetic of Drinfeld modular forms is sometimes very different from the elliptic case.

It has been realized that Drinfeld modular forms of level $\Gamma_1(\mathfrak{m})$ behave rather strangely. For example, for a positive integer M, the space of elliptic modular forms of level $\Gamma_1(M)$ is written as the direct sum of the spaces of level $\Gamma_0(M)$ with nebentypus. On the other hand, such a decomposition is not possible in general for the space of Drinfeld modular forms of level $\Gamma_1(\mathfrak{m})$, since representations of $(A/\mathfrak{m}A)^\times$ over $\mathbb{C}_\infty$ are not necessarily semisimple.

In [Hat2], we proved an unexpected triviality of the Hecke action on the t-ordinary part of the space of Drinfeld cuspforms of level $\Gamma_1(t^r)$. The aim

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of the present paper is to generalize it to the finite slope part for more general levels: it turns out that not the triviality but a constancy in some sense holds for Hecke eigensystems appearing in the finite $\wp$-slope part of the space of Drinfeld cuspforms of level $\Gamma_1(\mathfrak{n}\wp^r)$.

To state the main theorem, we fix some notation. Let $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ be the $\mathbb{C}_\infty$-vector space of Drinfeld cuspforms of level $\Gamma_1(\mathfrak{n}\wp^r)$ and weight k, on which the Hecke operator T_Q at Q acts for any $Q \in \Pi$. We write T_Q also as U_Q when $Q \mid \mathfrak{n}\wp^r$.

Let $\bar{K}$ be the algebraic closure of K in $\mathbb{C}_\infty$. Let $K_\wp$ be the completion of K at $\wp$ and let $\bar{K}_\wp$ be an algebraic closure of $K_\wp$. Let $v_\wp$ be the $\wp$-adic additive valuation on $\bar{K}_\wp$ satisfying $v_\wp(\wp) = 1$. We choose once and for all an embedding $\iota_\wp : \bar{K} \to \bar{K}_\wp$ of K-algebras. For any $a \in \mathbb{Q} \cup \{+\infty\}$, we say that an element $\alpha \in \bar{K}$ is of slope a if $v_\wp(\iota_\wp(\alpha)) = a$. For any $\alpha \in \bar{K}$, we denote by $m(\Gamma_1(\mathfrak{n}\wp^r), k, \wp, \alpha)$ the multiplicity of α as an eigenvalue of $U_\wp$ acting on $S_k(\Gamma_1(\mathfrak{n}\wp^r))$. For any $a \in \mathbb{Q} \cup \{+\infty\}$, we denote by $d(\Gamma_1(\mathfrak{n}\wp^r), k, a)$ the multiplicity of $U_\wp$-eigenvalues of slope a appearing in $S_k(\Gamma_1(\mathfrak{n}\wp^r))$. For any $\lambda = (\lambda_Q)_{Q\in\Pi} \in \bar{K}^\Pi$, we say that λ is a Hecke eigensystem appearing in $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ if there exists a nonzero element $f \in S_k(\Gamma_1(\mathfrak{n}\wp^r))$ satisfying $f|T_Q = \lambda_Q f$ for all $Q \in \Pi$.

Now our main theorem is as follows.

Theorem 1.1. *(1) (Corollary 5.2) For any $\alpha \in \bar{K}^\times$ and $a \in \mathbb{Q}$, we have*

$$m(\Gamma_1(\mathfrak{n}\wp^r), k, \wp, \alpha) = q^{(r-1)\deg(\wp)} m(\Gamma_1(\mathfrak{n}\wp), k, \wp, \alpha),$$

$$d(\Gamma_1(\mathfrak{n}\wp^r), k, a) = q^{(r-1)\deg(\wp)} d(\Gamma_1(\mathfrak{n}\wp), k, a).$$

In particular, the sets of nonzero $U_\wp$-eigenvalues (resp. finite slopes) appearing in $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ and $S_k(\Gamma_1(\mathfrak{n}\wp))$ are the same.

(2) (Theorem 5.3) Let $\lambda = (\lambda_Q)_{Q\in\Pi} \in \bar{K}^\Pi$ satisfy $\lambda_\wp \neq 0$. Then λ is a Hecke eigensystem appearing in $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ if and only if it is a Hecke eigensystem appearing in $S_k(\Gamma_1(\mathfrak{n}\wp))$.

This shows a stark contrast to the case of elliptic modular forms where Hecke eigensystems of finite p-slope vary in p-adic analytic families when the weight and the p-adic valuation of the level vary. Though in some cases we have $\wp$-adic continuous families of Drinfeld eigenforms varying the weight [Hat1], Theorem 1.1 indicates that varying the level only yields constant families of Hecke eigensystems appearing in the space of Drinfeld modular forms.

Let us give an idea of the proof of Theorem 1.1. Let Θ_r be the multiplicative group $1+\wp(A/\wp^r A)$. It is a commutative p-group acting on $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ via diamond operators. Let $\mathbb{C}_\infty[\Theta_r]$ be the group ring of Θ_r over $\mathbb{C}_\infty$. Then the key point of the proof is the freeness of the $\mathbb{C}_\infty[\Theta_r]$-module $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ (Proposition 4.2). Since $\mathbb{C}_\infty$ is of characteristic p, the ring $\mathbb{C}_\infty[\Theta_r]$ is local and any direct summand of the free $\mathbb{C}_\infty[\Theta_r]$-module $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ is also free. This applies in particular to generalized eigenspaces of each Hecke operator.

Since the Θ_r-fixed part of $\mathbb{C}_\infty[\Theta_r]$ is a $\mathbb{C}_\infty$-vector space of dimension one, this relates the multiplicity of a $U_\wp$-eigenvalue or the appearance of a Hecke eigensystem in $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ with that in $S_k(\Gamma_1(\mathfrak{n}\wp^r))^{\Theta_r} = S_k(\Gamma_1^p(\mathfrak{n},\wp^r))$, where

$$\Gamma_1^p(\mathfrak{n},\wp^r) = \left\{\gamma \in \Gamma_1(\mathfrak{n}) \;\middle|\; \gamma \bmod \wp^r \in \begin{pmatrix}\Theta_r & * \\ 0 & \Theta_r\end{pmatrix}\right\}.$$

Then the theorem follows from the fact that the inclusion $S_k(\Gamma_1(\mathfrak{n}\wp)) \to S_k(\Gamma_1^p(\mathfrak{n},\wp^r))$ induces a Hecke equivariant isomorphism on finite slope parts (Lemma 3.2).

The organization of this paper is as follows. In §2, we recall descriptions of Drinfeld cuspforms using harmonic cocycles and the Steinberg module [Tei]. In §3, we recall the definition of Hecke operators and diamond operators, and give the aforementioned isomorphism between the finite slope part of level $\Gamma_1(\mathfrak{n}\wp)$ and that of level $\Gamma_1^p(\mathfrak{n},\wp^r)$. In §4, we prove the key freeness of the $\mathbb{C}_\infty[\Theta_r]$-module $S_k(\Gamma_1(\mathfrak{n}\wp^r))$. In §5, we show Theorem 1.1.

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2. Combinatorial descriptions of Drinfeld cuspforms

In this section, we recall the descriptions of the space of Drinfeld cuspforms using harmonic cocycles and the Steinberg module due to Teitelbaum [Tei, p. 506], following the normalization of [Böc, §5.3] (see also [Hat1, §2]).

For any A-algebra B, we consider B^2 as the set of row vectors which admits a left action of $GL_2(B)$ by $\gamma \circ (x,y) = (x,y)\gamma^{-1}$. Let $\mathcal{T}$ be the Bruhat–Tits tree for $SL_2(K_\infty)$. We denote by $\mathcal{T}_0$ the set of vertices of $\mathcal{T}$ and by $\mathcal{T}_1^o$ the set of oriented edges of $\mathcal{T}$. By definition, the set $\mathcal{T}_0$ consists of $K_\infty^\times$-equivalence classes of $\mathcal{O}_{K_\infty}$-lattices in K_∞^2. For any $e \in \mathcal{T}_1^o$, the origin, the terminus and the opposite edge of e are denoted by $o(e)$, $t(e)$ and $-e$, respectively. Then the group $\{\pm 1\}$ acts on $\mathcal{T}_1^o$ by $(-1)e = -e$.

Let Γ be a congruence subgroup of $SL_2(A)$ which is p'-torsion free (that is, every element of Γ of finite order has p-power order). Consider its action $\circ$ on $\mathcal{T}$ via the natural inclusion $\Gamma \to GL_2(K_\infty)$. We say that a vertex or an oriented edge of $\mathcal{T}$ is Γ-stable if its stabilizer in Γ is trivial. We denote by $\mathcal{T}_0^{\Gamma\text{-st}}$ and $\mathcal{T}_1^{o,\Gamma\text{-st}}$ the sets of Γ-stable vertices and oriented edges, respectively. They are stable under the action of Γ. More generally, let $\tilde{\Gamma}$ be any subgroup of $SL_2(A)$ satisfying $\tilde{\Gamma} \rhd \Gamma$. Then for any $s \in \mathcal{T}_0 \sqcup \mathcal{T}_1^o$ and $\tilde{\gamma} \in \tilde{\Gamma}$, the element s is Γ-stable if and only if $\tilde{\gamma} \circ s$ is Γ-stable. Thus the sets $\mathcal{T}_0^{\Gamma\text{-st}}$ and $\mathcal{T}_1^{o,\Gamma\text{-st}}$ are stable under the action of $\tilde{\Gamma}$.

For any set S, we denote by $\mathbb{Z}[S]$ the free $\mathbb{Z}$-module with basis $\{[s] \mid s \in S\}$. Let

$$\mathbb{Z}[\bar{\mathcal{T}}_1^{o,\Gamma\text{-st}}] := \mathbb{Z}[\mathcal{T}_1^{o,\Gamma\text{-st}}]/\langle [e] + [-e] \mid e \in \mathcal{T}_1^{o,\Gamma\text{-st}}\rangle.$$

For any $v \in \mathcal{T}_0$, let $\langle v \rangle_\Gamma = v$ if v is Γ-stable and $\langle v \rangle_\Gamma = 0$ otherwise. Let

$$\partial_\Gamma : \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\Gamma\text{-st}}] \to \mathbb{Z}[\mathcal{T}_0^{\Gamma\text{-st}}], \quad e \mapsto \langle t(e)\rangle_\Gamma - \langle o(e)\rangle_\Gamma.$$

For $\tilde{\Gamma} \supseteq \Gamma$ as above, the map ∂_Γ is $\tilde{\Gamma}$-equivariant.

We refer to the kernel of the augmentation map

$$\mathrm{St} := \mathrm{Ker}(\mathbb{Z}[\mathbb{P}^1(K)] \to \mathbb{Z})$$

as the Steinberg module. We consider St as a left $\mathbb{Z}[\Gamma]$-module via

$$\gamma \circ (x : y) = (x : y)\gamma^{-1}, \quad (x : y) \in \mathbb{P}^1(K).$$

Then the left $\mathbb{Z}[\Gamma]$-module St is finitely generated and projective. Moreover, there exists a split exact sequence of left $\mathbb{Z}[\Gamma]$-modules

$$0 \longrightarrow \mathrm{St} \longrightarrow \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\Gamma\text{-st}}] \xrightarrow{\partial_\Gamma} \mathbb{Z}[\mathcal{T}_0^{\Gamma\text{-st}}] \longrightarrow 0 \tag{2.1}$$

([Böc, §5,3], [Ser, Ch. II, §2.9]).

Let $k \geqslant 2$ be an integer and let $H_{k-2}(\mathbb{C}_\infty)$ be the $\mathbb{C}_\infty$-subspace of the polynomial ring $\mathbb{C}_\infty[X,Y]$ consisting of homogeneous polynomials of degree $k-2$. It is equipped with a left action of $GL_2(\mathbb{C}_\infty)$ defined by $\gamma \circ (X,Y) = (X,Y)\gamma$. Let

$$V_k(\mathbb{C}_\infty) = \mathrm{Hom}_{\mathbb{C}_\infty}(H_{k-2}(\mathbb{C}_\infty), \mathbb{C}_\infty).$$

It is also equipped with a natural left action $\circ$ of $GL_2(\mathbb{C}_\infty)$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}_\infty)$, $\omega \in V_k(\mathbb{C}_\infty)$ and $P(X,Y) \in H_{k-2}(\mathbb{C}_\infty)$, the action $\circ$ is defined by

$$(\gamma \circ \omega)(P(X,Y)) = \omega(\gamma^{-1} \circ P(X,Y)) = \det(\gamma)^{2-k}\omega(P(dX - cY, -bX + aY)).$$

A harmonic cocycle of level Γ and weight k is a map $c : \mathcal{T}_1^o \to V_k(\mathbb{C}_\infty)$ satisfying the following conditions:

1. $c(\gamma \circ e) = \gamma \circ c(e)$ for any $\gamma \in \Gamma$ and $e \in \mathcal{T}_1^o$,
2. $c(-e) = -c(e)$ for any $e \in \mathcal{T}_1^o$,
3. $\sum_{e \in \mathcal{T}_1^o,\ v = t(e)} c(e) = 0$ for any $v \in \mathcal{T}_0$.

The $\mathbb{C}_\infty$-vector space of harmonic cocycles of level Γ and weight k is denoted by $C_k^{\mathrm{har}}(\Gamma)$. Similarly, the $\mathbb{C}_\infty$-vector space of maps $c : \mathcal{T}_1^o \to V_k(\mathbb{C}_\infty)$ satisfying the conditions (1) and (2) is denoted by $C_k^{\pm}(\Gamma)$.

Let $\mathcal{V}_k(\Gamma) = \mathrm{St} \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty)$ and

$$\mathcal{L}_{1,k}(\Gamma) = \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\Gamma\text{-st}}] \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty), \quad \mathcal{L}_{0,k}(\Gamma) = \mathbb{Z}[\mathcal{T}_0^{\Gamma\text{-st}}] \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty),$$

where the modules on the left of $\otimes$ are considered as right $\mathbb{Z}[\Gamma]$-modules via the action $s|_\gamma = \gamma^{-1} \circ s$. Then (2.1) yields an exact sequence of $\mathbb{C}_\infty$-vector spaces

$$0 \longrightarrow \mathcal{V}_k(\Gamma) \longrightarrow \mathcal{L}_{1,k}(\Gamma) \longrightarrow \mathcal{L}_{0,k}(\Gamma) \longrightarrow 0.$$

Let $\Lambda_1 \subseteq \mathcal{T}_1^{o,\Gamma\text{-st}}$ be a complete set of representatives of $\Gamma\backslash\mathcal{T}_1^{o,\Gamma\text{-st}}/\{\pm 1\}$. Then the $\mathbb{C}_\infty$-linear map

$$\Phi_\Gamma : C_k^{\pm}(\Gamma) \to \mathcal{L}_{1,k}(\Gamma), \quad c \mapsto \sum_{e \in \Lambda_1} [e] \otimes c(e)$$

is an isomorphism independent of the choice of Λ_1 [Hat1, p. 9]. Moreover, from [Tei, p. 506], we see that it induces a $\mathbb{C}_\infty$-linear isomorphism

$$\Phi_\Gamma : C_k^{\mathrm{har}}(\Gamma) \to \mathcal{V}_k(\Gamma).$$

We denote by $S_k(\Gamma)$ the $\mathbb{C}_\infty$-vector space of Drinfeld cuspforms of level Γ and weight k. For any rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K_\infty)$, let $f|_k\gamma$ be the rigid analytic function on Ω defined by

$$(f|_k\gamma)(z) = \det(\gamma)^{k-1}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Then we can associate with any $e \in \mathcal{T}_1^o$ an element $\mathrm{Res}_\Gamma(f)(e) \in V_k(\mathbb{C}_\infty)$ satisfying

$$\mathrm{Res}_\Gamma(f|_k\gamma)(e) = \gamma^{-1} \circ \mathrm{Res}_\Gamma(f)(\gamma \circ e) \tag{2.2}$$

for any $\gamma \in GL_2(K_\infty)$, which gives a $\mathbb{C}_\infty$-linear isomorphism

$$\mathrm{Res}_\Gamma : S_k(\Gamma) \to C_k^{\mathrm{har}}(\Gamma), \quad f \mapsto (e \mapsto \mathrm{Res}_\Gamma(f)(e))$$

([Tei, Theorem 16], [Böc, Theorem 5.10]).

Let $\tilde{\Gamma}$ be a subgroup of $SL_2(A)$ satisfying $\tilde{\Gamma} \rhd \Gamma$. Let $G = \tilde{\Gamma}/\Gamma$. For any right $\mathbb{Z}[\tilde{\Gamma}]$-module M and left $\mathbb{Z}[\tilde{\Gamma}]$-module N, the group $\tilde{\Gamma}$ acts from the right on $M \otimes_{\mathbb{Z}[\Gamma]} N$ by

$$(m \otimes n)\tilde{\gamma} := m\tilde{\gamma} \otimes \tilde{\gamma}^{-1} n$$

for any $m \in M$, $n \in N$ and $\tilde{\gamma} \in \tilde{\Gamma}$. This induces a right G-action on $M \otimes_{\mathbb{Z}[\Gamma]} N$ which is functorial on M and N. In particular, we have natural G-actions on $\mathcal{V}_k(\Gamma)$, $\mathcal{L}_{1,k}(\Gamma)$ and $\mathcal{L}_{0,k}(\Gamma)$. On the other hand, the group $\tilde{\Gamma}$ acts on $C_k^{\pm}(\Gamma)$, $C_k^{\mathrm{har}}(\Gamma)$ and $S_k(\Gamma)$ from the right by

$$c \mapsto c|_k\tilde{\gamma} := (e \mapsto \tilde{\gamma}^{-1} \circ c(\tilde{\gamma} \circ e)), \quad f \mapsto f|_k\tilde{\gamma},$$

which induce right G-actions on them.

Lemma 2.1. *The isomorphisms*

$$\Phi_\Gamma : C_k^{\pm}(\Gamma) \to \mathcal{L}_{1,k}(\Gamma), \quad \Phi_\Gamma \circ \mathrm{Res}_\Gamma : S_k(\Gamma) \to \mathcal{V}_k(\Gamma)$$

are G-equivariant.

Proof. Take any $\tilde{\gamma} \in \tilde{\Gamma}$. Let Λ_1 be a complete set of representatives of $\Gamma \backslash \mathcal{T}_1^{o,\Gamma\text{-st}}/\{\pm 1\}$. Since $\tilde{\gamma}^{-1}\Gamma\tilde{\gamma} = \Gamma$, the set $\tilde{\gamma} \circ \Lambda_1 = \{\tilde{\gamma} \circ e \mid e \in \Lambda_1\}$ is also a complete set of representatives of the same coset space. Since the map Φ_Γ is independent of the choice of Λ_1, we have

$$\begin{aligned}\Phi_\Gamma(c|_k\tilde{\gamma}) &= \sum_{e \in \Lambda_1} [e] \otimes \tilde{\gamma}^{-1} \circ c(\tilde{\gamma} \circ e) = \sum_{e \in \tilde{\gamma} \circ \Lambda_1} [\tilde{\gamma}^{-1} \circ e] \otimes \tilde{\gamma}^{-1} \circ c(e) \\ &= \sum_{e \in \tilde{\gamma} \circ \Lambda_1} [e]|_{\tilde{\gamma}} \otimes \tilde{\gamma}^{-1} \circ c(e) = \Phi_\Gamma(c)\tilde{\gamma}.\end{aligned}$$

Hence the first assertion follows.

By [BGP, Remark 1.4], the injection $\mathrm{St} \to \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\Gamma\text{-st}}]$ of (2.1) is $\tilde{\Gamma}$-equivariant. Thus the $\mathbb{C}_\infty$-subspace $\mathcal{V}_k(\Gamma) \subseteq \mathcal{L}_{1,k}(\Gamma)$ is stable under the action of G. Since $C_k^{\mathrm{har}}(\Gamma) \subseteq C_k^{\pm}(\Gamma)$ is also stable under the G-action, the latter assertion follows from the former and (2.2). □

3. Hecke operators and diamond operators

Let $\mathfrak{n} \in A$ be a nonzero element and let $\wp \in \Pi$ satisfy $\wp \nmid \mathfrak{n}$. For any integer $n \geqslant 1$, let $A_n = A/\wp^n A$. Let $r \geqslant 1$ be an integer and let

$$\Theta_r = 1 + \wp A_r,$$

which is a subgroup of the multiplicative group $A_r^\times$.

Let Θ be any subgroup of Θ_r. Let

$$\Gamma_1^\Theta(\mathfrak{n}, \wp^r) = \left\{ \gamma \in \Gamma_1(\mathfrak{n}) \;\middle|\; \gamma \bmod \wp^r \in \begin{pmatrix} \Theta & * \\ 0 & \Theta \end{pmatrix} \right\}.$$

Since Θ is a p-group, we see that $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$ is a congruence subgroup of $SL_2(A)$ which is p'-torsion free. When $\Theta = \Theta_r$, we denote $\Gamma_1^\Theta(\mathfrak{n}, \wp^r)$ by $\Gamma_1^p(\mathfrak{n}, \wp^r)$. Note that $\Gamma_1(\mathfrak{n}\wp^r) = \Gamma_1^{\{1\}}(\mathfrak{n}, \wp^r)$ and we have an isomorphism

$$\Gamma_1^\Theta(\mathfrak{n}, \wp^r)/\Gamma_1(\mathfrak{n}\wp^r) \to \Theta, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \bmod \wp^r. \tag{3.1}$$

Lemma 3.1. *The group ring $\mathbb{C}_\infty[\Theta]$ is an Artinian local ring. In particular, for any free $\mathbb{C}_\infty[\Theta]$-module of finite rank, its direct summand as a $\mathbb{C}_\infty[\Theta]$-module is free of finite rank.*

Proof. Since Θ is a commutative p-group, it is isomorphic to $\bigoplus_{i=1}^s \mathbb{Z}/p^{n_i}\mathbb{Z}$ with some nonnegative integers $n_1, \ldots, n_s$. Since the characteristic of $\mathbb{C}_\infty$ is p, we have an isomorphism of $\mathbb{C}_\infty$-algebras

$$\begin{aligned} \mathbb{C}_\infty[\Theta] &\simeq \mathbb{C}_\infty[X_1, \ldots, X_s]/(X_1^{p^{n_1}} - 1, \ldots, X_s^{p^{n_s}} - 1) \\ &= \mathbb{C}_\infty[X_1, \ldots, X_s]/((X_1 - 1)^{p^{n_1}}, \ldots, (X_s - 1)^{p^{n_s}}). \end{aligned}$$

This shows the former assertion. For the latter, such a direct summand is finitely generated and projective over the local ring $\mathbb{C}_\infty[\Theta]$. Thus it is free of finite rank. □

For any $d \in (A/\mathfrak{n}\wp^r A)^\times$, take any matrix $\eta_d \in SL_2(A)$ satisfying

$$\eta_d \equiv \begin{pmatrix} * & * \\ 0 & d \end{pmatrix} \bmod \mathfrak{n}\wp^r.$$

As in [Hat2, §2.2], we define the diamond operator $\langle d \rangle_{\mathfrak{n}\wp^r}$ acting on $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ by

$$f|\langle d \rangle_{\mathfrak{n}\wp^r} = f|_k \eta_d,$$

which is independent of the choice of η_d. Since

$$\eta_d \Gamma_1^\Theta(\mathfrak{n}, \wp^r) \eta_d^{-1} = \Gamma_1^\Theta(\mathfrak{n}, \wp^r),$$

the subspace $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$ is stable under the diamond operators.

Consider the natural isomorphism

$$\rho : (A/\mathfrak{n}\wp^r A)^\times \to (A/\mathfrak{n}A)^\times \times A_r^\times.$$

For any $d \in A_r^\times$, let $[1, d] = \rho^{-1}((1, d))$. Since the map

$$\Theta \to (A/\mathfrak{n}\wp^r A)^\times, \quad d \mapsto [1, d]$$

is a group homomorphism, we have a right Θ-action on $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ defined by

$$\Theta \to \mathrm{Aut}(S_k(\Gamma_1(\mathfrak{n}\wp^r)))^{\mathrm{op}}, \quad d \mapsto \langle [1, d] \rangle_{\mathfrak{n}\wp^r}.$$

By (3.1), we have

$$S_k(\Gamma_1(\mathfrak{n}\wp^r))^\Theta = S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r)). \tag{3.2}$$

Let $Q \in \Pi$. Write

$$\Gamma_1^\Theta(\mathfrak{n}, \wp^r) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_1^\Theta(\mathfrak{n}, \wp^r) = \coprod_{i \in I(Q)} \Gamma_1^\Theta(\mathfrak{n}, \wp^r)\xi_i.$$

The Hecke operator T_Q at Q acting on $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$ is defined by

$$f|T_Q = \sum_{i \in I(Q)} f|_k \xi_i.$$

When $Q \mid \mathfrak{n}\wp^r$, we also write U_Q for T_Q. Since the explicit description of T_Q given in [Hat1, §3.1] is independent of Θ, the natural inclusion

$$S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r)) \to S_k(\Gamma_1(\mathfrak{n}\wp^r))$$

is compatible with T_Q for any Q. Moreover, we can show that Hecke operators commute with each other. By [Hat2, Lemma 2.3], the diamond operators commute with all Hecke operators.

For a finite dimensional $\mathbb{C}_\infty$-vector space V, a $\mathbb{C}_\infty$-linear map $T : V \to V$ and $\alpha \in \mathbb{C}_\infty$, we denote by $V(T - \alpha)$ the generalized eigenspace of T belonging to α. It equals $\mathrm{Ker}((T - \alpha)^m)$ for any sufficiently large integer m. Let $\mathrm{EV}(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k, Q)$ be the set of eigenvalues of T_Q acting on $S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))$. Then we have

$$S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r)) = \bigoplus_{\alpha \in \mathrm{EV}(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k, Q)} S_k(\Gamma_1^\Theta(\mathfrak{n}, \wp^r))(T_Q - \alpha), \tag{3.3}$$

where each direct summand is stable under all Hecke operators and diamond operators. By [Hat1, Proposition 2.2], we have

$$\mathrm{EV}(\Gamma_1^\Theta(\mathfrak{n}, \wp^r), k, Q) \subseteq \bar{K}.$$

Consider the natural inclusion

$$\iota : S_k(\Gamma_1^p(\mathfrak{n}, \wp^r)) \to S_k(\Gamma_1^p(\mathfrak{n}, \wp^{r+1})).$$

For any $d \in (A/\mathfrak{n}\wp^{r+1}A)^\times$, we have

$$\langle d \rangle_{\mathfrak{n}\wp^{r+1}} \circ \iota = \iota \circ \langle d \bmod \mathfrak{n}\wp^r \rangle_{\mathfrak{n}\wp^r}.$$

By the explicit description of T_Q in [Hat1, §3.1], this implies that the map ι is compatible with all Hecke operators.

Lemma 3.2. *For any $\alpha \in \mathbb{C}_\infty^\times$, the map ι induces an isomorphism*

$$S_k(\Gamma_1^p(\mathfrak{n}, \wp^r))(U_\wp - \alpha) \to S_k(\Gamma_1^p(\mathfrak{n}, \wp^{r+1}))(U_\wp - \alpha)$$

which is compatible with all Hecke operators.

Proof. Write $\Gamma_r = \Gamma_1^p(\mathfrak{n}, \wp^r)$. Let $R(\wp) \subseteq A$ be a complete set of representatives of $A/\wp A$. We can write

$$\Gamma_{r+1} \begin{pmatrix} 1 & 0 \\ 0 & \wp \end{pmatrix} \Gamma_r = \coprod_{\beta \in R(\wp)} \Gamma_{r+1}\xi_\beta, \quad \xi_\beta = \begin{pmatrix} 1 & \beta \\ 0 & \wp \end{pmatrix}.$$

Then we have a $\mathbb{C}_\infty$-linear map

$$s : S_k(\Gamma_{r+1}) \to S_k(\Gamma_r), \quad f \mapsto \sum_{\beta \in R(\wp)} f|_k \xi_\beta$$

satisfying the commutative diagram

$$\begin{array}{ccc} S_k(\Gamma_r) & \xrightarrow{\iota} & S_k(\Gamma_{r+1}) \\ {\scriptstyle U_\wp}\downarrow & \overset{s}{\swarrow} & \downarrow{\scriptstyle U_\wp} \\ S_k(\Gamma_r) & \xrightarrow[\iota]{} & S_k(\Gamma_{r+1}). \end{array}$$

This implies $U_\wp = 0$ on $\mathrm{Coker}(\iota)$. Since ι commutes with $U_\wp$, it induces an injection

$$\iota_\alpha : S_k(\Gamma_r)(U_\wp - \alpha) \to S_k(\Gamma_{r+1})(U_\wp - \alpha).$$

By (3.3), we see that $\mathrm{Coker}(\iota_\alpha)$ is a $\mathbb{C}_\infty$-subspace of $\mathrm{Coker}(\iota)$. Since $\alpha \neq 0$, we have $\mathrm{Coker}(\iota_\alpha) = 0$ and the map ι_α is also surjective. Since ι is compatible with all Hecke operators, so is ι_α. □

4. Freeness under the Θ_r-action

In this section, write $\Gamma = \Gamma_1(\mathfrak{n}\wp^r)$ and $\tilde{\Gamma} = \Gamma_1^p(\mathfrak{n}, \wp^r)$, so that $\tilde{\Gamma} \rhd \Gamma$ and $\Theta_r = \tilde{\Gamma}/\Gamma$.

Consider the right $\mathbb{Z}[\tilde{\Gamma}]$-modules

$$\tilde{L}_1 := \mathbb{Z}[\bar{\mathcal{T}}_1^{o,\tilde{\Gamma}\text{-st}}] \quad \text{and} \quad \tilde{L}_0 := \mathbb{Z}[\mathcal{T}_0^{\tilde{\Gamma}\text{-st}}],$$

where the $\tilde{\Gamma}$-action is given by $s|_{\tilde{\gamma}} = \tilde{\gamma}^{-1} \circ s$ as before. Let $\tilde{\Lambda}_1 \subseteq \mathcal{T}_1^{o,\tilde{\Gamma}\text{-st}}$ and $\tilde{\Lambda}_0 \subseteq \mathcal{T}_0^{\tilde{\Gamma}\text{-st}}$ be complete sets of representatives of

$$\tilde{\Gamma}\backslash\mathcal{T}_1^{o,\tilde{\Gamma}\text{-st}}/\{\pm 1\} \quad \text{and} \quad \tilde{\Gamma}\backslash\mathcal{T}_0^{\tilde{\Gamma}\text{-st}},$$

respectively. By [Ser, Ch. II, §2.9, Theorem 13' (a)], the sets $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_0$ are finite. Moreover, we have

$$\tilde{L}_1 = \bigoplus_{e \in \tilde{\Lambda}_1} [e]\mathbb{Z}[\tilde{\Gamma}], \quad \tilde{L}_0 = \bigoplus_{v \in \tilde{\Lambda}_0} [v]\mathbb{Z}[\tilde{\Gamma}]. \tag{4.1}$$

Since the right $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}[\tilde{\Gamma}]$ is free of finite rank, we see that the sequence (2.1) for $\tilde{\Gamma}$ is a split exact sequence of right $\mathbb{Z}[\Gamma]$-modules. Hence we obtain an exact sequence of $\mathbb{C}_\infty$-vector spaces

(4.2)

$$0 \longrightarrow \mathcal{V}_k(\Gamma) \longrightarrow \tilde{L}_1 \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty) \longrightarrow \tilde{L}_0 \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty) \longrightarrow 0,$$

which is compatible with natural right Θ_r-actions.

Lemma 4.1. *For any $i \in \{0,1\}$, the right $\mathbb{C}_\infty[\Theta_r]$-module $\tilde{L}_i \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty)$ is free of finite rank.*

Proof. By (4.1), we have an isomorphism of right $\mathbb{C}_\infty[\Theta_r]$-modules

$$\tilde{L}_i \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty) \to \bigoplus_{s \in \tilde{\Lambda}_i} \mathbb{Z}[\tilde{\Gamma}] \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty).$$

Thus we reduce ourselves to showing that the right $\mathbb{C}_\infty[\Theta_r]$-module

$$\mathbb{Z}[\tilde{\Gamma}] \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty) = \mathrm{Ind}_\Gamma^{\tilde{\Gamma}} \mathrm{Res}_\Gamma^{\tilde{\Gamma}} V_k(\mathbb{C}_\infty)$$

is free of finite rank.

Let $\mathbf{1}$ be the trivial representation of Γ over $\mathbb{C}_\infty$. By the projection formula, we have a natural isomorphism of left $\mathbb{C}_\infty[\tilde{\Gamma}]$-modules

$$\pi : \mathrm{Ind}_\Gamma^{\tilde{\Gamma}} \mathrm{Res}_\Gamma^{\tilde{\Gamma}} V_k(\mathbb{C}_\infty) \to (\mathrm{Ind}_\Gamma^{\tilde{\Gamma}} \mathbf{1}) \otimes_{\mathbb{C}_\infty} V_k(\mathbb{C}_\infty)$$

which is given by

$$\mathbb{Z}[\tilde{\Gamma}] \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty) \to (\mathbb{Z}[\tilde{\Gamma}] \otimes_{\mathbb{Z}[\Gamma]} \mathbb{C}_\infty) \otimes_{\mathbb{C}_\infty} V_k(\mathbb{C}_\infty),$$
$$[\tilde{\gamma}] \otimes v \mapsto [\tilde{\gamma}] \otimes 1 \otimes \tilde{\gamma} \circ v.$$

On the target of the map π, we consider the trivial Θ_r-action on $V_k(\mathbb{C}_\infty)$ and a right Θ_r-action on $\mathbb{Z}[\tilde{\Gamma}] \otimes_{\mathbb{Z}[\Gamma]} \mathbb{C}_\infty$ defined by

$$([\tilde{\gamma}] \otimes 1)d = [\tilde{\gamma}\eta_{[1,d]}] \otimes 1$$

for any $d \in \Theta_r$. Then the map π is Θ_r-equivariant. Moreover, with the latter Θ_r-action, we have an isomorphism of right $\mathbb{C}_\infty[\Theta_r]$-modules

$$\mathbb{Z}[\tilde{\Gamma}] \otimes_{\mathbb{Z}[\Gamma]} \mathbb{C}_\infty \to \mathbb{C}_\infty[\Theta_r], \quad [\eta_{[1,d]}] \otimes 1 \mapsto [d].$$

Hence we obtain an isomorphism of right $\mathbb{C}_\infty[\Theta_r]$-modules

$$\mathbb{Z}[\tilde{\Gamma}] \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty) \simeq \mathbb{C}_\infty[\Theta_r]^{\oplus k-1}.$$

This concludes the proof. □

The following proposition is a generalization of [Hat2, Proposition 4.8] which treats the case of $\mathfrak{n} = 1$, $\wp = t$ and $k = 2$.

Proposition 4.2. *The $\mathbb{C}_\infty[\Theta_r]$-module $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ is free of finite rank.*

Proof. By Lemma 2.1, it is enough to show that the right $\mathbb{C}_\infty[\Theta_r]$-module $\mathcal{V}_k(\Gamma_1(\mathfrak{n}\wp^r))$ is free of finite rank. By (4.2) and Lemma 4.1, we see that the right $\mathbb{C}_\infty[\Theta_r]$-module $\mathcal{V}_k(\Gamma_1(\mathfrak{n}\wp^r))$ is a direct summand of a free $\mathbb{C}_\infty[\Theta_r]$-module of finite rank. Hence the proposition follows from Lemma 3.1. □

5. Constancy of Hecke eigensystems

For any $Q \in \Pi$ and $\alpha \in \bar{K}$, we denote by $m(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r), k, Q, \alpha)$ the multiplicity of α as an eigenvalue of T_Q acting on $S_k(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r))$. By definition, we have

$$m(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r), k, Q, \alpha) = \dim_{\mathbb{C}_\infty} S_k(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r))(T_Q - \alpha).$$

For any $a \in \mathbb{Q} \cup \{+\infty\}$, we denote by $d(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r), k, a)$ the multiplicity of eigenvalues of slope a of $U_\wp$ acting on $S_k(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r))$, so that

(5.1)
$$d(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r), k, a) = \sum_{\alpha \in \mathrm{EV}(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r), k, \wp),\ v_\wp(\iota_\wp(\alpha)) = a} m(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r), k, \wp, \alpha).$$

Proposition 5.1. *For any $Q \in \Pi$ and $\alpha \in \bar{K}$, we have*

$$m(\Gamma_1(\mathfrak{n}\wp^r), k, Q, \alpha) = |\Theta| m(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r), k, Q, \alpha).$$

In particular, for any $a \in \mathbb{Q} \cup \{+\infty\}$ we have

$$\mathrm{EV}(\Gamma_1(\mathfrak{n}\wp^r), k, Q) = \mathrm{EV}(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r), k, Q),$$
$$d(\Gamma_1(\mathfrak{n}\wp^r), k, a) = |\Theta| d(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r), k, a).$$

Proof. Since (3.3) is a decomposition as a $\mathbb{C}_\infty[\Theta_r]$-module, Proposition 4.2 and Lemma 3.1 imply that for any $\alpha \in \bar{K}$, the $\mathbb{C}_\infty[\Theta_r]$-module $S_k(\Gamma_1(\mathfrak{n}\wp^r))(T_Q - \alpha)$ is free of finite rank. Write

$$S_k(\Gamma_1(\mathfrak{n}\wp^r))(T_Q - \alpha) = \mathbb{C}_\infty[\Theta_r]^{\oplus m}$$

with some integer $m \geqslant 0$. By (3.2), this implies

$$S_k(\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r))(T_Q - \alpha) = (\mathbb{C}_\infty[\Theta_r]^{\Theta})^{\oplus m}.$$

Since $\dim_{\mathbb{C}_\infty} \mathbb{C}_\infty[\Theta_r]^{\Theta} = [\Theta_r : \Theta]$, the first assertion follows.

Since $\alpha \in \mathrm{EV}(\Gamma_1(\mathfrak{n}\wp^r), k, Q)$ if and only if $m(\Gamma_1(\mathfrak{n}\wp^r), k, Q, \alpha) \neq 0$ and similarly for $\Gamma_1^{\Theta}(\mathfrak{n}, \wp^r)$, the second assertion follows. Then (5.1) implies the third one. ☐

Note that $\Gamma_1^{p}(\mathfrak{n}, \wp) = \Gamma_1(\mathfrak{n}\wp)$.

Corollary 5.2. *For any $\alpha \in \bar{K}^\times$ and $a \in \mathbb{Q}$, we have*

$$m(\Gamma_1(\mathfrak{n}\wp^r), k, \wp, \alpha) = q^{(r-1)\deg(\wp)} m(\Gamma_1(\mathfrak{n}\wp), k, \wp, \alpha),$$
$$d(\Gamma_1(\mathfrak{n}\wp^r), k, a) = q^{(r-1)\deg(\wp)} d(\Gamma_1(\mathfrak{n}\wp), k, a).$$

Proof. For any $\alpha \in \bar{K}^\times$, Lemma 3.2 yields

$$m(\Gamma_1^{p}(\mathfrak{n}, \wp^r), k, \wp, \alpha) = m(\Gamma_1(\mathfrak{n}\wp), k, \wp, \alpha),$$
$$\mathrm{EV}(\Gamma_1^{p}(\mathfrak{n}, \wp^r), k, \wp) \backslash \{0\} = \mathrm{EV}(\Gamma_1(\mathfrak{n}\wp), k, \wp) \backslash \{0\}.$$

Since $|\Theta_r| = q^{(r-1)\deg(\wp)}$, Proposition 5.1 and (5.1) yield the corollary. ☐

Let $\lambda = (\lambda_Q)_{Q\in\Pi} \in \bar{K}^{\Pi}$. We say that λ is a Hecke eigensystem appearing in $S_k(\Gamma_1^{\Theta}(\mathfrak{n},\wp^r))$ if there exists a nonzero element $f \in S_k(\Gamma_1^{\Theta}(\mathfrak{n},\wp^r))$ satisfying $f|T_Q = \lambda_Q f$ for all $Q \in \Pi$.

Theorem 5.3. *Let $\lambda = (\lambda_Q)_{Q\in\Pi} \in \bar{K}^{\Pi}$ satisfy $\lambda_\wp \neq 0$. Then λ is a Hecke eigensystem appearing in $S_k(\Gamma_1(\mathfrak{n}\wp^r))$ if and only if it is a Hecke eigensystem appearing in $S_k(\Gamma_1(\mathfrak{n}\wp))$.*

Proof. As we have seen in §3, the natural inclusions

$$S_k(\Gamma_1(\mathfrak{n}\wp)) \to S_k(\Gamma_1^p(\mathfrak{n},\wp^r)) \to S_k(\Gamma_1(\mathfrak{n}\wp^r))$$

are compatible with all Hecke operators. This shows the "if" part of the theorem.

Conversely, suppose that λ is a Hecke eigensystem appearing in $S_0 := S_k(\Gamma_1(\mathfrak{n}\wp^r))$. Let $f_\lambda \in S_0$ be a nonzero Hecke eigenform which gives the Hecke eigensystem λ. Since Π is countably infinite, we can choose a bijection

$$\mathbb{Z}_{>0} \to \Pi, \quad i \mapsto Q_i.$$

Let n_1 be a positive integer such that

$$S_1 := S_0(T_{Q_1} - \lambda_{Q_1}) = \mathrm{Ker}((T_{Q_1} - \lambda_{Q_1})^{n_1} : S_0 \to S_0).$$

Since $f_\lambda \in S_1$, we have $S_1 \neq 0$.

Repeating this, we can inductively construct a decreasing sequence of nonzero $\mathbb{C}_\infty$-vector spaces

$$S_0 \supseteq S_1 \supseteq \cdots \supseteq S_i \supseteq S_{i+1} \supseteq \cdots$$

and a sequence of positive integers $(n_i)_{i\in\mathbb{Z}_{>0}}$ such that

$$S_i = S_{i-1}(T_{Q_i} - \lambda_{Q_i}) = \mathrm{Ker}((T_{Q_i} - \lambda_{Q_i})^{n_i} : S_{i-1} \to S_{i-1})$$

for any $i \in \mathbb{Z}_{>0}$. Since $\dim_{\mathbb{C}_\infty} S_0$ is finite, the sequence $\{S_i\}_{i\geqslant 0}$ is stationary. Thus there exists a positive integer j such that

$$S_j = \{f \in S_0 \mid f|(T_{Q_i} - \lambda_{Q_i})^{n_i} = 0 \text{ for any } i \in \mathbb{Z}_{>0}\}.$$

Note that for any $i \in \mathbb{Z}_{>0}$, the $\mathbb{C}_\infty$-vector space S_{i-1} is the direct sum of the generalized eigenspaces of T_{Q_i} acting on it. Since diamond operators commute with all Hecke operators, these direct summands are stable under the Θ_r-action. In particular, we see that S_i is a direct summand of S_{i-1} as a $\mathbb{C}_\infty[\Theta_r]$-module. By Proposition 4.2 and Lemma 3.1, the $\mathbb{C}_\infty[\Theta_r]$-module S_j is free of finite rank. Since $\dim_{\mathbb{C}_\infty} \mathbb{C}_\infty[\Theta_r]^{\Theta_r} = 1$, we have $S_j^{\Theta_r} \neq 0$.

By (3.2), we have

$$S_0^p := S_j^{\Theta_r} = \{f \in S_k(\Gamma_1^p(\mathfrak{n},\wp^r)) \mid f|(T_{Q_i} - \lambda_{Q_i})^{n_i} = 0 \text{ for any } i \in \mathbb{Z}_{>0}\}.$$

Since $T_{Q_1} - \lambda_{Q_1}$ is nilpotent on S_0^p, we have

$$S_1^p := \mathrm{Ker}(T_{Q_1} - \lambda_{Q_1} : S_0^p \to S_0^p) \neq 0,$$

on which $T_{Q_2} - \lambda_{Q_2}$ is nilpotent.

Thus we can inductively construct a decreasing sequence of nonzero $\mathbb{C}_\infty$-vector spaces

$$S_0^p \supseteq S_1^p \supseteq \cdots \supseteq S_i^p \supseteq S_{i+1}^p \supseteq \cdots$$

such that for any $i \in \mathbb{Z}_{>0}$, we have

$$S_i^p = \mathrm{Ker}(T_{Q_i} - \lambda_{Q_i} : S_{i-1}^p \to S_{i-1}^p).$$

Again the sequence is stationary and thus, for some positive integer l, we have

$$S_l^p = \{f \in S_0^p \mid f|(T_Q - \lambda_Q) = 0 \text{ for any } Q \in \Pi\}.$$

Since $S_l^p \neq 0$, it follows that λ is a Hecke eigensystem appearing in $S_k(\Gamma_1^p(\mathfrak{n}, \wp^r))$. Since $\lambda_\wp \neq 0$, Lemma 3.2 implies that the Hecke eigensystem λ also appears in $S_k(\Gamma_1(\mathfrak{n}\wp))$. This concludes the proof of the theorem. □

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