Dimension variation of Gouvêa-Mazur type for Drinfeld cuspforms of level $\Gamma_1(t)$

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Notation

- For any field F, denote by $\bar{F}$ its algebraic closure
- p: rational prime, q: p-power, $A = \mathbb{F}_q[t]$, $K_\infty = \mathbb{F}_q((1/t))$
- $\mathbb{C}_\infty$: the $(1/t)$-adic completion of $\overline{K_\infty}$
- $\Omega = \mathbb{C}_\infty \setminus K_\infty$: Drinfeld upper half plane
- $\Gamma_1(t) = \left\{ \gamma \in SL_2(A) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod t \right\}$
- $v_t : \mathbb{F}_q((t)) \to \mathbb{Z} \cup \{+\infty\}$: additive valuation, $v_t(t) = 1$, extended to $\overline{\mathbb{F}_q((t))} \to \mathbb{Q} \cup \{+\infty\}$

Drinfeld modular form

Definition

- A rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ is called Drinfeld modular form of weight k and level $\Gamma_1(t)$ if

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for any } z \in \Omega, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(t)$$

  and satisfies a holomorphy condition at cusps
- f is called Drinfeld cuspform if it vanishes at cusps
- $S_k(\Gamma_1(t))$: $\mathbb{C}_\infty$-vector space of DCFs of weight k and level $\Gamma_1(t)$

Slope

Definition

U: endomorphism of $S_k(\Gamma_1(t))$ defined by

$$Uf(z) = \frac{1}{t}\sum_{b\in\mathbb{F}_q} f\left(\frac{z+b}{t}\right)$$

- $S_k(\Gamma_1(t))$ has an $\mathbb{F}_q(t)$-structure preserved by U
- So we may think them over $\mathbb{F}_q(t)$, $\mathbb{F}_q((t))$ and $\overline{\mathbb{F}_q((t))}$

Definition

v_t of any eigenvalue of U is called slope, which is in $\mathbb{Q}_{\geq 0}\cup\{+\infty\}$

- $d(k,\alpha)$:= dimension of generalized eigenspace for $U \curvearrowright S_k(\Gamma_1(t))$ with eigenvalues of slope α

interlude: p-adic slope for elliptic modular forms

- For elliptic cuspform f of weight k and level $\Gamma_0(Np)$, we have analogous

$$Uf(z) = \frac{1}{p} \sum_{b=0,1,\ldots p-1} f\left(\frac{z+b}{p}\right),$$

slopes using normalized p-adic valuation and $d_0(k, \alpha)$

Gouvêa-Mazur conjecture, refuted by Buzzard-Calegari

For any integer $m \geq \alpha$,

$$\left.\begin{array}{l} k_1, k_2 \geq 2\alpha + 2 \\ k_1 \equiv k_2 \bmod p^m(p-1) \end{array}\right\} \overset{?}{\Rightarrow} d_0(k_1, \alpha) = d_0(k_2, \alpha)$$

interlude: p-adic family of eigenforms

- p-adic weight space $\mathcal{W}$: a rigid analytic space over $\mathbb{Q}_p$ with $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont.}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$
- "weight k, level $\Gamma_1(N) \cap \Gamma_0(p)$" $\leftrightarrow$ $(t \mapsto t^k) \in \mathcal{W}(\mathbb{Q}_p)$

Hida, Coleman, Coleman-Mazur,...

There exist families of elliptic eigenforms of finite slope parametrized by rigid analytic spaces over $\mathcal{W}$

Consequence (Coleman)

For any non-negative rational number α, $\exists m(\alpha) \in \mathbb{Z}_{\geq 0}$ such that

$$\left.\begin{array}{l} k_1, k_2 > \alpha + 1 \\ k_1 \equiv k_2 \bmod p^{m(\alpha)}(p-1) \end{array}\right\} \Rightarrow d_0(k_1, \alpha) = d_0(k_2, \alpha)$$

Why not mimic elliptic case?

- $\mathbb{C}_t$: t-adic completion of $\overline{\mathbb{F}_q((t))}$
- Why don't we consider an adic space $\mathfrak{W}$ with

$$\mathfrak{W}(\mathbb{C}_t) = \mathrm{Hom}_{\mathrm{cont.}}(\mathbb{F}_q[[t]]^\times, \mathbb{C}_t^\times),$$

- and try to find t-adic analytic families of Drinfeld eigenforms over $\mathfrak{W}$?

For Drinfeld case

parallel construction to elliptic case does not work (so far)

Why t-adic analytic family breaks down

Reason 1: scarce analytic characters (Jeong)

Only analytic characters $1 + t\mathbb{F}_q[[t]] \to \mathbb{C}_t^\times$ are $(t \mapsto t^c)$, $c \in \mathbb{Z}_p$

$\to$ No t-adic analytic interpolation of weights

Reason 2: no known characteristic power series (Buzzard)

$\mathbb{F}_q[[t]]^\times$ top. infinitely generated, $\mathfrak{W}$ locally non-Noetherian

$\to$ No known definition of characteristic power series

- Want to define it as a limit (of something)
- For convergence we use Noetherian assumption
- Key: "any submodule of complete module is closed" fails if non-Noether

Slope patterns

- Nonetheless there seem some patterns for slopes on $S_k(\Gamma_1(t))$

weight	slopes for $p = q = 2$
2	$0^{\mathbf{1}}$
3	$0^{\mathbf{1}}, +\infty^{\mathbf{1}}$
4	$0^{\mathbf{1}}, 1^{\mathbf{1}}, +\infty^{\mathbf{1}}$
5	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, +\infty^{\mathbf{1}}$
6	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, +\infty^{\mathbf{2}}$
7	$0^{\mathbf{1}}, 2^{\mathbf{1}}, \frac{5}{2}^{\mathbf{2}}, +\infty^{\mathbf{2}}$
8	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{3}}, +\infty^{\mathbf{2}}$
9	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, \frac{7}{2}^{\mathbf{2}}, +\infty^{\mathbf{3}}$
10	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{3}}, +\infty^{\mathbf{3}}$
11	$0^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{1}}, \frac{9}{2}^{\mathbf{4}}, +\infty^{\mathbf{3}}$
12	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{1}}, 4^{\mathbf{1}}, 5^{\mathbf{3}}, +\infty^{\mathbf{4}}$

Main theorem

Theorem (H.)

For any integer $m \geq \alpha$,

$$\left.\begin{array}{l} k_1, k_2 > \alpha + 1 \\ k_1 \equiv k_2 \bmod p^m \end{array}\right\} \Rightarrow d(k_1, \alpha) = d(k_2, \alpha)$$

Natural questions

- What if nebentypus and type allowed?
- What about higher tame level and $\wp$-adic case for $\deg(\wp) > 1$?
- Does it reflect existence of families of DMFs whatsoever?
- Are n-th smallest slopes periodic?
- Does anyone want to compute slopes for $g(X_1(\wp)) > 0$ case?

Proof: Bandini-Valentino formula and glissandoness

- Note: genus of $X_1(t)$ is zero and $\dim(S_k(\Gamma_1(t))) = k-1$

Theorem (Bandini-Valentino)

For a certain basis $\mathbf{c}_0^{(k)}, \dots, \mathbf{c}_{k-2}^{(k)}$ of $S_k(\Gamma_1(t))$, we have

$$U(\mathbf{c}_j^{(k)}) = (-t)^j \binom{k-2-j}{j} \mathbf{c}_j^{(k)} - t^j \sum_{h \in \mathbb{Z}, h \neq 0} \{*\}\, \mathbf{c}_{j+h(q-1)}^{(k)},$$

where $* = \binom{k-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1} \binom{k-2-j-h(q-1)}{j}$

- So the representing matrix $U^{(k)}$ of $U \curvearrowright S_k(\Gamma_1(t))$ with this basis is glissando, namely

entries of j-th column (starting with zeroth) are in $t^j \mathbb{F}_p$

Proof: Consequences of B-V formula

B-V formula and glissandoness imply:

- $d(k,0)=1$
- l-th smallest elementary divisor of $U^{(k)}$ is $\geq l-1$
- we have

$$U^{(k+p^m)} \equiv \left(\begin{array}{c|c|c} U^{(k)} & O & O \\ * & t^{k-1}D & O \end{array}\right) \bmod t^{p^m}$$

  with $D \in M_{p^m,p^m-k+1}(\mathbb{F}_q[[t]])$
- Let V be the matrix on RHS, then

$$\mathrm{Slopes}(V) \cap [0,k-1) = \mathrm{Slopes}(U^{(k)}) \cap [0,k-1),$$

  where $\mathrm{Slopes}(V)$ is the multiset of t-adic valuations of eigenvalues of V

Proof: Perturbation

Definition

For any $B \in M_m(\mathbb{F}_q[[t]])$, define reciprocal characteristic poly by

$$P_B(X) := \det(I - BX), \quad a_n(B) := \text{coeffcient of } X^n$$

Lemma (cf. Kedlaya's book on p-adic differential equations)

For any integers $m \geq 1$ and $n \geq 2$, we have

$$v_t(a_n(U^{(k+p^m)}) - a_n(V)) > m(n-1)$$

- $V = U^{(k+p^m)} + t^{p^m} W$ with some $W \in M_{k+p^m-1}(\mathbb{F}_q[[t]])$
- $a_n(V) = (-1)^n \sum$(prinicipal $(n \times n)$-minors of V)
- By Laplace expansion, LHS is controlled by elementary divisors of $U^{(k+p^m)}$, which are bounded below by glissandoness

Proof: Analyzing Newton polygons

- For $\alpha \in \mathbb{Q}_{\geq 0}$, multiplicity of slope α in $S_k(\Gamma_1(t))$ equals width of slope α segment of NP of $P_{U^{(k)}}(X)$

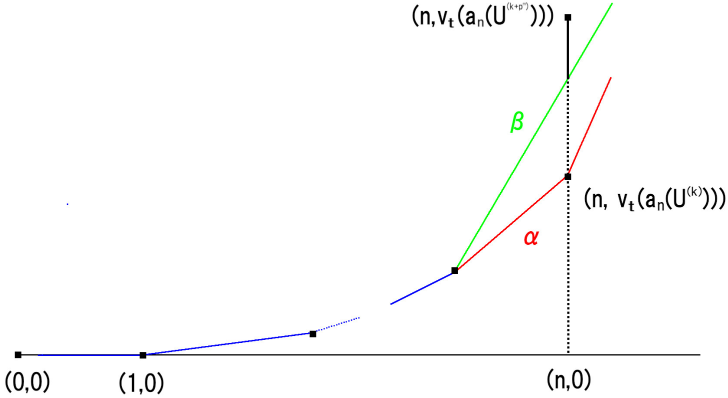
Want to show

NPs for k and $k + p^m$ agree on slope $\left\{ \begin{array}{c} < k-1 \\ \leq m \end{array} \right\}$ segments

- 1st segments: agree by $d(k,0) = 1$
- Suppose NPs agree up to N-th such segments
- Let $0 < \alpha \leq \beta$ be the slopes of the $(N+1)$-st segments, suppose $\alpha < k-1$ and $\alpha \leq m$
- Let n be the x-coordinate of the terminus of the lower $(N+1)$-st segment

Proof: Analyzing Newton polygons

- When slope α appears in NP for k,



Proof: Analyzing Newton polygons

- Picture $\Rightarrow v_t(a_n(U^{(k)})) \leq \alpha(n-1) \leq m(n-1)$
- Lemma $\Rightarrow v_t(a_n(U^{(k)})) < v_t(a_n(U^{(k+p^m)}) - a_n(V))$
- $\mathrm{Slopes}(V) \cap [0, k-1) = \mathrm{Slopes}(U^{(k)}) \cap [0, k-1)$ implies

$$v_t(a_n(V)) = v_t(a_n(U^{(k)}))$$

- These imply $v_t(a_n(U^{(k)})) = v_t(a_n(U^{(k+p^m)}))$ and thus

$$\alpha = \beta, \quad d(k+p^m, \alpha) \geq d(k, \alpha)$$

- When α appears in NP for $k + p^m$: can be treated similarly $\to$ get opposite inequality & $(N+1)$-st segments agree $\square$

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Thank you for your attention!