DIMENSION VARIATION OF GOUVÊA-MAZUR TYPE FOR DRINFELD CUSPFORMS OF LEVEL $\Gamma_1(t)$

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Abstract. Let p be a rational prime and $q > 1$ a p-power. Let $S_k(\Gamma_1(t))$ be the space of Drinfeld cuspforms of level $\Gamma_1(t)$ and weight k for $\mathbb{F}_q[t]$. For any non-negative rational number α, we denote by $d(k, \alpha)$ the dimension of the slope α generalized eigenspace for the U-operator acting on $S_k(\Gamma_1(t))$. In this paper, we prove a function field analogue of the Gouvêa-Mazur conjecture for this setting. Namely, we show that for any $\alpha \leqslant m$ and $k_1, k_2 > \alpha + 1$, if $k_1 \equiv k_2 \bmod p^m$, then $d(k_1, \alpha) = d(k_2, \alpha)$.

1. Introduction

Let p be a rational prime, $q > 1$ a p-power, $A = \mathbb{F}_q[t]$ and $\wp \in A$ a monic irreducible polynomial. For $K_\infty = \mathbb{F}_q((1/t))$, we denote by $\mathbb{C}_\infty$ the $(1/t)$-adic completion of an algebraic closure of K_∞. Then the Drinfeld upper half plane $\Omega = \mathbb{C}_\infty \setminus K_\infty$ has a natural structure of a rigid analytic variety over K_∞.

Let k be an integer and Γ a subgroup of $SL_2(A)$. Then a Drinfeld modular form of level Γ and weight k is a rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for any } z \in \Omega,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and a holomorphy condition at cusps. The notion of Drinfeld modular form can be considered as a function field analogue of that of elliptic modular form and the former often has properties which are parallel to the latter. However, despite that the theory of p-adic families of elliptic modular forms is highly developed and has been yielding many applications, $\wp$-adic properties of Drinfeld modular forms are not well-understood yet. A typical difficulty in the Drinfeld case seems that a naïve analogue of the universal character $\mathbb{Z}_p^\times \to \mathbb{Z}_p[[\mathbb{Z}_p^\times]]^\times$ is not locally analytic by [Jeo, Lemma 2.5] and thus similar constructions to those in the classical case including [AIP] will not immediately produce

Date: April 25, 2019.

an analytic family of invertible sheaves interpolating automorphic line bundles.

Still, there seem to exist interesting structures in $\wp$-adic properties of Drinfeld modular forms. In [BV1, BV2], Bandini-Valentino studied an analogue of the classical Atkin U-operator, which we also denote by U, acting on the space $S_k(\Gamma_1(t))$ of Drinfeld cuspforms of level $\Gamma_1(t)$ and weight k. The operator U is defined by

$$(Uf)(z) = \frac{1}{t}\sum_{\beta\in\mathbb{F}_q} f\left(\frac{z+\beta}{t}\right). \tag{1.1}$$

The normalized t-adic valuation of an eigenvalue of U is called slope. Note that here we adopt the different normalization from that of Bandini-Valentino, and as a result our notion of slope is smaller than theirs by one. For a non-negative rational number α, we denote by $d(k,\alpha)$ the dimension of the generalized eigenspace of U acting on $S_k(\Gamma_1(t))$ for the eigenvalues of slope α. Then they proposed a conjecture on a p-adic variation of $d(k,\alpha)$ with respect to k [BV2, Conjecture 6.1] which can be regarded as a function field analogue of the Gouvêa-Mazur conjecture [GM1, Conjecture 1]. In this paper, we will prove it.

Theorem 1.1. *(Theorem 2.10) Let $m \geqslant 0$ be an integer and α a non-negative rational number. Suppose $\alpha \leqslant m$. Then the dimension $d(k,\alpha)$ of the slope α generalized eigenspace in $S_k(\Gamma_1(t))$ satisfies*

$$k_1, k_2 > \alpha+1,\ k_1 \equiv k_2 \bmod p^m \Rightarrow d(k_1,\alpha) = d(k_2,\alpha).$$

We will also prove its variant for level $\Gamma_0(t)$ (Theorem 3.1).

For the proof, put

$$P^{(k)}(X) = \det(I - XU \mid S_k(\Gamma_1(t))).$$

First note that, as is mentioned in [Wan, §4, Remarks], the arguments of [GM2] and [Wan] can be generalized over suitable Drinfeld modular curves (including $X_1^{\Delta}(\mathfrak{n})$ of [Hat]). In particular, the characteristic power series of U acting on the spaces of $\wp$-adic overconvergent Drinfeld modular forms of weight k_1 and k_2 are congruent modulo $\wp^{p^m}$. As its analogue in our setting, we can show the congruence $P^{(k_1)}(X) \equiv P^{(k_2)}(X) \bmod t^{p^m}$ up to some factor. However, though with this we can prove Theorem 1.1 for $p \geqslant 3$, it is not enough to settle the case of $p = 2$ on which Bandini-Valentino stated their conjecture.

To go further, we investigate the formula of the representing matrix of U given by Bandini-Valentino [BV1, (3.1)] more closely. Luckily, the representing matrix is of very special form: each entry on the j-th column (with the normalization that the leftmost column is the zeroth) is an element of $\mathbb{F}_q t^j$. Thanks to this fact, we can give a lower bound of elementary divisors of the representing matrix (Lemma 2.2). Then a perturbation argument shows that the n-th coefficients of $P^{(k)}(X)$ and $P^{(k+p^m)}(X)$ are much more congruent than modulo t^{p^m} up to some factor of slope $\geqslant k-1$ (Corollary 2.7), which is enough to yield Theorem 1.1 for any p.

Acknowledgments. The author would like to thank Gebhard Böckle for informing him of Valentino's table computing characteristic polynomials of U, and Maria Valentino for pointing out an error in the author's previous computer calculation. He also would like to thank the anonymous referee for improving Lemma 2.2 and giving a suggestion to consider the case of $\Gamma_0(t)$. This work was supported by JSPS KAKENHI Grant Number JP17K05177.

2. Dimension variation

Let $k \geqslant 2$ be an integer. Put

$$\Gamma_1(t) = \left\{ \gamma \in GL_2(A) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod t \right\} \subseteq SL_2(A).$$

On the space $S_k(\Gamma_1(t))$ of Drinfeld cuspforms of level $\Gamma_1(t)$ and weight k, we consider the U-operator for t defined by (1.1). Note that we follow the usual normalization of the U-operator which differs from that of [BV1, §2.4] by $1/t$. Then Bandini-Valentino [BV1, (3.1)] explicitly describe the action of U with respect to some basis $\mathbf{c}_0^{(k)}, \ldots, \mathbf{c}_{k-2}^{(k)}$, which reads as follows with our normalization:

$$\begin{aligned} U(\mathbf{c}_j^{(k)}) =&(-t)^j \tbinom{k-2-j}{j} \mathbf{c}_j^{(k)} - t^j \sum_{h\in\mathbb{Z}, h\neq 0} \left\{ \tbinom{k-2-j-h(q-1)}{-h(q-1)} \right. \\ &\left. +(-1)^{j+1} \tbinom{k-2-j-h(q-1)}{j} \right\} \mathbf{c}_{j+h(q-1)}^{(k)}. \end{aligned} \tag{2.1}$$

Here it is understood that the binomial coefficient $\binom{c}{d}$ is zero if any of $c, d, c-d$ is negative and the terms involving $\mathbf{c}_{j+h(q-1)}^{(k)}$ are zero if $j+h(q-1) \notin [0, k-2]$. We denote by $U^{(k)} = (U_{i,j}^{(k)})_{0\leqslant i,j\leqslant k-2}$ the representing matrix of U for this basis. Then we have $U^{(k)} \in M_{k-1}(A)$. We identify the t-adic completion of A with $\mathbb{F}_q[[t]]$ naturally and consider $U^{(k)}$ as an element of $M_{k-1}(\mathbb{F}_q[[t]])$. Let v_t be the t-adic additive valuation normalized as $v_t(t) = 1$.

Definition 2.1. (1) Let $B = (B_{i,j})_{0\leqslant i\leqslant m-1, 0\leqslant j\leqslant n-1}$ be an element of $M_{m,n}(\mathbb{F}_q[[t]])$ and b a non-negative integer. We say B is b-glissando if $B_{i,j} \in \mathbb{F}_q t^{bj}$ for any i, j.

(2) For such B, write $B = B_0\mathrm{diag}(1, t^b, \dots, t^{b(n-1)})$, where $B_0 \in M_{m,n}(\mathbb{F}_q)$ and $\mathrm{diag}(1, t^b, \dots, t^{b(n-1)})$ is the diagonal matrix whose diagonal entries are as indicated. We say a non-negative integer j is a pivot number of B if in the row reduced form of B_0, a pivot is on the j-th column.

By (2.1), the matrix $U^{(k)}$ is 1-glissando. Moreover, the l-th smallest pivot number j_l of a b-glissando matrix satisfies $j_l \geqslant l-1$.

Lemma 2.2. *Let b be a non-negative integer. Let $B = (B_{i,j})_{0\leqslant i\leqslant m-1, 0\leqslant j\leqslant n-1}$ be a b-glissando matrix in $M_{m,n}(\mathbb{F}_q[[t]])$. Let $j_1 < \cdots < j_r$ be the pivot numbers of B. Let $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_u$ be the elementary divisors of B (namely, they are integers or $+\infty$ such that the $(i-1, i-1)$-entry of the Smith normal form of B has normalized t-adic valuation s_i). Then $s_l < +\infty$ if and only if $l \leqslant r$, and for any such l, we have $s_l = bj_l$.*

In particular, we have $s_l \geqslant b(l-1)$ for any l.

Proof. Let B_0 be as in Definition 2.1 (2) and B_0' its row reduced form. Then the Smith normal form of B agrees with that of $B_0'\mathrm{diag}(1, t^b, \dots, t^{b(n-1)})$. The latter product is of row echelon form such that the l-th pivot is t^{bj_l} and every entry of the l-th row is divisible by the pivot. This yields the lemma. □

For any element $P(X) = \sum_{n=0}^{\infty} p_n X^n \in \mathbb{F}_q[[t]][[X]]$, the Newton polygon of $P(X)$ is by definition the lower convex hull of the set

$$\{(n, v_t(p_n)) \mid n \geqslant 0\}.$$

Lemma 2.3. *For any $B \in M_m(\mathbb{F}_q[[t]])$ and any non-negative integer c, put*

$$P(X) = \det(I - t^c XB) = \sum_{n=0}^{m} p_n X^n \in \mathbb{F}_q[[t]][X].$$

Let $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_u$ be the elementary divisors of B.

(1) $v_t(p_n) \geqslant cn + \sum_{l=1}^{n} s_l$.
(2) Any slope of the Newton polygon of $P(X)$ is no less than c.
(3) If B is b-glissando, then we have $v_t(p_n) \geqslant cn + \frac{b}{2}n(n-1)$.

Proof. First note that, for the characteristic polynomial $Q(X) = \det(XI - t^c B)$, we have $P(X) = X^m Q(X^{-1})$ and thus p_n is, up to a sign, equal to the sum of the principal $n \times n$ minors of $t^c B$. Since the elementary divisors of $t^c B$ are $c + s_1, \dots, c + s_u$, this shows (1). Since $p_0 = 1$, the resulting inequality $v_t(p_n) \geqslant cn$ implies (2). By Lemma 2.2 and (1), we obtain (3). □

Now we put

$$P^{(k)}(X)=\det(I-XU^{(k)})=\sum_{n=0}^{k-1}a_n^{(k)}X^n$$

and $a_n^{(k)}=0$ for any $n\geqslant k$. Let $N^{(k)}$ be the Newton polygon of $P^{(k)}(X)$. For any non-negative rational number α, we denote by $d(k,\alpha)$ the dimension of the generalized U-eigenspace for the eigenvalues of normalized t-adic valuation α. Then $d(k,\alpha)$ is equal to the width of the segment of slope α in the Newton polygon $N^{(k)}$.

Lemma 2.4. $d(k,0)=1$.

Proof. By (2.1), we have $U_{0,0}^{(k)}=\binom{k-2}{0}=1$. On the other hand, since $U^{(k)}$ is 1-glissando, we have $v_t(U_{i,j}^{(k)})\geqslant j$ and

$$a_1^{(k)}=-\sum_{j=0}^{k-2}U_{j,j}^{(k)}\equiv -1 \bmod t.$$

Moreover, from Lemma 2.3 (3) we obtain $v_t(a_n^{(k)})>0$ for any $n\geqslant 2$. This yields the lemma. □

Lemma 2.5. *Let a and b be non-negative integers. Let $m\geqslant 1$ be an integer. Then we have*

$$\binom{a+p^m}{b}\equiv\binom{a}{b}+\binom{a}{b-p^m} \bmod p.$$

Here it is understood that $\binom{c}{d}=0$ if any of $c,d,c-d$ is negative.

Proof. This follows from

$$(X+1)^{a+p^m}\equiv (X+1)^a(X^{p^m}+1) \bmod p.$$

□

Proposition 2.6. *Let $m\geqslant 1$ be an integer. Then there exist 1-glissando matrices $C\in M_{p^m,k-1}(A)$ and $D\in M_{p^m,p^m-k+1}(A)$ satisfying*

$$U^{(k+p^m)}\equiv\left(\begin{array}{c|c|c} U^{(k)} & O & O\\ C & t^{k-1}D & O\end{array}\right) \bmod t^{p^m}.$$

Here it is understood that the middle blocks are empty if $p^m\leqslant k-1$.

Proof. Let j be an integer satisfying $0\leqslant j\leqslant k+p^m-2$. By (2.1), the element $U(\mathbf{c}_j^{(k+p^m)})$ is equal to

$$\begin{aligned}&(-t)^j\binom{k+p^m-2-j}{j}\mathbf{c}_j^{(k+p^m)}\\ &-t^j\sum_{h\in\mathbb{Z},h\neq 0}\left\{\binom{k+p^m-2-j-h(q-1)}{-h(q-1)}+(-1)^{j+1}\binom{k+p^m-2-j-h(q-1)}{j}\right\}\mathbf{c}_{j+h(q-1)}^{(k+p^m)}.\end{aligned}$$

Note that both of $U^{(k+p^m)}_{i,j}$ and $U^{(k)}_{i,j}$ are divisible by t^{p^m} for $j \geqslant p^m$. Since $U^{(k+p^m)}$ is 1-glissando, what we need to show is

(1) For any $j \leqslant \min\{k-2, p^m-1\}$ and $i \in [0, k-2]$, we have $U^{(k+p^m)}_{i,j} = U^{(k)}_{i,j}$, and
(2) If $k \leqslant p^m$, then for any $j \in [k-1, p^m-1]$ and $i \in [0, k-2]$, we have $U^{(k+p^m)}_{i,j} = 0$.

First we assume $j \leqslant \min\{k-2, p^m-1\}$. By Lemma 2.5, the element $U(\mathbf{c}^{(k+p^m)}_j)$ equals

$$
\begin{aligned}
&(-t)^j \left\{ \binom{k-2-j}{j} + \binom{k-2-j}{j-p^m} \right\} \mathbf{c}^{(k+p^m)}_j \\
&- t^j \sum_{\substack{h\in\mathbb{Z}, h\neq 0 \\ j+h(q-1)\in[0,k-2]}} \left\{ \binom{k-2-j-h(q-1)}{-h(q-1)} + \binom{k-2-j-h(q-1)}{-h(q-1)-p^m} \right. \\
&\left. +(-1)^{j+1} \left(\binom{k-2-j-h(q-1)}{j} + \binom{k-2-j-h(q-1)}{j-p^m} \right) \right\} \mathbf{c}^{(k+p^m)}_{j+h(q-1)} \\
&- t^j \sum_{\substack{h\in\mathbb{Z}, h\neq 0 \\ j+h(q-1)\geqslant k-1}} \left\{ \binom{k+p^m-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1} \binom{k+p^m-2-j-h(q-1)}{j} \right\} \mathbf{c}^{(k+p^m)}_{j+h(q-1)}.
\end{aligned}
$$

Hence $U(\mathbf{c}^{(k+p^m)}_j)$ agrees with

$$
\begin{aligned}
&\sum_{i=0}^{k-2} U^{(k)}_{i,j} \mathbf{c}^{(k+p^m)}_i + (-t)^j \binom{k-2-j}{j-p^m} \mathbf{c}^{(k+p^m)}_j \\
&- t^j \sum_{\substack{h\in\mathbb{Z}, h\neq 0 \\ j+h(q-1)\in[0,k-2]}} \left\{ \binom{k-2-j-h(q-1)}{-h(q-1)-p^m} + (-1)^{j+1} \binom{k-2-j-h(q-1)}{j-p^m} \right\} \mathbf{c}^{(k+p^m)}_{j+h(q-1)} \\
&- t^j \sum_{\substack{h\in\mathbb{Z}, h\neq 0 \\ j+h(q-1)\geqslant k-1}} \left\{ \binom{k+p^m-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1} \binom{k+p^m-2-j-h(q-1)}{j} \right\} \mathbf{c}^{(k+p^m)}_{j+h(q-1)}.
\end{aligned}
$$

Since $j < p^m$, we have $\binom{k-2-j}{j-p^m} = 0$. For the case of $j + h(q-1) \in [0, k-2]$, we also have $-h(q-1)-p^m \leqslant j-p^m < 0$ and $\binom{k-2-j-h(q-1)}{-h(q-1)-p^m} = \binom{k-2-j-h(q-1)}{j-p^m} = 0$. This proves (1).

Next we assume $k \leqslant p^m$ and $j \in [k-1, p^m-1]$. For any $i \in [0, k-2]$, the element $U^{(k+p^m)}_{i,j}$ is equal to

$$
-t^j \left\{ \binom{k+p^m-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1} \binom{k+p^m-2-j-h(q-1)}{j} \right\}
$$

if we can write $i = j + h(q-1)$ with some $h \neq 0$, and zero otherwise. Since $i \leqslant k-2$, in the former case we have $k-2-j-h(q-1) \geqslant 0$ and Lemma 2.5 implies

$$\binom{k-2-j-h(q-1)+p^m}{-h(q-1)} = \binom{k-2-j-h(q-1)}{-h(q-1)} + \binom{k-2-j-h(q-1)}{-h(q-1)-p^m},$$
$$\binom{k-2-j-h(q-1)+p^m}{j} = \binom{k-2-j-h(q-1)}{j} + \binom{k-2-j-h(q-1)}{j-p^m}.$$

Since $i = j+h(q-1) \in [0, k-2]$ and $j < p^m$, we have $\binom{k-2-j-h(q-1)}{-h(q-1)-p^m} = \binom{k-2-j-h(q-1)}{j-p^m} = 0$ as is seen above. Since $j \geqslant k-1$, we also have $\binom{k-2-j-h(q-1)}{-h(q-1)} = \binom{k-2-j-h(q-1)}{j} = 0$. This proves (2) and the proposition follows. □

Let $V \in M_{k+p^m-1}(A)$ be the matrix of the right-hand side of Proposition 2.6. Let D' be the upper $(p^m-k+1) \times (p^m-k+1)$ block of D if $k \leqslant p^m$ and $D' = O$ otherwise. Put

$$\tilde{P}(X) = \det(I - XV) = P^{(k)}(X) \det(I - t^{k-1} X D')$$

and write $\tilde{P}(X) = \sum_{n=0}^{k+p^m-1} \tilde{a}_n X^n$. We denote by $\tilde{N}$ the Newton polygon of $\tilde{P}(X)$.

Corollary 2.7. *Let m and n be integers satisfying $m \geqslant 1$ and $1 \leqslant n \leqslant k+p^m-1$. Then we have*

$$v_t(a_n^{(k+p^m)} - \tilde{a}_n) \geqslant p^m + \sum_{l=1}^{n-1} \min\{l-1, p^m\}.$$

Here the sum on the right-hand side is meant to be zero for $n = 1$.

Proof. Write

$$V = U^{(k+p^m)} + t^{p^m} W$$

with some $W \in M_{k+p^m-1}(A)$. Let $s_1 \leqslant \cdots \leqslant s_{k+p^m-1}$ be the elementary divisors of $U^{(k+p^m)}$. Since $U^{(k+p^m)}$ is 1-glissando, by Lemma 2.2 we obtain $s_l \geqslant l-1$ for any l. Then [Ked, Theorem 4.4.2] shows

$$v_t(a_n^{(k+p^m)} - \tilde{a}_n) \geqslant p^m + \sum_{l=1}^{n-1} \min\{s_l, p^m\} \geqslant p^m + \sum_{l=1}^{n-1} \min\{l-1, p^m\}.$$

□

Lemma 2.8. *Let $j_0 \geqslant 0$ be an integer. Let m and n be positive integers. Then we have*

$$p^m + \sum_{l=1}^{n-1} \min\{j_0 + l - 1, p^m\} > \begin{cases} m(n-1) & (j_0 = 0) \\ mn & (j_0 > 0). \end{cases}$$

Proof. We denote the left-hand side of the inequality by L. The case $n = 1$ follows from $L = p^m > m > 0$. For $n \geqslant 2$, first we assume $n - 2 \geqslant p^m - j_0$. Note that in this case $j_0 + l - 1 < p^m$ if and only if $l \leqslant p^m - j_0$. If $p^m \leqslant j_0$, then the minimum in the sum of the lemma is always p^m and thus $L = p^m n$. Since $n \geqslant 1$ and $p^m > m$ for $m \geqslant 1$, we have $p^m n > mn$ and the lemma follows for this case. If $p^m > j_0$, then we have

$$\begin{aligned} L &= p^m + \sum_{l=1}^{p^m - j_0} (j_0 + l - 1) + p^m(n - p^m + j_0 - 1) \\ &= \frac{1}{2}(p^m - j_0)(p^m + j_0 - 1) + p^m(n - p^m + j_0) \\ &= \frac{1}{2}p^m(2n - 1 - (p^m - j_0)) + \frac{1}{2}j_0(p^m - j_0 + 1). \end{aligned}$$

Since we are assuming $n - 2 \geqslant p^m - j_0 > 0$, we obtain $L \geqslant \frac{1}{2}p^m(n+1)$. For $m \geqslant 1$, we have $\frac{1}{2}p^m \geqslant m$ and

$$L \geqslant \frac{1}{2}p^m(n + 1) \geqslant m(n + 1) > mn.$$

Next we assume $n - 2 < p^m - j_0$. In this case, put $\varepsilon = 0$ if $j_0 = 0$ and $\varepsilon = 1$ otherwise. Then L equals

$$p^m + \sum_{l=1}^{n-1}(j_0 + l - 1) \geqslant p^m + \sum_{l=1}^{n-1}(\varepsilon + l - 1) = p^m + \frac{1}{2}(n - 1)(n - 2 + 2\varepsilon).$$

Since $\varepsilon^2 = \varepsilon$, the right-hand side is greater than $m(n - 1 + \varepsilon)$ if and only if

$$\left(n - \left(m - \varepsilon + \frac{3}{2}\right)\right)^2 + 2p^m - m(m + 1) - \frac{1}{4} > 0.$$

Since m, n and ε are integers, the first term is no less than $\frac{1}{4}$. Since we can show $2p^m > m(m + 1)$ for any p and $m \geqslant 1$, the lemma also follows for this case. $\square$

Lemma 2.9. *The part of the Newton polygon $\tilde{N}$ of $\tilde{P}(X)$ of slope less than $k - 1$ agrees with that of $N^{(k)}$.*

Proof. For any $Q(X) \in \mathbb{F}_q[[t]][X]$ and any non-negative rational number α, the Newton polygon of $Q(X)$ has a segment of slope α and width l if and only if it has exactly l roots of normalized t-adic valuation $-\alpha$. By Lemma 2.3 (2), every root of the polynomial $\det(I - t^{k-1}XD')$ has normalized t-adic valuation no more than $-(k - 1)$. Thus, for $\tilde{P}(X)$ and $P^{(k)}(X)$, the sets of roots of normalized t-adic valuation more than $-(k - 1)$ agree including multiplicities. This shows the lemma. $\square$

Theorem 2.10. *Let k and m be integers satisfying $k \geqslant 2$ and $m \geqslant 0$. Let α be a non-negative rational number satisfying $\alpha \leqslant m$ and $\alpha < k-1$. Then we have $d(k+p^m, \alpha) = d(k, \alpha)$.*

Proof. As in the proof of [Wan, Lemma 4.1], let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of slopes of the Newton polygons $N^{(k+p^m)}$ and $N^{(k)}$ which is no more than m and less than $k-1$, and renumber them so that $\alpha_i < \alpha_{i+1}$ for any i. It is enough to show $d(k+p^m, \alpha_i) = d(k, \alpha_i)$ for any i.

Suppose the contrary, and take the smallest slope $\alpha = \alpha_i$ in this set satisfying $d(k+p^m, \alpha) \neq d(k, \alpha)$. By Lemma 2.4, we have $\alpha_1 = 0$ and $d(k+p^m, 0) = d(k, 0) = 1$. Thus we may assume $m \geqslant 1$, $r \geqslant i \geqslant 2$ and $\alpha > 0$.

By Lemma 2.9, the Newton polygons $N^{(k)}$, $N^{(k+p^m)}$ and $\tilde{N}$ agree with each other on the part of slope less than α. We choose $k' \in \{k, k+p^m\}$ such that the slope α occurs in $N^{(k')}$ and let k'' be the other.

Let $(n, v_t(a_n^{(k')}))$ be the right endpoint of the segment of $N^{(k')}$ of slope α, and Q its left endpoint. Note that Q is a common vertex of the Newton polygons $N^{(k)}$, $N^{(k+p^m)}$ and $\tilde{N}$. Since the Newton polygon $N^{(k')}$ has a segment of slope zero, we have $n \geqslant 2$ and

$$v_t(a_n^{(k')}) \leqslant \alpha(n-1) \leqslant m(n-1).$$

Then Corollary 2.7 and Lemma 2.8 imply

$$v_t(a_n^{(k')}) < v_t(a_n^{(k+p^m)} - \tilde{a}_n). \tag{2.2}$$

If $k' = k$, then Lemma 2.9 shows $v_t(a_n^{(k')}) = v_t(a_n^{(k)}) = v_t(\tilde{a}_n)$ and from (2.2) we obtain $v_t(a_n^{(k+p^m)}) = v_t(\tilde{a}_n) = v_t(a_n^{(k)})$. Thus the Newton polygon $N^{(k+p^m)}$ has a segment of finite slope β with left endpoint Q. Since α is the smallest, we have $\beta \geqslant \alpha$. The equality $v_t(a_n^{(k+p^m)}) = v_t(a_n^{(k)})$ implies $\alpha = \beta$ and $d(k, \alpha) \leqslant d(k+p^m, \alpha)$. In particular, the slope α also occurs in $N^{(k+p^m)}$.

If $k' = k+p^m$, then (2.2) gives $v_t(\tilde{a}_n) = v_t(a_n^{(k+p^m)})$. Thus the Newton polygon $\tilde{N}$ has a segment of finite slope γ with left endpoint Q. Then this equality implies $\gamma \leqslant \alpha < k-1$. By Lemma 2.9, the Newton polygon $N^{(k)}$ also has a segment of slope γ with left endpoint Q. Since α is the smallest, we have $\gamma = \alpha$, and the equality above also implies that the width of the segment of slope α in $\tilde{N}$ is no less than that in $N^{(k+p^m)}$. Thus Lemma 2.9 again shows $d(k, \alpha) \geqslant d(k+p^m, \alpha)$. In particular, the slope α also occurs in $N^{(k)}$. Combining these two cases, we obtain $d(k, \alpha) = d(k+p^m, \alpha)$, which is the contradiction. This concludes the proof of Theorem 2.10. □

3. Variant for $\Gamma_0(t)$

We put

$$\Gamma_0(t) = \left\{ \gamma \in GL_2(A) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod t \right\}.$$

By a similar argument, we can show a variant of Theorem 2.10 for the Drinfeld cuspforms of level $\Gamma_0(t)$, as follows. Let $k \geqslant 2$ be an integer and $w, e \in \mathbb{Z}/(q-1)\mathbb{Z}$. Consider the character

$$\chi_e : \mathbb{F}_q^\times \to \mathbb{C}_\infty^\times, \quad d \mapsto d^e.$$

A Drinfeld cuspform of level $\Gamma_0(t)$, weight k, type w and nebentypus character χ_e is a rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = \chi_e(d)(ad-bc)^{-w}(cz+d)^k f(z) \text{ for any } z \in \Omega, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(t)$$

which vanishes at cusps. They form a $\mathbb{C}_\infty$-subspace $S_{k,w}(\Gamma_0(t), \chi_e)$ of $S_k(\Gamma_1(t))$ which is stable under the U-action. Then $S_{k,w}(\Gamma_0(t), \chi_e) \neq 0$ only if $k \bmod q-1 = 2w - e$. For any non-negative rational number α, we denote by $d(k, w, e, \alpha)$ the dimension of the generalized U-eigenspace of $S_{k,w}(\Gamma_0(t), \chi_e)$ for the eigenvalues of normalized t-adic valuation α. Since we have

$$S_k(\Gamma_1(t)) = \bigoplus_{w,e \in \mathbb{Z}/(q-1)\mathbb{Z}} S_{k,w}(\Gamma_0(t), \chi_e), \quad d(k, \alpha) = \sum_{w,e \in \mathbb{Z}/(q-1)\mathbb{Z}} d(k, w, e, \alpha),$$

the following theorem gives a refinement of Theorem 1.1.

Theorem 3.1. *Let w, e be elements of $\mathbb{Z}/(q-1)\mathbb{Z}$. Let $m \geqslant 0$ be an integer and α a non-negative rational number satisfying $\alpha \leqslant m$. Then we have*

$$k_1, k_2 > \alpha + 1,\ k_1 \equiv k_2 \bmod p^m(q-1) \Rightarrow d(k_1, w, e, \alpha) = d(k_2, w, e, \alpha).$$

Proof. It is enough to show $d(k + p^m(q-1), w, e, \alpha) = d(k, w, e, \alpha)$ for any integer $k \geqslant 2$ and non-negative rational number α satisfying $\alpha \leqslant m$ and $\alpha < k - 1$. We may assume $k \bmod q-1 = 2w - e$. Let $j_0 \in \{0, \ldots, q-2\}$ be the representative of $w - 1$. Put

$$J_{k,w} = \{ j \in \mathbb{Z} \mid 0 \leqslant j \leqslant k-2,\ j \equiv j_0 \bmod q-1 \}, \quad d_{k,w} = \sharp J_{k,w}.$$

Then $S_{k,w}(\Gamma_0(t), \chi_e)$ is spanned by $\{\mathbf{c}_j^{(k)} \mid j \in J_{k,w}\}$ [BV2, §4.3] and the representing matrix $U^{(k,w,e)} = (U_{i,j}^{(k,w,e)})_{0 \leqslant i,j \leqslant d_{k,w}-1}$ of U acting on $S_{k,w}(\Gamma_0(t), \chi_e)$ with respect to this basis is the principal submatrix of $U^{(k)}$ given by

$$U_{i,j}^{(k,w,e)} = U_{j_0+(q-1)i, j_0+(q-1)j}^{(k)}.$$

Thus we can write $U^{(k,w,e)} = t^{j_0} B$ with some $(q-1)$-glissando matrix B, and by Lemma 2.2 the l-th smallest elementary divisor s_l of the matrix $U^{(k,w,e)}$ satisfies

$$s_l \geqslant j_0 + (q-1)(l-1) \geqslant j_0 + l - 1. \tag{3.1}$$

First we consider the case $m = 0$. By Lemma 2.3 (2), every slope for $S_{k,w}(\Gamma_0(t), \chi_e)$ is no less than j_0 and thus $d(k + p^m(q-1), w, e, 0) = d(k, w, e, 0) = 0$ unless $j_0 = 0$. When $j_0 = 0$, we see as in the proof of Lemma 2.4 that $d(k, w, e, 0) = 1$ for any $k \geqslant 2$ satisfying $k \bmod q-1 = 2w - e$. Hence the theorem follows for $m = 0$.

Now we assume $m \geqslant 1$. Since $J_{k,w} \subseteq J_{k+(q-1)p^m,w}$, Proposition 2.6 implies that there exist matrices C, D satisfying

$$U^{(k+p^m(q-1),w,e)} \equiv V^{(k,w,e)} \bmod t^{p^m}, \quad V^{(k,w,e)} = \left(\begin{array}{c|c|c} U^{(k,w,e)} & O & O \\ C & t^{k-1}D & O \end{array} \right).$$

We denote by $a_n^{(k,w,e)}$ and $\tilde{a}_n^{(k,w,e)}$ the n-th coefficients of

$$\det(I - XU^{(k,w,e)}), \quad \det(I - XV^{(k,w,e)}),$$

respectively. Then the Newton polygon $N^{(k,w,e)}$ of the former agrees with that of the latter on the part of slope less than $k-1$. Moreover, [Ked, Theorem 4.4.2], (3.1) and Lemma 2.8 yield

$$v_t(a_n^{(k+p^m(q-1),w,e)} - \tilde{a}_n^{(k,w,e)}) > \begin{cases} m(n-1) & (j_0 = 0) \\ mn & (j_0 > 0). \end{cases}$$

This enables us to show the theorem just as in the proof of Theorem 2.10: when $j_0 = 0$, the first slope is zero with multiplicity one and the proof works verbatim. When $j_0 > 0$, consider the set of slopes of $N^{(k,w,e)}$ and $N^{(k+p^m(q-1),w,e)}$ which is no more than m and less than $k-1$. Let α be the minimal slope in this set satisfying $d(k + p^m(q-1), w, e, \alpha) \neq d(k, w, e, \alpha)$. Let $k' \in \{k, k + p^m(q-1)\}$ be such that the segment of slope α appears in $N^{(k',w,e)}$ and $(n, v_t(a_n^{(k',w,e)}))$ the right endpoint of this segment. Then we have

$$v_t(a_n^{(k',w,e)}) \leqslant \alpha n \leqslant mn < v_t(a_n^{(k+p^m(q-1),w,e)} - \tilde{a}_n^{(k,w,e)}).$$

With this inequality, the proof works verbatim also for this case. □

Remark 3.2. The space $S_{k,w}(GL_2(A))$ of Drinfeld cuspforms of level $GL_2(A)$, weight k and type w admits an action of the operator T_t given by

$$(T_t f)(z) = t^{k-1} f(tz) + \frac{1}{t} \sum_{\beta \in \mathbb{F}_q} f\left(\frac{z+\beta}{t}\right).$$

It is known that every eigenvalue of T_t acting on $S_{k,w}(GL_2(A))$ appears also as an eigenvalue of U acting on $S_{k,w}(\Gamma_0(t))$ (see for example [BV2, Proposition 3.3]). Bandini-Valentino [BV2, §3.5] expect that, with our normalization, the set of all finite slopes for the latter except $\frac{k-2}{2}$ is equal to the set of t-adic valuations of eigenvalues for the former, including multiplicities. If this expectation holds true, then Theorem 3.1 will also give a dimension variation of generalized T_t-eigenspaces.

4. Remarks

Computations using (2.1) with Pari/GP indicate that the slopes appearing in $S_k(\Gamma_1(t))$ have some patterns (see also [BV2, §6]). The below is a table of the case $p = q = 2$, where the bold numbers denote multiplicities.

k	slopes	k	slopes
2	$0^{\mathbf{1}}$	13	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, 4^{\mathbf{1}}, \frac{11}{2}^{\mathbf{4}}, +\infty^{\mathbf{4}}$
3	$0^{\mathbf{1}}, +\infty^{\mathbf{1}}$	14	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, 5^{\mathbf{1}}, 6^{\mathbf{5}}, +\infty^{\mathbf{4}}$
4	$0^{\mathbf{1}}, 1^{\mathbf{1}}, +\infty^{\mathbf{1}}$	15	$0^{\mathbf{1}}, 2^{\mathbf{1}}, \frac{5}{2}^{\mathbf{2}}, 6^{\mathbf{1}}, \frac{13}{2}^{\mathbf{4}}, +\infty^{\mathbf{5}}$
5	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, +\infty^{\mathbf{1}}$	16	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{3}}, 7^{\mathbf{5}}, +\infty^{\mathbf{5}}$
6	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, +\infty^{\mathbf{2}}$	17	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, \frac{7}{2}^{\mathbf{2}}, \frac{15}{2}^{\mathbf{6}}, +\infty^{\mathbf{5}}$
7	$0^{\mathbf{1}}, 2^{\mathbf{1}}, \frac{5}{2}^{\mathbf{2}}, +\infty^{\mathbf{2}}$	18	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{3}}, 8^{\mathbf{5}}, +\infty^{\mathbf{6}}$
8	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{3}}, +\infty^{\mathbf{2}}$	19	$0^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{1}}, \frac{9}{2}^{\mathbf{2}}, 8^{\mathbf{1}}, \frac{17}{2}^{\mathbf{6}}, +\infty^{\mathbf{6}}$
9	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, \frac{7}{2}^{\mathbf{2}}, +\infty^{\mathbf{3}}$	20	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{1}}, 4^{\mathbf{1}}, 5^{\mathbf{1}}, 8^{\mathbf{1}}, 9^{\mathbf{7}}, +\infty^{\mathbf{6}}$
10	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{3}}, +\infty^{\mathbf{3}}$	21	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, 4^{\mathbf{1}}, \frac{11}{2}^{\mathbf{2}}, 8^{\mathbf{1}}, \frac{19}{2}^{\mathbf{6}}, +\infty^{\mathbf{7}}$
11	$0^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{1}}, \frac{9}{2}^{\mathbf{4}}, +\infty^{\mathbf{3}}$	22	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, 5^{\mathbf{1}}, 6^{\mathbf{1}}, 8^{\mathbf{1}}, 9^{\mathbf{1}}, 10^{\mathbf{7}}, +\infty^{\mathbf{7}}$
12	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{1}}, 4^{\mathbf{1}}, 5^{\mathbf{3}}, +\infty^{\mathbf{4}}$	23	$0^{\mathbf{1}}, 2^{\mathbf{1}}, \frac{5}{2}^{\mathbf{2}}, 6^{\mathbf{1}}, 8^{\mathbf{1}}, 10^{\mathbf{1}}, \frac{21}{2}^{\mathbf{8}}, +\infty^{\mathbf{7}}$

From the table, it seems that only small denominators are allowed for slopes: In the author's computation, as is already mentioned in [BV2, §1], the only case a non-trivial denominator appears is the case of $p = 2$ and the denominator is at most 2. Moreover, it seems likely that the finite slopes of $S_k(\Gamma_1(t))$ are less than $k - 1$, and that for any n, the n-th smallest finite slope of $S_k(\Gamma_1(t))$ is bounded independently of k (say, by q^{n-1}). If the latter observations hold in general, then combined with Theorem 2.10 it follows that for any n, the n-th smallest finite slopes of $S_k(\Gamma_1(t))$ are periodic of p-power period with respect to k including multiplicities. For example, it seems from the table that the third smallest finite slopes of $S_k(\Gamma_1(t))$ in the case of $p = q = 2$ are the repetition of

$$2^{\mathbf{1}}, \frac{5}{2}^{\mathbf{2}}, 3^{\mathbf{3}}, \frac{7}{2}^{\mathbf{2}}, 2^{\mathbf{1}}, 4^{\mathbf{1}}, 3^{\mathbf{1}}, 4^{\mathbf{1}}.$$

This could be thought of as a function field analogue of Emerton's theorem [Eme] which asserts that the minimal slopes of $S_k(\Gamma_0(2))$ are periodic of period 8.

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