RAMIFICATION OF CRYSTALLINE REPRESENTATIONS

SHIN HATTORI

Abstract. This is a survey on integral p-adic Hodge theory, especially on the Fontaine-Laffaille theory, and a ramification bound for crystalline representations due to Abrashkin and Fontaine, based on the author's talks at the spring school "Classical and p-adic Hodge theories".

Contents

1. Introduction

Let p be a rational prime, k a perfect field of characteristic p and $W = W(k)$ the Witt ring for k. Put $W_n = W/p^nW$ for any $n \in \mathbb{Z}_{>0}$. We write σ for the p-th power Frobenius maps of various p-torsion rings, and also

Date: September 17, 2014.

for the natural lift $(x_0, x_1, \dots) \mapsto (x_0^p, x_1^p, \dots)$ to W of the Frobenius map on k. Put $K_0 = \mathrm{Frac}(W)$. Let K be a finite totally ramified extension of K_0 of degree e with integer ring $\mathcal{O}_K$, maximal ideal m_K and uniformizer π_K. We fix an algebraic closure $\bar{K}$ of K and let $\bar{k}$ be the residue field of $\bar{K}$, which is an algebraic closure of k. Let v_p and v_K be the p-adic valuations on $\bar{K}$ normalized as $v_p(p) = 1$ and $v_K(\pi_K) = 1$, respectively. Put $G_K = \mathrm{Gal}(\bar{K}/K)$. Let C be the completion of $\bar{K}$ and $\mathcal{O}_C$ its integer ring. Let K^{ur} be the maximal unramified extension of K inside $\bar{K}$ and $I_K = \mathrm{Gal}(\bar{K}/K^{\mathrm{ur}})$ the inertia subgroup of G_K. We denote the category of $\mathbb{Z}_p$-modules with a continuous G_K-action for the p-adic topology by $\underline{\mathrm{Rep}}_{\mathbb{Z}_p}(G_K)$.

Recall that p-adic Hodge theory has the following two aspects:

(i) to study, or classify if possible, nice p-adic G_K-representations (such as Hodge-Tate, de Rham, semi-stable and crystalline representations), using semi-linear algebraic data (such as filtered (φ, N)-modules).
(ii) to study a relation between p-adic etale cohomology of algebraic varieties over K and other cohomology theories on the differential side, such as de Rham, log-crystalline and crystalline cohomology.

Integral p-adic Hodge theory is its variant of $\mathbb{Z}_p$- or p-power torsion coefficients. Namely, its aim can be summarized as:

(i) to study, or classify if possible, subquotients of nice p-adic representations, using semi-linear algebraic data which are *in a sense* subquotients of the data attached to p-adic representations (such as filtered (φ, N)-modules).
(ii) to study a relation between etale cohomology of $\mathbb{Z}_p$- or p-power torsion coefficients of algebraic varieties over K and other cohomology theories on the differential side, such as de Rham, log-crystalline and crystalline cohomology.

Example 1.1. (i) Let A be an Abelian variety over K of dimension g. Let $T_p(A) = \varprojlim_n A[p^n](\bar{K})$ be the Tate module of A and put $V_p(A) = T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The module $T_p(A)$ is a free $\mathbb{Z}_p$-module of rank $2g$ on which G_K acts continuously (for the p-adic topology). Similarly, the module $V_p(A)$ is a p-adic G_K-representation of dimension $2g$, and moreover it is de Rham with Hodge-Tate weights in $[0, 1]$. The module $T_p(A)$ is a G_K-stable $\mathbb{Z}_p$-lattice in $V_p(A)$, and the G_K-module $A[p^n](\bar{K})$ is isomorphic to the quotient $T_p(A)/p^n T_p(A)$. These subquotients of $V_p(A)$ are typical examples of what integral p-adic Hodge theory studies on the aspect (i).

More generally, let V be a de Rham G_K-representation. Then we want to study any G_K-stable $\mathbb{Z}_p$-lattice T in V and the quotient T/T' for any such lattices $T \supseteq T'$ in V.

(ii) Let X be a proper smooth scheme over K. Then the etale cohomology groups $H^i_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Z}_p)$ and $H^i_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$ are what we want to study on the aspect (ii).

The integral theory captures more subtle information than the p-adic counterpart, such as ramification of Galois representations. We can pass from the theory of $\mathbb{Z}_p$-coefficients both to that of $\mathbb{Q}_p$-coefficients by inverting p, and to that of $\mathbb{F}_p$-coefficients by reducing modulo p. For example, let V be a semi-stable G_K-representation with associated filtered (φ, N)-module D and T a G_K-stable $\mathbb{Z}_p$-lattice in V. By integral p-adic Hodge theory, we can associate with T a semilinear algebraic data M. The data M is a finitely generated module over a coefficient ring which is a W-algebra, and M is endowed with additional structures (such as a Frobenius map). We can recover the filtered (φ, N)-module D via $M \otimes_W K_0$, and the data M/pM has information of the G_K-representation T/pT. In some cases M is defined as a W-lattice in D as we will see in §2, while it is not the case in general.

Downsides are the following: In the integral case, the coefficient ring of semi-linear algebraic data is not a field, unlike the coefficient field K_0 of filtered (φ, N)-modules. Actually, it is not even a complete discrete valuation ring but a larger W-algebra, unless the base field K is absolutely unramified (namely, $K = K_0$), and the theory becomes more complicated compared to the p-adic theory. Moreover, it is often very hard to describe explicitly the semilinear algebraic data associated with a given integral G_K-representation.

In this article, we will focus on the classical integral theory of Fontaine-Laffaille [FL82], which treats the subquotients of crystalline G_K-representations with Hodge-Tate weights in $[0, p-1]$ for the case where K is absolutely unramified. We introduce basic definitions and state main theorems of the theory in §2. We also give a sketch of proofs for some of the main theorems (Theorem 2.9 and Theorem 2.19) in §4. In fact, now we have an integral theory without these three restrictions (crystalline representations, small range of Hodge-Tate weights, absolutely unramified base field) thanks to works of many people including Breuil, Kisin and Liu. We will discuss such recent developments of the integral theory in §3.

The author would like to thank Yoshiyasu Ozeki, Takeshi Saito and Yuichiro Taguchi for helpful comments on an earlier draft of the article.
The interested reader is recommended also to consult other survey articles [Bre02, BMs02, BC14+], to which this article owes very much.

2. Fontaine-Laffaille modules

Let the notation be as in §1. Let V be a p-adic G_K-representation and $D = D^*_{\mathrm{crys}}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_K]}(V, B_{\mathrm{crys}})$ the associated filtered φ-module. The module D is a K_0-vector space satisfying $\dim_{K_0}(D) \leq \dim_{\mathbb{Q}_p}(V)$. We say V is crystalline if the equality holds. Put $D_K = D \otimes_{K_0} K$. The Hodge-Tate weights are the integers i such that $\mathrm{gr}^i D_K^\bullet \neq 0$. Note that here we adopt the convention such that the cyclotomic character $\mathbb{Q}_p(1)$ has the Hodge-Tate weight one.

Remark 2.1. This dual convention to the usual $D_{\mathrm{crys}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_K}$ is preferred in integral p-adic Hodge theory. A reason for this dual preference seems to be the following: The module $D^*_{\mathrm{crys}}(V)$ is a Hom group, and so is its quasi-inverse $V^*_{\mathrm{crys}}(D) = \mathrm{Hom}_{K_0,\varphi,\mathrm{Fil}}(D, B_{\mathrm{crys}})$. In the integral theory, we use a similar Hom group to construct Galois representations from semilinear algebraic data, as we will see later. One of the advantages of Hom and Ext compared to $\otimes$ and Tor is that we can study them in a more explicit way, by using the Yoneda extension. This makes the analysis of the associated Galois representations easier especially in the integral theory, where the coefficient ring of semilinear algebraic data is not a field and we cannot always resort to dimension calculation or flatness. However, in many cases we can develop covariant integral theories and they are also useful (for example, see §2.3).

Now we concentrate on the case of $K = K_0$. Let V be a crystalline G_{K_0}-representation with Hodge-Tate weights in $[0, p-1]$ and put $D = D^*_{\mathrm{crys}}(V)$. The module D is a K_0-vector space of dimension $\dim_{\mathbb{Q}_p}(V)$ endowed with a decreasing filtration $\{D^i\}_{i\in\mathbb{Z}}$ and a σ-semilinear bijection $\varphi : D \to D$ satisfying the admissibility condition.

The idea of the integral p-adic Hodge theory of Fontaine-Laffaille is to classify G_{K_0}-stable $\mathbb{Z}_p$-lattices in V and their quotients by φ-stable W-lattices M in D and their quotients. We have to impose on M an integral version of admissibility; namely, the conditions that $\varphi(M \cap D^i) \subseteq p^i M$ for any i and that $\sum_{i\in\mathbb{Z}} p^{-i}\varphi(M \cap D^i) = M$. In fact, these conditions originated from a structure of integral crystalline cohomology [Fon83, 1.3, Proposition]. This idea leads to the following definitions.

2.1. **Definition of Fontaine-Laffaille modules.**

Definition 2.2. (i) A Fontaine-Laffaille module over W is a triplet $(M, \{M^i\}_{i\in\mathbb{Z}}, \{\varphi^i_M\}_{i\in\mathbb{Z}})$, which we denote abusively by M, consisting of

- a W-module M,
- a decreasing filtration $\{M^i\}_{i\in\mathbb{Z}}$ of M by W-submodules satisfying $M = \bigcup_{i\in\mathbb{Z}} M^i$ and $\bigcap_{i\in\mathbb{Z}} M^i = 0$,
- a family of σ-semilinear maps $\{\varphi^i_M : M^i \to M\}_{i\in\mathbb{Z}}$ which makes the diagram

$$\begin{tikzcd} M^{i+1} \arrow[r, "\varphi_M^{i+1}"] \arrow[d, hook] & M \arrow[d, "\times p"] \\ M^i \arrow[r, "\varphi_M^i"'] & M \end{tikzcd}$$

commutative for any i. We often drop the subscript M in φ^i_M if no confusion occurs.

(ii) A morphism of Fontaine-Laffaille modules $f : M \to N$ over W is a W-linear map $f : M \to N$ between the underlying W-modules satisfying $f(M^i) \subseteq N^i$ and $f \circ \varphi^i_M = \varphi^i_N \circ f$ for any i.

(iii) A sequence of Fontaine-Laffaille modules $0 \to L \to M \to N \to 0$ is exact if the sequence on the underlying W-modules $0 \to L \to M \to N \to 0$ is exact, and also the sequence induced on the i-th filtration $0 \to L^i \to M^i \to N^i \to 0$ is exact for any i.

We denote the category of Fontaine-Laffaille modules over W by $\underline{\mathrm{MF}}_W$. We also denote by $\underline{\mathrm{MF}}_k$ its full subcategory consisting of those killed by p. With the above notion of exact sequence, the categories $\underline{\mathrm{MF}}_W$ and $\underline{\mathrm{MF}}_k$ are exact categories in the sense of Quillen [Qui73]. Thus we can define Ext^n groups via the Yoneda extension and we have long exact sequences of Hom and Ext for these categories. Though Yoneda Ext^n groups in an exact category may be a proper class in general, we can see that $\mathrm{Ext}^1_{\underline{\mathrm{MF}}_W}(M,N)$ and $\mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M,N)$ are sets, since any equivalence class of such an extension is represented by a Fontaine-Laffaille module whose underlying set is $N \times M$.

For any W-module M, put $M_\sigma = M \otimes_{W,\sigma^{-1}} W$. We identify the data $\{\varphi^i_M\}_{i\in\mathbb{Z}}$ in the above definition with a family of W-linear maps $\{\varphi^i_M : M^i \to M_\sigma\}_{i\in\mathbb{Z}}$ making the diagram there commutative. Note that if M is of finite length, then so is M_σ and $\mathrm{lg}_W(M) = \mathrm{lg}_W(M_\sigma)$.

We define a category of torsion Fontaine-Laffaille modules satisfying a torsion version of the admissibility condition, as follows.

Definition 2.3. (i) For any integers $a \le b$, we denote by $\underline{\mathrm{MF}}^{[a,b]}_W$ the full subcategory of $\underline{\mathrm{MF}}_W$ consisting of M such that $M^a = M$ and $M^{b+1} = 0$.

(ii) We denote by $\underline{\mathrm{MF}}^{\mathrm{f}}_{W,\mathrm{tor}}$ the full subcategory of $\underline{\mathrm{MF}}_W$ consisting of M such that
- M is a W-module of finite length,
- $\sum_{i\in\mathbb{Z}} \varphi^i(M^i) = M_\sigma$.

The full subcategory of $\underline{\mathrm{MF}}^{[a,b]}_W$ defined by the same conditions is denoted by $\underline{\mathrm{MF}}^{\mathrm{f},[a,b]}_{W,\mathrm{tor}}$.

(iii) We denote by $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{W,\mathrm{tor}}$ the full subcategory of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$ consisting of M such that
- There exists no non-zero quotient $M \to N$ in $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$ satisfying $N = N^{p-1}$.

This category is denoted in [FL82, Théorème 6.1] by $\underline{\mathrm{MF}}^{\mathrm{f},p'}_{\mathrm{tor}}$. Note that $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-2]}_{W,\mathrm{tor}} \subseteq \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{W,\mathrm{tor}}$.

(iv) Similar full subcategories of $\underline{\mathrm{MF}}_k$ are denoted by $\underline{\mathrm{MF}}^{[a,b]}_k$, $\underline{\mathrm{MF}}^{\mathrm{f}}_k$, $\underline{\mathrm{MF}}^{\mathrm{f},[a,b]}_k$ and $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_k$.

Finally, we define the category of free objects over W, which classifies G_{K_0}-stable $\mathbb{Z}_p$-lattices in crystalline G_{K_0}-representations.

Definition 2.4. For any integers $a \leq b$, we denote by $\underline{\mathrm{MF}}_W^{\mathrm{fd},[a,b]}$ the full subcategory of $\underline{\mathrm{MF}}_W^{[a,b]}$ consisting of M such that

- M is a free W-module of finite rank,
- M^i is a direct summand of the W-module M for any i,
- $\sum_{i\in\mathbb{Z}} \varphi^i(M^i) = M_\sigma$.

The sign fd stands for the French word *fortement divisible*, or strongly divisible (Definition 2.10).

What is striking for the category $\underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[a,b]}$ is the following.

Proposition 2.5 ([FL82], 1.10 (b))**.** *Any morphism $f : M \to N$ of the category $\underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[a,b]}$ is strict with filtrations. Namely, we have*

$$f(M^i) = f(M) \cap N^i$$

for any i.

Proof. First let M be any object of $\underline{\mathrm{MF}}_W^{[a,b]}$ and define a W-module $\bar{M}$ by the exact sequence

$$0 \longrightarrow \bigoplus_{l=a+1}^b M^l \xrightarrow{\theta_M} \bigoplus_{l=a}^b M^l \longrightarrow \bar{M} \longrightarrow 0,$$

where θ_M is given by

$$(x_{a+1}, \dots, x_b) \mapsto (x_{a+1}, -px_{a+1} + x_{a+2}, \dots, -px_{b-1} + x_b, -px_b).$$

Then the map

$$\bigoplus_{l=a}^b M^l \to M_\sigma;\ (y_a, \dots, y_b) \mapsto \sum_{l=a}^b \varphi^l(y_l)$$

induces a W-linear map $\varphi_M : \bar{M} \to M_\sigma$. These constructions are functorial, and the functor $M \mapsto \bar{M}$ is exact by the snake lemma. If the underlying W-module M is of finite length, then we have the equality

$$\lg_W(M) = \lg_W(\bar{M}).$$

Moreover, such M is contained in $\underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[a,b]}$ if and only if φ_M is surjective, and from the above length equality, it is equivalent to saying that φ_M is injective or bijective.

To prove the proposition, first we assume that f is injective, and identify M with a submodule of N. By the commutative diagram

$$\begin{tikzcd} \bar{M} \arrow[r] \arrow[d, "\wr"'] & \bar{N} \arrow[d, "\wr"] \\ M_\sigma \arrow[r, hook] & N_\sigma \end{tikzcd}$$

the induced map $\bar{M} \to \bar{N}$ is injective (note that here we cannot use the exactness of $M \mapsto \bar{M}$, since the cokernel object N/M is well-defined only if

$M^l = M \cap N^l$ for any l). On the other hand, suppose that there exists an element $x \in (N^i \cap M) \setminus M^i$. Take $j < i$ such that $x \in M^j \setminus M^{j+1}$. Since M is p-power torsion, we have $p^s x \notin M^{j+1}$ and $p^{s+1}x \in M^{j+1}$ for some integer $s \geq 0$. Replacing x by $p^s x$, we may assume $px \in M^{j+1}$. Put

$$y = (0, \dots, 0, x, -px, 0, \dots, 0) \in \bigoplus_{l=a}^{b} M^l,$$

where x is at the j-th entry. Then $y \notin \mathrm{Im}(\theta_M)$ since $x \notin M^{j+1}$, while $y \in \mathrm{Im}(\theta_N)$ since $x \in N^i \subseteq N^{j+1}$. Thus the image of y in $\bar{M}$ is non-zero, while the image in $\bar{N}$ is zero. This contradicts the injectivity of $\bar{M} \to \bar{N}$.

For a general f, put $L = f(M)$ and $L^i = f(M^i)$. Since $\mathrm{Ker}(M^i \to L^i) = \mathrm{Ker}(f) \cap M^i$ and $f \circ \varphi^i_M = \varphi^i_N \circ f$, the map φ^i_M induces a map $\varphi^i_L : L^i \to L$ for any i. Then the triplet $L = (L, \{L^i\}_{i \in \mathbb{Z}}, \{\varphi^i_L\}_{i \in \mathbb{Z}})$ is an object of $\underline{\mathrm{MF}}^{[a,b]}_W$ whose underlying W-module is of finite length, and the natural maps $M \to L \to N$ are morphisms of this category. Now we have the commutative diagram

$$\begin{tikzcd} \bar{M} \arrow[r] \arrow[d, "\wr"'] & \bar{L} \arrow[d] \\ M_\sigma \arrow[r, two heads] & L_\sigma, \end{tikzcd}$$

which yields the surjectivity of the map $\bar{L} \to L_\sigma$. This implies that $L \in \underline{\mathrm{MF}}^{\mathrm{f},[a,b]}_{W,\mathrm{tor}}$. Applying the former half of the proof to the injection $L \to N$, we obtain $f(M^i) = f(M) \cap N^i$. □

Corollary 2.6. *(i) (*[FL82]*, Proposition 1.8) The category* $\underline{\mathrm{MF}}^{\mathrm{f},[a,b]}_{W,\mathrm{tor}}$ *is Abelian.*

(ii) ([Win84]*, Proposition 1.4.1 (ii)) For any object* $M \in \underline{\mathrm{MF}}^{\mathrm{f},[a,b]}_{W,\mathrm{tor}}$*, the* W*-submodule* $M^i \subseteq M$ *is a direct summand for any* i*.*

Proof. For the first assertion, let $f : M \to N$ be a morphism of $\underline{\mathrm{MF}}^{\mathrm{f},[a,b]}_{W,\mathrm{tor}}$ and put $\mathrm{Ker}(f)^i = \mathrm{Ker}(f) \cap M^i$. Then, by using the exact sequence

$$0 \longrightarrow \mathrm{Ker}(f) \longrightarrow M \longrightarrow L \longrightarrow 0$$

with L as in the proof of Proposition 2.5, we can show that the triplet

$$(\mathrm{Ker}(f), \{\mathrm{Ker}(f)^i\}_{i \in \mathbb{Z}}, \{\varphi^i_M|_{\mathrm{Ker}(f)^i}\}_{i \in \mathbb{Z}})$$

is an object of $\underline{\mathrm{MF}}^{\mathrm{f},[a,b]}_{W,\mathrm{tor}}$, just as in the proof of the latter part of Proposition 2.5. Moreover, since the map φ^i_N induces a map $\bar{\varphi}^i_N : N^i/f(M^i) \to N/f(M)$, Proposition 2.5 implies that the Fontaine-Laffaille module

$$(\mathrm{Coker}(f), \{N^i/f(M^i)\}_{i \in \mathbb{Z}}, \{\bar{\varphi}^i_N\}_{i \in \mathbb{Z}})$$

is well-defined, and it is an object of $\underline{\mathrm{MF}}^{\mathrm{f},[a,b]}_{W,\mathrm{tor}}$ similarly. These constructions give the kernel and cokernel of f in $\underline{\mathrm{MF}}^{\mathrm{f},[a,b]}_{W,\mathrm{tor}}$. To show this category is Abelian amounts to showing that the filtrations $\{f(M^i)\}_{i\in\mathbb{Z}}$ (coimage filtration) and $\{f(M)\cap N^i\}_{i\in\mathbb{Z}}$ (image filtration) on $f(M)$ coincide, which is Proposition 2.5.

On the other hand, applying Proposition 2.5 to the map $p^n : M \to M$ yields $p^n M^i = M^i \cap p^n M$ for any n, which implies the second assertion by an Ext calculation. □

2.2. **Associated Galois representations.** In this subsection, we will attach to any object $M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$ a G_{K_0}-module $T^*_{\mathrm{crys}}(M)$ in a similar way to $V^*_{\mathrm{crys}}(D)$, using A_{crys} instead of B_{crys}. Recall that we have the projective limit ring

$$R = \varprojlim(\mathcal{O}_C/p\mathcal{O}_C \leftarrow \mathcal{O}_C/p\mathcal{O}_C \leftarrow \cdots),$$

where every transition map is the p-th power map. For any element $x = (x_0, x_1, \ldots) \in R$, take any lift $\hat{x}_l \in \mathcal{O}_C$ of x_l and put $x^{(m)} = \lim_{l\to\infty} \hat{x}_{m+l}^{p^l} \in \mathcal{O}_C$, which is independent of the choice of lifts. Then the ring R is a complete valuation ring of characteristic p with algebraically closed fraction field whose valuation is given by $v_R(x) = v_p(x^{(0)})$ for any element $x \in R$. This ring admits a natural action of G_{K_0} induced by the action on each entry. Moreover, using the natural inclusion $W(\bar{k}) \to \mathcal{O}_C$, we consider R as a k-algebra by the map $k \to R$ given by $a \mapsto ([a], [a^{1/p}], [a^{1/p^2}], \ldots)$. Then the Witt ring $W(R)$ has a natural W-algebra structure.

We also have a natural W-linear, G_{K_0}-equivariant surjection $\theta : W(R) \to \mathcal{O}_C$ sending $(r_0, r_1, \ldots) \in W(R)$ to $\sum_{l=0}^{\infty} p^l r_l^{(l)}$. Choose a system of p-power roots of $-p$ in $\mathcal{O}_C$ satisfying $((-p)^{1/p^{l+1}})^p = (-p)^{1/p^l}$ for any l. Put

$$\beta = (-p, (-p)^{1/p}, (-p)^{1/p^2}, \cdots) \in R$$

and $\xi = p + [\beta] \in W(R)$. Then we have $\mathrm{Ker}(\theta) = (\xi)$. The ring A_{crys} is by definition

$$A_{\mathrm{crys}} = W(R)[\frac{\xi^n}{n!} \mid n \in \mathbb{Z}_{\geq 0}]^\wedge,$$

where $\wedge$ means the p-adic completion.

The natural G_{K_0}-action on R induces a G_{K_0}-action on A_{crys}. Other additional structures on A_{crys} are defined as follows: The p-th power map $R \to R$ induces a natural map $\varphi : W(R) \to W(R)$ by $(r_0, r_1, \ldots) \mapsto (r_0^p, r_1^p, \ldots)$ and it extends to a ring endomorphism $\varphi : A_{\mathrm{crys}} \to A_{\mathrm{crys}}$. Note that

$$\begin{aligned}\varphi(\xi) = p + [\beta^p] = p + (\xi - p)^p &= p(1 + (-1)^p p^{p-1} + \sum_{l=1}^{p-1} \frac{1}{p}\binom{p}{l} \xi^l (-p)^{p-l} + \frac{\xi^p}{p}) \\ &= p(1 + \frac{\xi^p}{p} + p(\cdots)).\end{aligned}$$

Let $\mathrm{Fil}^i A_{\mathrm{crys}}$ be the closure of the ideal of A_{crys} generated by $\xi^n/n!$ for any $n \geq i$. Then we have $\varphi(\mathrm{Fil}^i A_{\mathrm{crys}}) \subseteq p^i A_{\mathrm{crys}}$ for $i = 0, 1, \ldots, p-1$. Put

$\varphi_i = p^{-i}\varphi|_{\mathrm{Fil}^i A_{\mathrm{crys}}}$ for such i. The above equality implies

$$\varphi_1(\xi) = 1 + \frac{\xi^p}{p} \bmod pA_{\mathrm{crys}}. \tag{1}$$

All these structures are compatible with the G_{K_0}-action. By putting

$$A^i_{\mathrm{crys}} = \begin{cases} A_{\mathrm{crys}} & (i<0) \\ \mathrm{Fil}^i A_{\mathrm{crys}} & (0 \le i \le p-1) \\ 0 & (i \ge p) \end{cases} \quad \text{and} \quad \varphi^i_{A_{\mathrm{crys}}} = \begin{cases} p^{-i}\varphi_0 & (i<0) \\ \varphi_i & (0 \le i \le p-1) \\ 0 & (i \ge p) \end{cases}$$

we consider the ring A_{crys} as a Fontaine-Laffaille module over W with a natural G_{K_0}-action. On the other hand, since $A_{\mathrm{crys}}/\mathrm{Fil}^i A_{\mathrm{crys}}$ is p-torsion free for any i, the natural map $\mathrm{Fil}^i A_{\mathrm{crys}}/p^n \mathrm{Fil}^i A_{\mathrm{crys}} \to A_{\mathrm{crys}}/p^n A_{\mathrm{crys}}$ is injective. Put $\mathrm{Fil}^i(A_{\mathrm{crys}}/p^n A_{\mathrm{crys}}) = \mathrm{Fil}^i A_{\mathrm{crys}}/p^n \mathrm{Fil}^i A_{\mathrm{crys}}$. We also have the induced map $\varphi_i : \mathrm{Fil}^i(A_{\mathrm{crys}}/p^n A_{\mathrm{crys}}) \to A_{\mathrm{crys}}/p^n A_{\mathrm{crys}}$ from $\varphi_i : \mathrm{Fil}^i A_{\mathrm{crys}} \to A_{\mathrm{crys}}$. Using these induced structures, we consider $A_{\mathrm{crys}}/p^n A_{\mathrm{crys}}$ and

$$A_{\mathrm{crys},\infty} = A_{\mathrm{crys}} \otimes_W (K_0/W) \simeq \varinjlim_n A_{\mathrm{crys}}/p^n A_{\mathrm{crys}}$$

as Fontaine-Laffaille modules over W with a natural G_{K_0}-action similarly, by setting i-th filtration to be zero for any $i \ge p$.

Let M be an object of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$. We define

$$T^*_{\mathrm{crys}}(M) = \mathrm{Hom}_{\underline{\mathrm{MF}}_W}(M, A_{\mathrm{crys},\infty}).$$

This is a W-module with a G_{K_0}-action induced by the natural action on $A_{\mathrm{crys},\infty}$. This construction is functorial on M, and we obtain a functor

$$T^*_{\mathrm{crys}} : \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}} \to \underline{\mathrm{Rep}}_{\mathbb{Z}_p}(G_{K_0}).$$

Remark 2.7. In [FL82, Théorème 3.3], an associated G_{K_0}-module with M is defined as $\underline{U}(M) = \mathrm{Ext}^1_{\underline{\mathrm{MF}}_W}(M, S_{\mathrm{FL}})$ by using the subring $S_{\mathrm{FL}} = W(R)[\frac{\xi^p}{p}]$ of A_{crys} which they denote by S [FL82, Lemme 5.4] and it is identified by [FL82, Lemme 3.8] with $\mathrm{Hom}_{\underline{\mathrm{MF}}_W}(M, S_{\mathrm{FL},\infty})$, where $S_{\mathrm{FL},\infty} = S_{\mathrm{FL}} \otimes_W (K_0/W)$. In fact, we can show that the natural map $S_{\mathrm{FL},\infty} \to A_{\mathrm{crys},\infty}$ induces an isomorphism of G_{K_0}-modules $\underline{U}(M) \simeq T^*_{\mathrm{crys}}(M)$ (Remark 4.13).

Note that $M \mapsto M \otimes_W W(\bar{k})$ defines a functor

$$\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}} \to \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W(\bar{k}),\mathrm{tor}}.$$

The effect of this base extension on the associated Galois representation is the following. Let I_{K_0} be the inertia subgroup of G_{K_0}, which we identify with the absolute Galois group of $\hat{K}^{\mathrm{ur}}_0 = \mathrm{Frac}(W(\bar{k}))$. By using the natural embedding $W(\bar{k}) \to \mathcal{O}_C$, we identify the associated I_{K_0}-module $T^*_{\mathrm{crys}}(M')$ of an object $M' \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W(\bar{k}),\mathrm{tor}}$ with

$$\mathrm{Hom}_{\underline{\mathrm{MF}}_{W(\bar{k})}}(M', A_{\mathrm{crys},\infty}).$$

From this we can see:

Lemma 2.8 ([FL82], 3.11). *For any object $M \in \underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[0,p-1]}$, the natural map*

$$T^*_{\mathrm{crys}}(M)|_{I_{K_0}} \to T^*_{\mathrm{crys}}(M \otimes_W W(\bar{k}))$$

is an isomorphism of I_{K_0}-modules.

Now the first main theorem of this article is the following:

Theorem 2.9. *(i) ([FL82], Théorème 3.3 (ii)) The functor T^*_{crys} is exact and faithful.*

(ii) ([FL82], Théorème 3.3 (i)) For any $M \in \underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[0,p-1]}$, the $\mathbb{Z}_p$-module $T^*_{\mathrm{crys}}(M)$ is of finite length. Moreover, the invariant factors of the $\mathbb{Z}_p$-module $T^*_{\mathrm{crys}}(M)$ and the W-module M are the same.

(iii) ([FL82], Théorème 6.1 (ii)) The restriction of T^*_{crys} to the full subcategory $\underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[0,p-1]'}$ is full.

(iv) The essential image of the restricted functor $T^*_{\mathrm{crys}} : \underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[0,p-1]'} \to \underline{\mathrm{Rep}}_{\mathbb{Z}_p}(G_{K_0})$ is stable under subquotients.

Next we consider the case where M is free over W. Let r be an integer satisfying $0 \leq r \leq p-1$ and M an object of $\underline{\mathrm{MF}}_{W}^{\mathrm{fd},[0,r]}$. Then the associated G_{K_0}-representation with M is defined by

$$\hat{T}^*_{\mathrm{crys}}(M) = \mathrm{Hom}_{\underline{\mathrm{MF}}_W}(M, A_{\mathrm{crys}}).$$

Note that M/p^nM is an object of $\underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[0,r]}$. From Theorem 2.9 (ii) and $M \simeq \varprojlim_n M/p^nM$, we can show that $\hat{T}^*_{\mathrm{crys}}(M)$ is a free $\mathbb{Z}_p$-module of rank equal to $\mathrm{rank}_W(M)$.

Moreover, put $D = M \otimes_W K_0$, $D^i = M^i \otimes_W K_0$ and $\varphi = \varphi^0 \otimes 1$. These define on D a structure of a filtered φ-module over K_0, which is admissible by a theorem of Laffaille [Laf80, Théorème 3.2]. From the definition, we can show that the natural map

$$\hat{T}^*_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to V^*_{\mathrm{crys}}(D) = \mathrm{Hom}_{K_0,\varphi,\mathrm{Fil}}(D, B^+_{\mathrm{crys}})$$

is an isomorphism of G_{K_0}-representations.

Conversely, let us start from a crystalline G_{K_0}-representation V with Hodge-Tate weights in $[0,r]$. Put $D = D^*_{\mathrm{crys}}(V)$, which is a filtered φ-module over K_0.

Definition 2.10. A strongly divisible lattice in D is a W-submodule M of D such that

- M is free of finite rank and $M \otimes_W K_0 = D$,
- $\varphi(M \cap D^i) \subseteq p^i M$ for any i,
- $\sum_{i\in\mathbb{Z}} \frac{1}{p^i}\varphi(M \cap D^i) = M_\sigma$.

For such M, put $M^i = M \cap D^i$ and $\varphi^i = \frac{1}{p^i}\varphi|_{M^i}$. With these structures we consider M as an object of $\underline{\mathrm{MF}}_W^{\mathrm{fd},[0,r]}$, and the structure of a filtered φ-module on D can be recovered by $D^i = M^i \otimes_W K_0$ and $\varphi = \varphi^0 \otimes 1$. Then we can show the following.

Theorem 2.11. *(i) ([FL82], Théorème 8.4 (i)) For any $r \in \{0, \ldots, p-1\}$ and any $M \in \underline{\mathrm{MF}}_W^{\mathrm{fd},[0,r]}$, the p-adic G_{K_0}-representation $V = \hat{T}^*_{\mathrm{crys}}(M) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline with Hodge-Tate weights in $[0,r]$ satisfying $D^*_{\mathrm{crys}}(V) \simeq M \otimes_W K_0$ as filtered φ-modules over K_0.*

(ii) ([Bre99], Proposition 3) Suppose $r < p-1$. Let V be a crystalline G_{K_0}-representation with Hodge-Tate weights in $[0,r]$ and put $D = D^*_{\mathrm{crys}}(V)$. Then the functor $\hat{T}^*_{\mathrm{crys}}$ induces an anti-equivalence between the category of strongly divisible lattices in D and that of G_{K_0}-stable $\mathbb{Z}_p$-lattices in V.

In fact, the first assertion is now a part of the "weakly admissible implies admissible" theorem of Colmez-Fontaine [CF00, Théorème A]. The second assertion follows from Theorem 2.9, except the essential surjectivity. For this, by the theorem of Laffaille [Laf80, Théorème 3.2], D contains a strongly divisible lattice M. Using Theorem 2.9 (iv), we can modify M to produce the strongly divisible lattice corresponding with a given G_{K_0}-stable $\mathbb{Z}_p$-lattice in V.

Definition 2.12. A G_{K_0}-representation $\bar{T} \in \underline{\mathrm{Rep}}_{\mathbb{Z}_p}(G_{K_0})$ is said to be torsion crystalline with Hodge-Tate weights in $[0,r]$ if there exist a crystalline G_{K_0}-representation V with Hodge-Tate weights in $[0,r]$ and G_{K_0}-stable $\mathbb{Z}_p$-lattices $T \supseteq T'$ in V such that $\bar{T}$ is isomorphic as a G_{K_0}-module to T/T'.

Corollary 2.13 ([BMs02], Theorem 3.1.3.3)**.** *For any $r \in \{0, \ldots, p-2\}$, the functor T^*_{crys} induces an anti-equivalence between the category $\underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[0,r]}$ and the full subcategory of $\underline{\mathrm{Rep}}_{\mathbb{Z}_p}(G_{K_0})$ consisting of torsion crystalline G_{K_0}-representations with Hodge-Tate weights in $[0,r]$.*

For this, first we can show by using Corollary 2.6 (ii) that any object $M \in \underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[a,b]}$ has a resolution in $\underline{\mathrm{MF}}_W$

$$0 \longrightarrow L' \longrightarrow L \longrightarrow M \longrightarrow 0$$

with objects $L, L' \in \underline{\mathrm{MF}}_W^{\mathrm{fd},[a,b]}$ [Win84, Proposition 1.6.3]. This and Theorem 2.11 show that the functor in Corollary 2.13 is well-defined, since we have $T^*_{\mathrm{crys}}(M) \simeq \hat{T}^*_{\mathrm{crys}}(L')/\hat{T}^*_{\mathrm{crys}}(L)$. Theorem 2.9 and Theorem 2.11 imply that it is an anti-equivalence.

As we will see in §4, the proof in [FL82] of Theorem 2.9 is based on a classification of the simple objects of the category $\underline{\mathrm{MF}}_k^{\mathrm{f},[0,p-1]}$ and an explicit description of their associated Galois representations, for the case where k is algebraically closed (Propositions 4.7 and 4.9). They also have the following interesting corollary. Let h be a positive integer. We define a character $\theta_h : I_{K_0} \to \mathbb{F}_{p^h}^\times$ by

$$\theta_h(g) = \frac{g(p^{1/(p^h-1)})}{p^{1/(p^h-1)}} \bmod m_{\bar{K}} \in \mu_{p^h-1}(\bar{k}) = \mathbb{F}_{p^h}^\times,$$

which is independent of the choice of a (p^h-1)-st root $p^{\frac{1}{p^h-1}}$ of p. It factors through the quotient I_{K_0}/P_{K_0} by the wild inertia subgroup P_{K_0}. The character θ_h is called the fundamental character of level h.

Theorem 2.14 ([FL82], Théorème 5.3)**.** *Let r be an integer satisfying $0 \leq r \leq p-1$ and M an object of $\underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[0,r]}$. Then the semi-simplification of the I_{K_0}-module $T^*_{\mathrm{crys}}(M)|_{I_{K_0}}$ is a direct sum of characters of the form*

$$\theta_h^{i_0+pi_1+\cdots+p^{h-1}i_{h-1}}$$

with integers $h \geq 1$ and $i_j \in [0,r]$.

In fact, each character appearing in the semi-simplification comes from a Jordan-Hölder factor N of the Fontaine-Laffaille module $M \otimes_W W(\bar{k})$ over $W(\bar{k})$, and the i_j's in the theorem are exactly the integers i satisfying $N^i \neq N^{i+1}$. Thus we have a complete understanding of the semi-simplification of $T^*_{\mathrm{crys}}(M)|_{I_{K_0}}$ in terms of M. As for the extension structure of these characters, we will prove an estimate of ramification of $T^*_{\mathrm{crys}}(M)$ (Theorem 2.19).

2.3. **Relation to crystalline cohomology.** Let X be a proper smooth scheme over W and put $X_n = X \times_W \mathrm{Spec}(W_n)$. For any $r \in \{0, 1, \ldots, p-1\}$, we can give the r-th de Rham cohomology group $H^r_{\mathrm{dR}}(X_n) = H^r(X_n, \Omega^\bullet_{X_n})$ a structure as an object of $\underline{\mathrm{MF}}_W^{[0,r]}$, as follows. The i-th filtration for $0 \leq i \leq r$ is the usual Hodge filtration:

$$H^r_{\mathrm{dR}}(X_n)^i = H^r(X_n, \sigma_{\geq i}\Omega^\bullet_{X_n}),$$

where $\sigma_{\geq i}\Omega^\bullet_{X_n}$ is the truncation $(0 \to \cdots \to 0 \to \Omega^i_{X_n} \to \Omega^{i+1}_{X_n} \to \cdots)$ of the de Rham complex of X_n over W_n. Note that we have the isomorphism

$$H^r(X_n, \sigma_{\geq i}\Omega^\bullet_{X_n}) \simeq H^r((X_n/W_n)_{\mathrm{crys}}, \mathcal{J}^{[i]}_{X_n/W_n}),$$

where the right-hand side is the r-th crystalline cohomology group of the i-th divided power of the natural ideal sheaf $\mathcal{J}_{X_n/W_n}$ on the crystalline site of X_n/W_n [Bert74, Chapitre V, Corollaire 2.3.7]. Hence we obtain a crystalline Frobenius map $\varphi^0 : H^r_{\mathrm{dR}}(X_n) \to H^r_{\mathrm{dR}}(X_n)$, and by [FMe87, II, 2.3] we can also define a σ-semilinear map

$$\varphi^i : H^r(X_n, \sigma_{\geq i}\Omega^\bullet_{X_n}) \to H^r_{\mathrm{dR}}(X_n).$$

Theorem 2.15 ([FMe87], II, Corollary 2.7 (ii))**.** *With these structures, $H^r_{\mathrm{dR}}(X_n)$ is an object of $\underline{\mathrm{MF}}_{W,\mathrm{tor}}^{\mathrm{f},[0,r]}$.*

For any $M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,r]}_{W,\mathrm{tor}}$, the map $\mathrm{Fil}^l A_{\mathrm{crys}} \otimes_W M^{i-l} \to A_{\mathrm{crys}} \otimes_W M$ is injective, since $A_{\mathrm{crys}}/\mathrm{Fil}^l A_{\mathrm{crys}}$ is p-torsion free and the W-submodule M^{i-l} of M is a direct summand by Corollary 2.6 (ii). Using this, we give $A_{\mathrm{crys}} \otimes_W M$ the tensor product structure as a Fontaine-Laffaille module. Namely, we put

$$(A_{\mathrm{crys}} \otimes_W M)^i = \sum_{l \in \mathbb{Z}} A^l_{\mathrm{crys}} \otimes_W M^{i-l}$$

and $\varphi^i_{A_{\mathrm{crys}} \otimes_W M} = \sum_{l \in \mathbb{Z}} \varphi^l_{A_{\mathrm{crys}}} \otimes \varphi^{i-l}_M$. We define

$$T_{\mathrm{crys}}(M) = ((A_{\mathrm{crys}} \otimes_W M)^r)^{\varphi^r = 1}(-r),$$

where $(-r)$ means the Tate twist. Note that we can show $T_{\mathrm{crys}}(M) \simeq T^*_{\mathrm{crys}}(M)^\vee$ [Bre98b, Proposition 3.2.1.7]. Then we can state a torsion version of the comparison isomorphism between etale and de Rham cohomology.

Theorem 2.16 ([FMe87], III, Remark 6.4)**.** *For any $r \in \{0, \ldots, p-2\}$, there exists a natural isomorphism of G_{K_0}-modules*

$$T_{\mathrm{crys}}(H^r_{\mathrm{dR}}(X_n)) \simeq H^r_{\mathrm{et}}(X \times_W \mathrm{Spec}(\bar{K}), \mathbb{Z}/p^n\mathbb{Z}).$$

From this and Theorem 2.14, we obtain the following corollary, which settles a conjecture of Serre [Ser72, 1.13] on tame characters appearing in torsion etale cohomology of proper smooth schemes over $\mathcal{O}_K$, for the case where K is absolutely unramified.

Corollary 2.17 ([FMe87], I, 3.2)**.** *Let X be as above. Then the semi-simplification of the I_{K_0}-module $H^r_{\mathrm{et}}(X \times_W \mathrm{Spec}(\bar{K}), \mathbb{Z}/p^n\mathbb{Z})|_{I_{K_0}}$ is a direct sum of characters of the form*

$$\theta_h^{-(i_0 + p i_1 + \cdots + p^{h-1} i_{h-1})}$$

with integers $h \geq 1$ and $i_j \in [0, r]$.

2.4. **Ramification of torsion crystalline representations.** For any finite Galois extension L/K in $\bar{K}$ and any $j \in \mathbb{R}_{\geq -1}$, we have the j-th upper numbering ramification subgroups $\mathrm{Gal}(L/K)^j \subseteq \mathrm{Gal}(L/K)$ and $G^j_K = \varprojlim_{L/K} \mathrm{Gal}(L/K)^j \subseteq G_K$ [Ser68, Chapitre IV]. Put $G^{(j)}_K = G^{j-1}_K$. We also define G^{j+}_K to be the closure of the union $\bigcup_{j'>j} G^{j'}_K$, and similarly for $G^{(j+)}_K$. Then $G^{(0+)}_K$ is equal to the inertia subgroup I_K, and G^{0+}_K is equal to the wild inertia subgroup P_K. For any finite Galois extension L/K in $\bar{K}$, the value

$$u_{L/K} = \inf\{j \in \mathbb{R}_{>0} \mid G^{(j)}_K \subseteq G_L\}$$

measures the extent of ramification of L/K. For example, $u_{L/K} = 0$ if and only if L/K is unramified and $u_{L/K} = 1$ if and only if it is tamely ramified and not unramified. It is also related to the different $\mathcal{D}_{L/K}$ [Fon85, Proposition 1.3], and used as a local term to calculate the different and the discriminant of a finite extension of number fields. In algebraic number theory, imposing local constraints in terms of ramification everywhere often yields the finiteness, and even the non-existence, of number fields. For example, the Hermite-Minkowski theorem states that for any integer d and any finite set S of primes, there exist only finitely many number fields of degree d which is unramified outside S, and that there exists no nontrivial unramified extension over $\mathbb{Q}$.

More generally, a G_K-representation V is said to be unramified (*resp.* tamely ramified) if I_K (*resp.* P_K) acts trivially on V. We want to study ramification (*resp.* wild ramification) of V, namely how far V is from unramified (*resp.* tamely ramified) ones. For any p-adic G_K-representation V of finite local monodromy (in other words, when I_K acts on V through a finite quotient), the Artin and Swan conductors

$$\mathrm{Art}(V)=\sum_{j>0} j\dim_{\mathbb{Q}_p} V^{G_K^{(j+)}}/V^{G_K^{(j)}},\ \mathrm{Sw}(V)=\sum_{j>0} j\dim_{\mathbb{Q}_p} V^{G_K^{j+}}/V^{G_K^{j}},$$

which are integers by the Hasse-Arf theorem, measure the extent of ramification and wild ramification of V, respectively. They are also used, for any number field F with absolute Galois group G_F and any p-adic G_F-representation V which factors through a finite quotient, to compute the global Artin conductor of V

$$\mathrm{Art}(V)=\prod_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_F)} \mathfrak{p}^{\mathrm{Art}(V|_{G_{F_\mathfrak{p}}})},$$

which appears in the functional equation of the Artin L-function of V.

If V is not of finite local monodromy (say V is Hodge-Tate with a non-zero Hodge-Tate weight), these definitions of conductors do not work well. When V is de Rham, one way to study ramification is to pass to the Weil-Deligne representation $D_{\mathrm{pst}}(V)$, by which any crystalline representation is considered unramified. For any number field F and any p-adic G_F-representation V which is unramified outside a finite set of places and de Rham at any place dividing p, by using $D_{\mathrm{pst}}(V)$ at places over p and Grothendieck's monodromy theorem at places over primes $l\neq p$, we can attach to V a G_{F_v}-representation of finite local monodromy for any finite place v of F and can define the global Artin conductor $\mathrm{Art}(V)$ for V.

For the case where $F=\mathbb{Q}$ and V is of dimension two over a coefficient field which is a finite extension of $\mathbb{Q}_p$, Fontaine-Mazur [FMa95] conjectured that such V is modular, namely it is isomorphic to the p-adic $G_{\mathbb{Q}}$-representation attached to an elliptic modular form of level equal to the global Artin conductor $\mathrm{Art}(V)$ and weight determined by the Hodge-Tate weights of $V|_{G_{\mathbb{Q}_p}}$. Such modularity theorems often give finiteness or non-existence results for global objects, since the set of normalized Hecke eigenforms of fixed level and weight is finite, and even empty for some cases. One of the examples of such results is the proof of Fermat's last theorem [TW95, Wil95] via proving modularity of elliptic curves over $\mathbb{Q}$ (the Shimura-Taniyama conjecture). The Fontaine-Mazur conjecture was proved in many cases by works of Kisin [Kis10] and Emerton [Eme11+].

Another way to study ramification of a p-adic G_K-representation V, which we will consider in this article, is to take a G_K-stable $\mathbb{Z}_p$-lattice T in V and consider its quotients T/p^nT. Define the finite extension L_n/K by $G_{L_n} = \mathrm{Ker}(G_K \to \mathrm{Aut}(T/p^nT))$. Then by a theorem of Sen [Sen72], there exists a constant c such that for any n we have

$$e(n-c) \leq u_{L_n/K} \leq e(n+c).$$

Though $u_{L_n/K}$ itself depends on the choice of a lattice T, we may consider that such an estimate of ramification reflects how good V is. In fact, for the case where V is crystalline with Hodge-Tate weights in $[0,1]$, the G_K-module T/p^nT is isomorphic to the one attached to a finite flat (commutative) group scheme over $\mathcal{O}_K$, and we have the following upper bound of ramification:

Theorem 2.18. *(*[Fon85]*, Théorème A and Corollaire) Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ which is killed by p^n. Then the subgroup $G_K^{(j)}$ acts trivially on the G_K-module $\mathcal{G}(\bar{K})$ for any $j > e(n+1/(p-1))$. Namely, if we define a finite Galois extension L/K by $G_L = \mathrm{Ker}(G_K \to \mathrm{Aut}(\mathcal{G}(\bar{K})))$, then we have*

$$u_{L/K} \leq e(n + \frac{1}{p-1}).$$

Moreover, we have $v_K(\mathcal{D}_{L/K}) < e(n+\frac{1}{p-1})$, where $\mathcal{D}_{L/K}$ is the different of L/K.

Fontaine's proof makes an essential use of the fact that $\mathcal{G}$ is locally of complete intersection over $\mathcal{O}_K$, while the special case of $n=1$ and $e=1$ was obtained independently by Abrashkin [Abr87] via the theory of Honda systems (§2.5). Combining this theorem with the Odlyzko bound of the root discriminant of a number field, Abrashkin and Fontaine deduced that there exists no Abelian variety with everywhere good reduction over $\mathbb{Q}$, and even over a couple of other small number fields [Abr87, Fon85]. Note that Faltings [Fal83] proved that for any number field F, any finite set S of places of F and any positive integer g, there exist finitely many isomorphism classes of Abelian varieties over F of dimension g which has good reduction outside S. These generalizations of the Hermite-Minkowski theorem to Abelian varieties had originally been conjectured by Shafarevich.

On the other hand, Fontaine also conjectured that, for any proper smooth scheme X over $\mathcal{O}_K$, the subgroup $G_K^{(j)}$ acts trivially on the G_K-module

$$H^i_{\mathrm{et}}(X \times_{\mathcal{O}_K} \mathrm{Spec}(\bar{K}), \mathbb{Z}/p^n\mathbb{Z})$$

for any $j > e(n+i/(p-1))$ [Fon85, Remarques 2.2]. Though this conjecture is still wide open, Abrashkin and Fontaine independently gave a proof for the case where $K = K_0$, $i < p-1$ and $n=1$. Theorem 2.16 reduces it to the following theorem on ramification of Galois representations associated with Fontaine-Laffaille modules, which is the second main theorem of this article.

Theorem 2.19 ([Abr89b], §2, Assertion 8.1 and [Fon93], Théorème 2)**.** *Let $r \in \{0, \dots, p-2\}$ be an integer and M an object of the category $\underline{\mathrm{MF}}_k^{\mathrm{f},[0,r]}$. Then the subgroup $G_{K_0}^{(j)}$ acts trivially on the G_{K_0}-module $T^*_{\mathrm{crys}}(M)$ for any $j > 1 + r/(p-1)$. Namely, if we define a finite extension L/K_0 by $G_L = \mathrm{Ker}(G_{K_0} \to \mathrm{Aut}(T^*_{\mathrm{crys}}(M)))$, then we have*

$$u_{L/K_0} \leq 1 + \frac{r}{p-1},\ v_{K_0}(\mathcal{D}_{L/K_0}) < 1 + \frac{r}{p-1}.$$

Moreover, this also holds for $r = p-1$ and $M \in \underline{\mathrm{MF}}_k^{\mathrm{f},[0,p-1]'}$.

For $0 \leq r \leq p-2$, Corollary 2.13 implies that the same assertions as Theorem 2.19 hold for any torsion crystalline G_{K_0}-representation $\bar{T}$ with Hodge-Tate weights in $[0, r]$ such that $p\bar{T} = 0$. Note that $u_{L/K_0} = 0$ for $r = 0$, since Corollary 4.6 shows that the G_{K_0}-module $T^*_{\mathrm{crys}}(M)$ is unramified for any $M \in \underline{\mathrm{MF}}_k^{\mathrm{f},[0,0]}$.

2.5. **Classification of finite flat group schemes.** Another application of Fontaine-Laffaille modules is a classification theory of finite flat group schemes killed by some p-power over W. First recall the classification of finite group schemes over k via Dieudonné modules. Let $\bar{\mathcal{G}}$ be a finite group scheme over k which is killed by some p-power. Let $\bar{\mathcal{G}}^{(p)}$ be the pull-back of $\bar{\mathcal{G}}$ by the Frobenius map $\mathrm{Spec}(\sigma) : \mathrm{Spec}(k) \to \mathrm{Spec}(k)$. Let $F_{\bar{\mathcal{G}}} : \bar{\mathcal{G}} \to \bar{\mathcal{G}}^{(p)}$ and $V_{\bar{\mathcal{G}}} : \bar{\mathcal{G}}^{(p)} \to \bar{\mathcal{G}}$ be the (relative) Frobenius and the Verschiebung homomorphisms of $\bar{\mathcal{G}}$. On the other hand, a Dieudonné module over W is a W-module M endowed with a σ-semilinear endomorphism F and a σ^{-1}-semilinear endomorphism V satisfying $FV = VF = p$. Then there exists an anti-equivalence

$$\bar{\mathcal{G}} \mapsto \underline{M}^*_k(\bar{\mathcal{G}})$$

from the category of finite group schemes over k which are killed by some p-power to that of Dieudonné modules over W whose underlying W-module is of finite length, in a way that F and V correspond with $F_{\bar{\mathcal{G}}}$ and $V_{\bar{\mathcal{G}}}$, respectively [Fon77, III, Théoremè 1]. Moreover, $\underline{M}^*_k$ induces an anti-equivalence between the category of p-divisible groups over W and that of Dieudonné modules over W whose underlying W-module is free of finite rank [Fon77, Proposition 6.1].

The crystalline Dieudonné theory of Grothendieck-Messing [Gro74, Mes72] suggested that finite flat group schemes and p-divisible groups $\mathcal{G}$ over W should be classified by a pair of the Dieudonné module associated with the special fiber $\bar{\mathcal{G}} = \mathcal{G} \times_W \mathrm{Spec}(k)$ and its submodule (the Hodge filtration). Fontaine constructed such a classification via Honda systems, which were named after a work of Honda [Hon70] (see [Fon77, IV, Théorème 1] for p-divisible groups, and [Con99, Fon75] for finite flat group schemes, where we need the additional assumption that group schemes are unipotent or $p \neq 2$). Using this, Fontaine-Laffaille also obtained the following:

Theorem 2.20 ([FL82], 9.11 and Proposition 9.12)**.** *Suppose $p \neq 2$. Then there exists an anti-equivalence $\underline{\mathrm{ILM}}$ from the category of finite flat group schemes over W killed by some p-power to the category $\underline{\mathrm{MF}}^{\mathrm{f},[0,1]}_{W,\mathrm{tor}}$, with a natural isomorphism of G_{K_0}-modules*

$$\mathcal{G}(\bar{K}) \simeq T^*_{\mathrm{crys}}(\underline{\mathrm{ILM}}(\mathcal{G})).$$

The same assertion holds for any p if we restrict to the category of finite flat unipotent group schemes and $\underline{\mathrm{MF}}^{\mathrm{f},[0,1]'}_{W,\mathrm{tor}}$.

The Fontaine-Laffaille module $M = \underline{\mathrm{ILM}}(\mathcal{G})$ for a finite flat group scheme $\mathcal{G}$ over W is defined as follows: the underlying W-module M is $M = \underline{M}^*_k(\bar{\mathcal{G}}) \otimes_{W,\sigma} W$ for the special fiber $\bar{\mathcal{G}}$. The first filtration M^1 is the image by $V : \underline{M}^*_k(\bar{\mathcal{G}}) \to \underline{M}^*_k(\bar{\mathcal{G}}) \otimes_{W,\sigma} W$ of the Hodge filtration $L \subseteq \underline{M}^*_k(\bar{\mathcal{G}})$. The restricted map $V|_L$ is injective, and the Frobenius structure on M is defined by $\varphi^0 = F$ and $\varphi^1 = V^{-1}$.

3. Further developments

3.1. **Applications and generalizations of the Fontaine-Laffaille theory.**

Typical applications of the Fontaine-Laffaille theory are as follows:

(i) Studying crystalline deformation rings, especially to prove modularity of Galois representations for number fields unramified over p [BeK13, CHT08, DFG04, FMa95, Fuj06+, Ram93, Tay03, Tay06, Tay08, TW95, Wes04, Wil95].

(ii) Studying the reduction modulo p of G_{K_0}-stable $\mathbb{Z}_p$-lattices in crystalline G_{K_0}-representations, and of Galois representations attached to modular forms of level prime to p [BMz02, Dim05], including a generalization of Serre's conjecture [Ser87] on modularity of mod p $G_{\mathbb{Q}}$-representations to totally real number fields unramified over p [Gee11].

(iii) Proving a rareness of global objects satisfying strong local constraints everywhere [Jos99], including a vanishing of Hodge numbers for algebraic varieties over $\mathbb{Q}$ with everywhere good reduction [Abr89b, Fon93], which will be explained in this volume.

Integral p-adic Hodge theory itself has also been highly developed and used for applications where original theories stated above are not available—under ramification of the base field K or with larger range of Hodge-Tate weights.

First we remark that the theory of Honda systems was generalized to the case where the base field is ramified by Fontaine himself [Fon77] to obtain a classification of p-divisible groups over $\mathcal{O}_K$ for the case of $e < p-1$, and also their classification up to isogeny for any e. For finite flat group schemes over $\mathcal{O}_K$, a classification via Honda systems was obtained by Conrad [Con99] for the case of $e < p-1$ and applied to the proof of a case of the Shimura-Taniyama conjecture which was not treated by Taylor-Wiles [CDT99].

Note that Theorem 2.9 does not hold for $r = p-1$, a counterexample with $p = 2$ being given by the isomorphism of G_{K_0}-modules $\mu_2 \to \mathbb{Z}/2\mathbb{Z}$. By considering a category of filtered G_{K_0}-modules, Abrashkin constructed a modified version of the functor T^*_{crys} which is fully faithful [Abr89a] and deduced the above vanishing of Hodge numbers for weight $p-1$ [Abr89b].

The assumption $r \leq p-1$ is crucial for the Fontaine-Laffaille theory, since the map $p^{-i}\varphi$ on $\mathrm{Fil}^i A_{\mathrm{crys}}$ is well-defined only for $i \leq p-1$. This was one of the difficulties for studying subquotients of crystalline representations with larger range of Hodge-Tate weights. Integral theories without any restriction on Hodge-Tate weights have been developed in two ways: theories of Wach modules and Kisin modules.

3.2. **Wach modules.** Consider the infinite extension $K(\zeta_{p^\infty}) = \bigcup_n K(\zeta_{p^n})$ of K obtained by adjoining all the p-power roots of unity. It is a Galois extension over K which is arithmetically profinite with Galois group Γ_K, and thus the theory of fields of norms of Fontaine-Wintenberger [Win83] can be applied. This theory shows that the Galois group $G_{K(\zeta_{p^\infty})} = \mathrm{Gal}(\bar{K}/K(\zeta_{p^\infty}))$ is isomorphic to the absolute Galois group of the field of Laurent power series $l((\pi))$ over the residue field l of $K(\zeta_{p^\infty})$. Thus the category of $G_{K(\zeta_{p^\infty})}$-modules is equivalent to that of $G_{l((\pi))}$-modules.

Fontaine [Fon90] showed that the latter is also equivalent to a category of φ-modules over the p-adic completion $\mathcal{O}_{\mathcal{E}}$ of the ring of Laurent power series $W(l)[[\pi]][1/\pi]$. Here, for any ring B endowed with a ring endomorphism $\varphi : B \to B$, a φ-module over B is a B-module M with a φ-semilinear map $\varphi_M : M \to M$ which we often denote abusively by φ. We can define a natural Frobenius map φ on the ring $\mathcal{O}_{\mathcal{E}}$, and we consider φ-modules over $\mathcal{O}_{\mathcal{E}}$ with respect to this map. The ring $\mathcal{O}_{\mathcal{E}}$ has a natural Γ_K-action, and he also showed that the category of G_K-modules which are finite over $\mathbb{Z}_p$ (*resp.* $\mathbb{Q}_p$) is equivalent to a category of such φ-modules over $\mathcal{O}_{\mathcal{E}}$ (*resp.* $\mathcal{E} = \mathrm{Frac}(\mathcal{O}_{\mathcal{E}})$) with a compatible Γ_K-action, which are called (φ, Γ_K)-modules.

For the case of $K = K_0$, we can study crystalline G_{K_0}-representations and their subquotients with this construction: The theory of Wach modules [Berg04, Col99, Wac96, Wac97] tells us that for any crystalline G_{K_0}-representation V with non-negative Hodge-Tate weights, there exists a nice (φ, Γ_{K_0})-stable $W[[\pi]][1/p]$-lattice $\mathbb{N}(V)$ in its associated (φ, Γ_{K_0})-module $\mathbb{D}(V)$ over $\mathcal{E}$ such that $V \mapsto \mathbb{N}(V)$ gives an equivalence of categories. Moreover, there also exists an equivalence between the category of G_{K_0}-stable $\mathbb{Z}_p$-lattices in V and a category of (φ, Γ_{K_0})-stable $W[[\pi]]$-lattices in $\mathbb{N}(V)$.

Wach modules are typically used for a study of the reduction modulo p of crystalline G_{K_0}-representations with larger range of Hodge-Tate weights [Berg12, BLZ04, CD11, Dou10]. Another interesting feature of Wach modules is a relation to Iwasawa theory [Ben00, BB08, Berg03, LLZ10, LLZ11, LeZ12, LoZ13].

The theory of Wach modules generalizes to a case where K is absolutely ramified [KR09] by using the Lubin-Tate extension for K instead of the cyclotomic extension $K_0(\zeta_{p^\infty})$, to give a classification of G_K-stable $\mathcal{O}_F$-lattices in a class of crystalline G_K-representations with coefficients in a finite extension F of $\mathbb{Q}_p$ which is called F-analytic. Such a Lubin-Tate version of (φ, Γ_{K_0})-modules has also been investigated in an attempt to generalize the p-adic Langlands correspondence for $GL_2(\mathbb{Q}_p)$ to p-adic local fields other than $\mathbb{Q}_p$ [Berg13+, FX13].

3.3. **Breuil modules.** On the other hand, for any integer r satisfying $0 \leq r \leq p-1$, let us think of generalizing the Fontaine-Laffaille theory to semi-stable representations, by naïvely adding a monodromy operator N to Fontaine-Laffaille modules $M \in \underline{\mathrm{MF}}_W^{\mathrm{fd},[0,r]}$. Then dividing the usual relation $N\varphi = p\varphi N$ by p^{i+1} yields $N\varphi^{i+1} = \varphi^i N$ for any i, which forces us to impose the Griffiths transversality $N(M^{i+1}) \subseteq M^i$ on M, and thus $N(D^{i+1}) \subseteq D^i$ on the filtered (φ, N)-module $D = M \otimes_W K_0$. However, admissible filtered (φ, N)-modules over K do not satisfy it in general. This difficulty was resolved by Breuil [Bre97], who proved that the category of filtered (φ, N)-modules over K is equivalent to a category of filtered (φ, N)-modules over a ring S_{K_0} satisfying the Griffiths transversality. Here the ring S_{K_0} is defined as follows: Let π_K be a uniformizer of K and $E(u) \in W[u]$ the minimal polynomial of π_K over W. Then we put

$$S = W[u][\frac{E(u)^n}{n!} \mid n \in \mathbb{Z}_{\geq 0}]^\wedge,\ S_{K_0} = S \otimes_W K_0,$$

where $\wedge$ means the p-adic completion. The ring S has a natural divided power filtration $\mathrm{Fil}^i S$ for $i \geq 0$, a σ-semilinear Frobenius endomorphism φ sending u to u^p, a derivation N satisfying $N(u) = -1$, and a φ-semilinear map $\varphi_i : \mathrm{Fil}^i S \to S$ for $0 \leq i \leq p-1$, which define a similar structure on S_{K_0}. For any filtered (φ, N)-module D over K, the equivalence is given by $D \mapsto \mathcal{D} = D \otimes_{K_0} S_{K_0}$ with the induced φ, N and a filtration defined using the isomorphism $\mathcal{D}/E(u)\mathcal{D} \simeq D_K$. Note that the ring $S/p^n S$ had also appeared in a work of Kato [Kat94].

Using this construction, Breuil generalized the Fontaine-Laffaille theory to the case of semi-stable representations with Hodge-Tate weights in $[0, r]$. Let D be a filtered (φ, N)-module over K satisfying $D_K^0 = D_K$ and $D_K^{r+1} = 0$. His idea is to consider S-lattices in $\mathcal{D} = D \otimes_{K_0} S_{K_0}$ and their subquotients, instead of W-lattices in D itself. In other words, we consider a category of filtered S-modules $\mathcal{M}$ with a Frobenius structure and a derivation, which are called Breuil modules. It turned out that we have to keep only the last filtration $\mathrm{Fil}^r \mathcal{M}$ in the definition of filtered S-modules, to make a variant of Proposition 2.5 hold. Then Theorem 2.9 and Theorem 2.11 were generalized to this semi-stable variant by Breuil [Bre98a, Bre99] (when $e = 1,\ r < p-1$) and Caruso [Car06] (when $er < p-1$). Theorem 2.14 and Theorem 2.16 were also generalized under these assumptions [Bre98a, Bre98b, Car06, Car08], which yield the aforementioned conjecture of Serre on tame characters appearing in torsion etale cohomology, even for the case of semi-stable reduction. Note that this control on tame characters was applied to show the non-existence or a rareness of global objects having good properties everywhere [Oze11, Oze13b, OT14]. Moreover, a classification of finite flat group schemes and p-divisible groups over $\mathcal{O}_K$ via Breuil modules was also obtained by Breuil himself [Bre00] for $p \geq 3$.

Breuil modules have also been applied to modularity theorems [Sav04, Sav05], including the complete proof of the Shimura-Taniyama conjecture [BCDT01], to a study of the reduction modulo p of semi-stable representations [CS09, CS10] and to that of generalized Serre conjectures [EGH13, Gee11, GLS12].

3.4. **Breuil-Kisin classification of finite flat group schemes.** Another breakthrough on integral p-adic Hodge theory was also brought by Breuil. In search of a classification of finite flat group schemes over $\mathcal{O}_K$ which is valid for $p = 2$, he proposed the idea of a classification via φ-modules over $W[[u]]$ and prove such a classification for the case where group schemes are killed by $p \geq 3$ [Bre98+, Bre02].

One of the advantages of this theory is that we can control the G_K-module $\mathcal{G}(\bar{K})$ associated with a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ very easily. In fact, choose a system of p-power roots $(\pi_n)_{n\geq 0}$ of π_K satisfying $\pi_0 = \pi_K$, $\pi_{n+1}^p = \pi_n$ and put $K_\infty = \bigcup_n K(\pi_n)$. This is an arithmetically profinite non-Galois extension over K and thus, via the theory of fields of norms [Win83], the category of G_{K_∞}-modules is equivalent to a category of φ-modules over $\mathcal{O}_\mathcal{E} = W[[u]][1/u]^\wedge$. Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ which is killed by $p \geq 3$, and let M be the φ-module over $\mathcal{O}_\mathcal{E}/p\mathcal{O}_\mathcal{E} = k((u))$ associated with the G_{K_∞}-module $\mathcal{G}(\bar{K})|_{G_{K_\infty}}$. Then Breuil proved that $\mathcal{G}$ is classified by a φ-stable $k[[u]]$-lattice $\mathfrak{M}$ in M such that the cokernel of the map $1 \otimes \varphi : k[[u]] \otimes_{\varphi, k[[u]]} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)$, and thus we can recover $\mathcal{G}(\bar{K})|_{G_{K_\infty}}$ from $\mathfrak{M}$. We do not lose information by the restriction to G_{K_∞}, since he also proved that the restriction functor $\mathcal{G}(\bar{K}) \mapsto \mathcal{G}(\bar{K})|_{G_{K_\infty}}$ from the category of G_K-modules attached to finite flat group schemes over $\mathcal{O}_K$ killed by some p-power to that of G_{K_∞}-modules is fully faithful.

Kisin established such a classification valid for any finite flat group schemes killed by some p-power and any p-divisible groups over $\mathcal{O}_K$ for $p \geq 3$ [Kis06, Kis09a], and combining it with his modified Taylor-Wiles patching argument, he generalized modularity theorems to the case where the base number field is highly ramified over p [Kis09a]. The φ-module $\mathfrak{M}$ is often referred to as the Breuil-Kisin module associated with $\mathcal{G}$.

Breuil-Kisin's classification has been generalized in various ways. Firstly, the classification theory of finite flat group schemes and p-divisible groups over $\mathcal{O}_K$ was completed, after a work of Kisin [Kis09b] treating the case where $p = 2$ and group schemes are unipotent, independently by Kim

[Kim12], Lau [Lau10+] and Liu [Liu13]. We also have a similar classification of finite flat group schemes and p-divisible groups over regular rings [Kim13+, Lau10, VZ10], whereas the theory of displays [Zin02] provides a classification theory of formal p-divisible groups applicable for more general base rings, via much more complicated semilinear algebraic data.

The Breuil-Kisin classification of finite flat group schemes turned out to be useful also for studying the Hecke operators at places dividing p on the space of p-adic modular forms, via an analysis of canonical subgroups [Hat13, Hat14a, Kas13, Tia11+].

3.5. **Kisin modules and $(\varphi, \hat{G})$-modules.** Secondly, for any $r \geq 0$, Kisin studied φ-modules $\mathfrak{M}$ over $W[[u]]$ such that the cokernel of the map $1 \otimes \varphi_{\mathfrak{M}} : W[[u]] \otimes_{\varphi, W[[u]]} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$, which are often called Kisin modules, and also those endowed with an endomorphism N on $(\mathfrak{M}/u\mathfrak{M}) \otimes_W K_0$, which are called (φ, N)-modules over $W[[u]]$. He proved that the category of admissible filtered (φ, N)-modules D satisfying $D_K^0 = D_K$ can be considered as a full subcategory of the isogeny category of (φ, N)-modules over $W[[u]]$. This new classification of semi-stable G_K-representations was used to prove the existence of potentially semi-stable deformation rings [Kis08], which led to proving many cases of the Breuil-Mézard conjecture and the Fontaine-Mazur conjecture [Kis09c] and to a construction of the p-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ [Kis10].

Kisin's theory also enables us to have a control on G_{K_∞}-stable $\mathbb{Z}_p$-lattices in semi-stable G_K-representations. Moreover, for $r \leq p-1$, by tensoring the ring S over $W[[u]]$ along the map φ with Kisin modules, we obtain Breuil modules without derivation, and to recover not only the G_{K_∞}-action but the G_K-action amounts to giving it a derivation. Using these features, Liu generalized Theorem 2.11 to semi-stable representations with Hodge-Tate weights in $[0, p-2]$ which is valid for any K, in terms of Breuil modules [Liu08] (the unipotent case with Hodge-Tate weights in $[0, p-1]$ was shown by [Gao14+]). Kisin's theory was generalized to the case where the residue field k is imperfect [BT08], using a framework of p-adic Hodge theory for imperfect residue fields [Bri06].

On the other hand, since K_∞/K is not Galois, it is not so easy to study the action of the whole Galois group G_K on semi-stable G_K-representations via Kisin's theory, unless the range of Hodge-Tate weights is small. Liu overcame this difficulty, by considering the Galois closure $K_\infty(\zeta_{p^\infty})/K$ of K_∞, its Galois group $\hat{G} = \mathrm{Gal}(K_\infty(\zeta_{p^\infty})/K)$ and a category of φ-modules $\mathfrak{M}$ over $W[[u]]$ with a $\hat{G}$-action on a scalar extension $\mathfrak{M} \otimes_{W[[u]], \varphi} \hat{\mathcal{R}}$, which are called $(\varphi, \hat{G})$-modules [Liu07, Liu10]. He proved an anti-equivalence between a category of $(\varphi, \hat{G})$-modules and the category of G_K-stable $\mathbb{Z}_p$-lattices in semi-stable G_K-representations generalizing Theorem 2.11, without any restriction on Hodge-Tate weights nor on the base field K. It was also applied to a study of generalized Serre conjectures [GLS14]. For the case of $K = K_0$, the G_{K_0}-stable $\mathbb{Z}_p$-lattices in crystalline G_{K_0}-representations are classified both by Wach modules and $(\varphi, \hat{G})$-modules, and the attached Wach module to such a lattice can be recovered from the attached $(\varphi, \hat{G})$-module [Liu13+]. More generally, we can develop a variant of the theory of (φ, Γ_{K_0})-modules using $\hat{G}$ instead of Γ_{K_0} [Car13, Oze13a, Tav11].

Using these developments on integral p-adic Hodge theory, Theorem 2.19 was also generalized for subquotients of semi-stable representations, which have a worse ramification bound [BMs02, Hat09, Car13, CL11], and it was applied to a vanishing of Hodge numbers of algebraic varieties over $\mathbb{Q}$ with everywhere good reduction except semi-stable reduction at 3 [Abr13]. Abrashkin recently found a new method to prove such a ramification bound of G_K-representations, by using a non-canonical isomorphism between two complete discrete valuation fields of equal characteristic [Abr12+, Abr14+].

3.6. **Ramification theory.** The classical ramification theory as reviewed in §2.4 has been typically applied to:

(i) Definitions and calculations of invariants of global objects via local terms concerning ramification, such as the global Artin conductor and the Grothendieck-Ogg-Shafarevich formula to compute Euler-Poincaré characteristics of l-adic sheaves over algebraic curves [Gro77].
(ii) Proving the non-existence or a rareness of global objects satisfying strong local constraints everywhere, such as Abelian varieties [Abr87, BrK01, BrK04, BrK14, Fon85, Schf03, Schf05, Schf12, Ver13], algebraic varieties [Abr89b, Abr13, Fon93] and Galois representations [Bru99, Bru05, Jos99, Moo00, Moo03, MT03, Tat94], generalizing the Hermite-Minkowski theorem.
(iii) The theory of fields of norms of Fontaine-Wintenberger, to give an isomorphism between the absolute Galois group of an arithmetically profinite extension of a complete discrete valuation field of mixed characteristic with perfect residue field and that of a complete discrete valuation field of equal characteristic [Win83].
(iv) The theory of close local fields, including a comparison of absolute Galois groups of complete discrete valuation fields of mixed and equal characteristics with the same perfect residue field modulo ramification subgroups [Del84], especially to deduce the local Langlands correspondence of reductive groups for the equal characteristic case from the mixed characteristic case [ABPS13+, Gan13+] by combining with [Kaz86].
(v) A local version of the Grothendieck conjecture, which states that any local field can be recovered from its absolute Galois group and ramification subgroups [Abr00, Abr10, Moc97].

In the classical ramification theory as in [Ser68], it is crucial that the residue field k is perfect and thus for any finite extension L/K, the $\mathcal{O}_K$-algebra $\mathcal{O}_L$ is generated by a single element. In general, for any finite Galois extension L/K, we can define the i-th lower numbering ramification subgroup

$$\mathrm{Gal}(L/K)_i = \mathrm{Ker}(\mathrm{Gal}(L/K) \to \mathrm{Aut}(\mathcal{O}_L/m_L^{i+1})).$$

For the case where k is perfect, we can prove, by using the monogenicity of $\mathcal{O}_L$, that the j-th upper numbering ramification subgroup $\mathrm{Gal}(L/K)^j = \mathrm{Gal}(L/K)_{\psi_{L/K}(j)}$ is compatible with quotients, where $\psi_{L/K}$ is the Herbrand function. This enables us to take the projective limit to define the ramification subgroup G_K^j of G_K. However, when k is imperfect, the lower numbering ramification subgroups are not necessarily compatible with quotients even after renumbering.

After various attempts to define appropriate ramification subgroups of G_K when k is imperfect, including especially a work of Kato for Abelian extensions [Kat89], different definitions were proposed independently by Abbes-Saito [AS02, AS03] (geometric definition via connected components of analytic varieties), Borger [Bor04] (reducing to the case of perfect k via the universal residual perfection), Kedlaya [Ked07a] (differential definition via (φ, ∇)-modules over the Robba ring) and Zhukov [Abr02, Zhu03] (for higher dimensional local fields, via induction from lower dimensional ones).

Each definition has its own advantages and applications. Abbes-Saito's definition was used to define characteristic cycles of etale sheaves on higher dimensional algebraic varieties to compute their Euler-Poincaré characteristics [Sai09, Sai14+]. One of the advantages of their definition is that it also gives a ramification theory of finite flat algebras over $\mathcal{O}_K$. With this framework, Theorem 2.18 can be shown easily from the fact that multiplicative groups are minimal in the category of finite flat group schemes over $\mathcal{O}_K$ [Hat06]. This advantage was also used to prove the overconvergence of canonical subgroups of Abelian varieties [AM04, Hat13, Hat14a, Tia10, Tia12] and a generalization of the theory of close local fields to the case of imperfect k [Hat14b].

Kedlaya's definition is more flexible about base extensions than Abbes-Saito's. One of its consequences is that we can show the integrality of the Swan conductor generalizing the classical Hasse-Arf theorem much more easily [Ked07a], via a reduction to the case of perfect k by adding p-basis to k and "rotate" in the enlarged space to make ramification concentrate on the uniformizer direction. This differential version of ramification theory was used to prove a conjecture of Shiho [Shi02, Conjecture 3.1.8] on semistable reduction of overconvergent F-isocrystals on smooth varieties over a perfect field of characteristic p [Ked07b, Ked08, Ked09, Ked11]. In Zhukov's definition, ramification subgroups are multi-indexed, and it is suitable for a detailed study of the structure of G_K. It was used for a proof of a two dimensional analogue of the Grothendieck conjecture [Abr03] and a generalization of the theory of fields of norms to higher dimensional local fields [Abr07]. Note that the theory of higher fields of norms was also obtained in different ways for more general settings [And06, Schl06] and generalized to the theory of perfectoid spaces [Schz12].

Among these different definitions, the comparison between Abbes-Saito's and Kedlaya's has been well studied. Note that a differential interpretation of the classical ramification theory had been investigated, for example, by [Cre00, Ked05, Mat02, Tsu98]. For the case of imperfect k, after works of Matsuda [Mat04] and Chiarellotto-Pulita [CP09], Xiao proved that these two definitions (and also Borger's for the equal characteristic case) yield the same Artin and Swan conductors for almost all the cases [Xia10, Xia12], from which the integrality of the conductors for Abbes-Saito's definition follows.

4. Sketch of proofs

In this section, we sketch the proofs of Theorem 2.9 and Theorem 2.19, following [FL82] and [Fon93]. The key ideas for the proofs are twofold: First, by a dévissage argument, we reduce a large part of the proofs to a study of simple objects in the category $\underline{\mathrm{MF}}_{\bar{k}}^{\mathrm{f},[0,p-1]}$, which we can classify completely. Second, passing from the period ring $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$ to $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ by using the first projection $\mathrm{pr}_1 : R \to \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$, we reduce a study of the G_{K_0}-module $T^*_{\mathrm{crys}}(M)$ to a study of equations over W and their solutions in $\mathcal{O}_{\bar{K}}$. The latter enables us especially to prove the aforementioned estimate of ramification. Note that, as we remarked before (Remark 2.7), the original arguments in [FL82] used the subring S_{FL} instead of A_{crys}, while almost verbatim arguments are valid also for our period ring A_{crys} which is more standard these days. To mediate these two period rings and also to make the notation simpler, in this article we will mainly work on the ring

$$A = \mathrm{Im}(S_{\mathrm{FL}}/pS_{\mathrm{FL}} \to A_{\mathrm{crys}}/pA_{\mathrm{crys}}).$$

4.1. **Cutting off higher divided powers.** Put $R^{\mathrm{DP}} = A_{\mathrm{crys}}/pA_{\mathrm{crys}}$. We consider this ring as an object of $\underline{\mathrm{MF}}_{k}^{[0,p-1]}$ as before. Note that a divided power structure on this ring is induced from that of A_{crys}: we put the l-th divided power $\gamma_l(x)$ of $x \in \mathrm{Fil}^1 R^{\mathrm{DP}} = \mathrm{Fil}^1 A_{\mathrm{crys}}/p\mathrm{Fil}^1 A_{\mathrm{crys}}$ by

$$\gamma_l(x) = \gamma_l(\hat{x}) \bmod pA_{\mathrm{crys}} = \frac{\hat{x}^l}{l!} \bmod pA_{\mathrm{crys}}$$

with any lift $\hat{x}$ of x in $\mathrm{Fil}^1 A_{\mathrm{crys}}$. The ring R^{DP} is also the divided power envelope of the surjection $\mathrm{pr}_0 : R \to \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ by [BO78, Remarks 3.20 (8)].

Recall that we have defined the element $\beta = (-p, (-p)^{1/p}, \ldots)$ of the ring R before. The surjection $\mathrm{pr}_1 : R \to \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ induces a σ^{-1}-semilinear isomorphism of rings $R/\beta^p R \to \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ sending β to $(-p)^{1/p}$. Since $\beta^p = p!\gamma_p(\beta) = 0$ in the ring R^{DP}, we can consider the ring R^{DP} as an $(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$-algebra by the map $\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}} \simeq R/\beta^p R \to R^{\mathrm{DP}}$. For any divided power ideal I of a ring and $x \in I$, we put $\delta(x) = (p-1)!\gamma_p(x)$ and write $\delta^l(x) =$

$(\delta \circ \cdots \circ \delta)(x)$ for the l-times iteration of δ. Then we can describe the structure of the ring R^{DP} as follows:

Lemma 4.1. *The ring homomorphism*

$$(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})[Y_1, Y_2, \dots,]/(Y_1^p, Y_2^p, \dots) \to R^{\mathrm{DP}}$$

sending Y_l to $\delta^l(\beta)$ is a σ-semilinear bijection, by which we identify both sides.

Proof. Put $\xi = p + [\beta] \in W(R)$ as before. It is enough to show a similar $W(R)$-linear map

$$W(R)[Y_1, Y_2, \dots]/(\xi^p - pY_1, Y_1^p - pY_2, \dots) \to W(R)^{\mathrm{DP}} = W(R)[\frac{\xi^n}{n!} \mid n \in \mathbb{Z}_{\geq 0}]$$

defined by $Y_l \mapsto \delta^l(\xi)$ is an isomorphism. From the definition, we see that the map is surjective, and isomorphic after inverting p. Thus it is enough to show that the ring on the left-hand side is p-torsion free. Put

$$B_l = W(R)[Y_1, \dots, Y_l]/(\xi^p - pY_1, \dots, Y_{l-1}^p - pY_l).$$

For any ring B without p-torsion and any element $x \in B$, if the image of x is a regular element of B/pB, then B/xB is p-torsion free. Since the image of ξ^p is a regular element of R, we see inductively that $B_l[Y_{l+1}]/(Y_l^p - pY_{l+1})$ is p-torsion free. Taking the inductive limit yields the claim. $\square$

Using this identification, we can explicitly describe additional structures of R^{DP} such as G_{K_0}-action, filtration and Frobenius maps (see [FL82, 5.9], where they consider the subring $S_{\mathrm{FL}} = W(R)[\frac{\xi^p}{p}]$ of A_{crys} with similar explicit description modulo p). The action of $g \in G_{K_0}$ is written as

$$g(\sum_{l_1,\dots,l_n} a_{l_1,\dots,l_n} Y_1^{l_1} \cdots Y_n^{l_n}) = \sum_{l_1,\dots,l_n} g(a_{l_1,\dots,l_n}) Y_1^{l_1} \cdots Y_n^{l_n},$$

where $a_{l_1,\dots,l_n} \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. Indeed, from the definition of the divided power structure on R^{DP}, we have $\delta(x) = \delta(\hat{x}) \bmod p$ for any $x \in \mathrm{Fil}^1 R^{\mathrm{DP}}$, where $\hat{x}$ is any lift of x in $\mathrm{Fil}^1 A_{\mathrm{crys}}$. Putting $g(\beta) = \beta \underline{\varepsilon}^a$ with $\underline{\varepsilon} = (1, \zeta_p, \zeta_{p^2}, \dots)$ and $a \in \mathbb{Z}_p$, we have

$$\begin{aligned} g(\delta(\beta)) &= \delta(g(\xi)) \bmod p = \delta(\xi + [\beta]([\underline{\varepsilon}]^a - 1)) \bmod p \\ &= \delta(\xi) + \sum_{l=1}^{p-1} p^{-1} \binom{p}{l} \xi^l ([\beta]([\underline{\varepsilon}]^a - 1))^{p-l} + [\beta]^p \delta([\underline{\varepsilon}]^a - 1) \bmod p \\ &= \delta(\beta) + \beta^p \sum_{l=1}^{p-1} p^{-1} \binom{p}{l} (\underline{\varepsilon}^a - 1)^{p-l} + \beta^p \delta(\underline{\varepsilon}^a - 1) = \delta(\beta), \end{aligned}$$

since $\beta^p = 0$ in R^{DP}. Thus we obtain $g(Y_1) = Y_1$ and $g(Y_l) = Y_l$ for any l.

Moreover, the ideal $\mathrm{Fil}^i R^{\mathrm{DP}}$ of R^{DP} is written as

$$\mathrm{Fil}^i R^{\mathrm{DP}} = ((-p)^{i/p}, Y_l \mid l \geq 1) \ (i \leq p-1), \quad \mathrm{Fil}^p R^{\mathrm{DP}} = (Y_l \mid l \geq 1).$$

Recall that we consider R^{DP} as an object of $\underline{\mathrm{MF}}_k^{[0,p-1]}$ by putting $(R^{\mathrm{DP}})^i = \mathrm{Fil}^i R^{\mathrm{DP}}$ for $0 \leq i \leq p-1$ and $(R^{\mathrm{DP}})^p = 0$. For any $i \in \{0, \ldots, p-1\}$, write any element x in the i-th filtration as

$$x = x_0 + \sum_{l \geq 1} x_l Y_l + (\text{higher terms})$$

with $x_0 \in (-p)^{i/p}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ and $x_l \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. Let $\hat{x}_0$ be any lift of x_0 in $\mathcal{O}_{\bar{K}}$. From the equality (1) in §2.2, we obtain

$$\varphi^i(x) = \begin{cases} \frac{\hat{x}_0^p}{(-p)^i}(1+Y_1)^i & (i \leq p-2), \\ \frac{\hat{x}_0^p}{(-p)^i}(1+Y_1)^i + x_1^p & (i = p-1,\ p \geq 3), \\ \frac{\hat{x}_0^p}{(-p)^i}(1+Y_1)^i + \sum_{l \geq 1} x_l^p & (i = p-1,\ p = 2). \end{cases}$$

Remark 4.2. In particular, we have $\varphi^1((-p)^{1/p}) = 1 + Y_1$ in the ring R^{DP}. This implies that, contrary to what is stated in [Wac97, 2.1.3.4], there exists no non-trivial ideal I of R^{DP} stable under φ^{p-1} such that $\varphi^1((-p)^{1/p}) \equiv 1 \bmod I$. Indeed, this forces $Y_1 \in I$ and thus I contains $\varphi^{p-1}(Y_1) = 1$.

Now we put

$$A = (\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})[Y_1]/(Y_1^p) \subseteq R^{\mathrm{DP}}$$

and give it the induced structure as an object of $\underline{\mathrm{MF}}_k^{[0,p-1]}$ from R^{DP}. Namely, we put the i-th filtration A^i of A as

$$A^i = ((-p)^{i/p}, Y_1) \ (i \leq p-1), \quad A^p = 0$$

and the i-th Frobenius map of A as

$$\varphi^i(x_0 + x_1 Y_1 + \sum_{l=2}^{p-1} x_l Y_1^l) = \begin{cases} \frac{\hat{x}_0^p}{(-p)^i}(1+Y_1)^i & (i \leq p-2), \\ \frac{\hat{x}_0^p}{(-p)^i}(1+Y_1)^i + x_1^p & (i = p-1). \end{cases}$$

Then we have $\varphi^i(\mathrm{Fil}^i R^{\mathrm{DP}}) \subseteq A$ for any $i \leq p-1$. Consider the quotient R^{DP}/A as an object of $\underline{\mathrm{MF}}_k^{[0,p-1]}$ with the induced structure from R^{DP}. It satisfies $(R^{\mathrm{DP}}/A)^{p-1} = R^{\mathrm{DP}}/A$ and $\varphi^{p-1} = 0$ on R^{DP}/A. Then the following lemma enables us to replace R^{DP} by the simpler ring A to study $T^*_{\mathrm{crys}}(M)$ for the case of $pM = 0$.

Lemma 4.3. *For any object $M \in \underline{\mathrm{MF}}_k^{\mathrm{f},[0,p-1]}$, we have*

$$\mathrm{Hom}_{\underline{\mathrm{MF}}_k}(M, R^{\mathrm{DP}}/A) = \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, R^{\mathrm{DP}}/A) = 0.$$

In particular, we have

$$T^*_{\mathrm{crys}}(M) = \mathrm{Hom}_{\underline{\mathrm{MF}}_k}(M, A), \quad \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, R^{\mathrm{DP}}) = \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, A).$$

This follows from the lemma below:

Lemma 4.4. *Let $r \in \{0, \ldots, p-1\}$ be an integer and M an object of $\underline{\mathrm{MF}}_k^{\mathrm{f},[0,r]}$. Let N be an object of $\underline{\mathrm{MF}}_k$ such that $N^r = N$ and that for any $x \in N$, the iteration $(\varphi_N^r \circ \cdots \circ \varphi_N^r)(x)$ of any sufficiently large times is zero. Then we have $\mathrm{Hom}_{\underline{\mathrm{MF}}_k}(M, N) = \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, N) = 0$.*

Proof. Fix a k-basis $e_1, \ldots, e_h$ of M with $e_n \in M^{i_n} \setminus M^{i_n+1}$ and write as

$$\varphi_M^{i_n}(e_n) = \sum_{m=1}^{h} c_{m,n} e_m$$

with $P = (c_{m,n}) \in GL_h(k)$. Let $f : M \to N$ be a morphism of $\underline{\mathrm{MF}}_k$ and put $f(e_n) = x_n \in N^{i_n} = N^r = N$. Then the equality $\varphi_N^i \circ f = f \circ \varphi_M^i$ implies

$$(x_1, \ldots, x_h) = \Phi(x_1, \ldots, x_h) P^{-1}$$

with $\Phi = (\varphi_N^{i_1}, \ldots, \varphi_N^{i_h})$. By assumption, the l-th iteration $\Phi^l(x_1, \ldots, x_h)$ is zero for any sufficiently large l and thus we obtain $f = 0$ by recursion.

Next consider an extension in $\underline{\mathrm{MF}}_k$

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0.$$

Choose a lift $\hat{e}_n$ of e_n to E^{i_n}. We have

$$\varphi_E^{i_n}(\hat{e}_n) = \sum_{m=1}^{h} c_{m,n} \hat{e}_m + b_n$$

with some $b_n \in N$. Then the extension splits if and only if there exists $a_n \in N^{i_n}$ satisfying

$$\varphi_E^{i_n}(\hat{e}_n + a_n) = \sum_{m=1}^{h} c_{m,n}(\hat{e}_m + a_m),$$

which is equivalent to

$$b_n + \varphi_N^{i_n}(a_n) = \sum_{m=1}^{h} c_{m,n} a_m.$$

Thus we obtain

$$(a_1, \ldots, a_h) = (b_1, \ldots, b_h) P^{-1} + \Phi(a_1, \ldots, a_h) P^{-1}.$$

By recursion, we see that the equation has the unique solution

$$(a_1, \ldots, a_h) = \sum_{l=0}^{\infty} \Phi^l(b_1, \ldots, b_h)(P^{-1})^{\sigma^l}(P^{-1})^{\sigma^{l-1}} \cdots P^{-1},$$

since the right-hand side is a finite sum. □

Let F be any algebraic extension of K_0 in $\bar{K}$. We put

$$A_F = (\mathcal{O}_F / p\mathcal{O}_F)[Y_1]/(Y_1^p) \subseteq A = A_{\bar{K}}$$

and give this subring the induced structure as an object of $\underline{\mathrm{MF}}_k^{[0,p-1]}$ from A. For $M \in \underline{\mathrm{MF}}_k^{\mathrm{f},[0,p-1]}$, we define

$$T^*_{\mathrm{crys},F}(M) = \mathrm{Hom}_{\underline{\mathrm{MF}}_k}(M, A_F).$$

This is a subset of $T^*_{\mathrm{crys}}(M)$ on which the absolute Galois group G_F acts trivially.

For $r \in \{0, \ldots, p-2\}$ and $M \in \underline{\mathrm{MF}}_k^{\mathrm{f},[0,r]}$, the module $T^*_{\mathrm{crys},F}(M)$ can also be obtained by using a far simpler ring than A_F. Put

$$\mathfrak{b}_F = \{x \in \mathcal{O}_F \mid v_p(x) > \frac{r}{p-1}\}.$$

We consider the quotient $\mathcal{O}_F/\mathfrak{b}_F$ as an object of $\underline{\mathrm{MF}}_k^{[0,r]}$ by putting

$$(\mathcal{O}_F/\mathfrak{b}_F)^i = \{x \in \mathcal{O}_F \mid v_p(x) \geq \frac{i}{p}\}/\mathfrak{b}_F, \quad \varphi^i(x) = \frac{\hat{x}^p}{(-p)^i} \bmod \mathfrak{b}_F$$

for any $0 \leq i \leq r$ and $(\mathcal{O}_F/\mathfrak{b}_F)^{r+1} = 0$. Here $\hat{x}$ denotes any lift of x in $\mathcal{O}_F$.

Lemma 4.5. *For any $r \in \{0, \ldots, p-2\}$ and any $M \in \underline{\mathrm{MF}}_k^{\mathrm{f},[0,r]}$, the surjection $A_F \to \mathcal{O}_F/\mathfrak{b}_F$ defined by $Y_1 \mapsto 0$ induces an isomorphism*

$$T^*_{\mathrm{crys},F}(M) \to \mathrm{Hom}_{\underline{\mathrm{MF}}_k}(M, \mathcal{O}_F/\mathfrak{b}_F).$$

Proof. Since $r < p-1$, the map $\varphi^r_{A_F}$ kills Y_1. Moreover, for any $\hat{x} \in \mathcal{O}_F$ satisfying $v_p(\hat{x}) = r/(p-1) + \varepsilon$ with some $\varepsilon > 0$, we have $v_p(\hat{x}^p/(-p)^r) \geq r/(p-1) + p\varepsilon$. Thus the ideal $I = \mathfrak{b}_F A_F + (Y_1)$ of A_F, which is contained in the r-th filtration A^r_F, is stable under $\varphi^r_{A_F}$ and, for any $x \in I$, the iteration $(\varphi^r_{A_F} \circ \cdots \circ \varphi^r_{A_F})(x)$ of any sufficiently large times is zero. Thus, by considering I as an object of $\underline{\mathrm{MF}}_k$ with the induced structure from A_F, Lemma 4.4 yields $\mathrm{Hom}_{\underline{\mathrm{MF}}_k}(M, I) = \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, I) = 0$. Since $I = \mathrm{Ker}(A_F \to \mathcal{O}_F/\mathfrak{b}_F)$, the long exact sequence of Hom and Ext implies the lemma. □

For $r = 0$, the ideal $\mathfrak{b}_{\bar{K}}$ is equal to the maximal ideal $m_{\bar{K}}$ and we obtain the following corollary.

Corollary 4.6. *For any $M \in \underline{\mathrm{MF}}_k^{\mathrm{f},[0,0]}$, the G_{K_0}-module $T^*_{\mathrm{crys}}(M)$ is unramified.*

4.2. **Classification of simple objects.** Let h be a positive integer and i a map $\mathbb{Z}/h\mathbb{Z} \to \mathbb{Z}$. We write $i_n = i(n)$ for any $n \in \mathbb{Z}/h\mathbb{Z}$. We suppose that i is of period h. This means that h is the minimum among the positive integers h' satisfying $i_{n+h'} = i_n$ for any n.

For such h and i, we define an object $M(h; i)$ of the category $\underline{\mathrm{MF}}_{\bar{k}}^{\mathrm{f}}$ by

$$M(h; i) = \bigoplus_{n \in \mathbb{Z}/h\mathbb{Z}} \bar{k} e_n$$

with a basis $\{e_n\}_{n \in \mathbb{Z}/h\mathbb{Z}}$ and

$$M(h; i)^l = \bigoplus_{i_n \geq l} \bar{k} e_n, \quad \varphi^{i_n}(e_n) = e_{n-1}.$$

In particular, $e_n \in M(h;i)^{i_n} \setminus M(h;i)^{i_n+1}$. When $h = 1$ and $i(0) = j \in \mathbb{Z}$, we write $M(h;i)$ as $M(1;j)$.

We define an $\mathbb{F}_{p^h}$-action on this object by

$$[a] \in \mathrm{End}_{\underline{\mathrm{MF}}_{\bar{k}}}(M(h;i)), \quad [a](e_n) = a^{p^{-n}} e_n$$

for any $a \in \mathbb{F}_{p^h}$. Then we can show the following proposition.

Proposition 4.7. *(i) ([FL82], Proposition 4.4) An object $M \in \underline{\mathrm{MF}}^{\mathrm{f}}_{\bar{k}}$ is simple if and only if M is isomorphic to $M(h;i)$ for some $h \in \mathbb{Z}_{>0}$ and $i : \mathbb{Z}/h\mathbb{Z} \to \mathbb{Z}$ of period h.*

(ii) ([FL82], Lemme 4.9) Let (h, i) be a pair of a positive integer h and a map $i : \mathbb{Z}/h\mathbb{Z} \to \mathbb{Z}$ of period h. Let (h', i') be a similar pair. Then $\mathrm{Hom}_{\underline{\mathrm{MF}}_{\bar{k}}}(M(h;i), M(h';i')) = 0$ unless $h = h'$ and $i = i'$ up to a shift. In the latter case the two objects are isomorphic, and $\mathrm{End}_{\underline{\mathrm{MF}}_{\bar{k}}}(M(h;i)) \simeq \mathbb{F}_{p^h}$ via $a \mapsto [a]$.

(iii) ([FL82], Lemme 6.3, 6.4) $\mathrm{Ext}^1_{\underline{\mathrm{MF}}_{\bar{k}}}(M, M(1;p-1)) = 0$ for any $M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{\bar{k}}$.

The assertions (i) and (ii) follow from a semilinear algebraic argument whose key point is that any σ^h-semilinear automorphism of a finite dimensional $\bar{k}$-vector space has a non-zero fixed vector. The last assertion is shown by dévissage, after explicitly constructing a split of any extension of $M(h;i)$ by $M(1;p-1)$ for $i : \mathbb{Z}/h\mathbb{Z} \to \{0, \dots, p-1\}$ as in the proof of Lemma 4.4.

Proposition 4.7 (iii) implies that an object M of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{\bar{k}}$ is in the full subcategory $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{\bar{k}}$ if and only if every Jordan-Hölder factor of M is not isomorphic to $M(1;p-1)$. This leads to the following corollary.

Corollary 4.8 ([FL82], Théorème 6.1 (i))**.** *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$. Then M is an object of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{W,\mathrm{tor}}$ if and only if so are M' and M''.

Proof. For this, it is enough to show that if $M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{W,\mathrm{tor}}$ then $M \otimes_W W(\bar{k}) \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{W(\bar{k}),\mathrm{tor}}$. Suppose that there exists a non-zero quotient $M \otimes_W W(\bar{k}) \to N$ satisfying $N = N^{p-1}$. Then we can find a finite extension k'/k in $\bar{k}$ such that this quotient is a base extension of a quotient $M \otimes_W W(k') \to N'$ in $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W(k'),\mathrm{tor}}$ satisfying $N' = (N')^{p-1}$. Since k'/k is finite, the latter quotient can be viewed as a quotient in $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$. In this category $M \otimes_W W(k')$ is the direct sum of finite copies of M. The image N_0 of a copy of M in N' is non-zero, and satisfies $N_0 = N_0^{p-1}$ by Proposition 2.5, which is the contradiction. $\square$

Next we study the Galois representations associated with the simple objects of $\underline{\mathrm{MF}}_{\bar{k}}^{\mathrm{f},[0,p-1]}$, which is one of the key ingredients of [FL82].

Proposition 4.9 ([FL82], Théorème 5.1)**.** *Let h be a positive integer and $i : \mathbb{Z}/h\mathbb{Z} \to \{0, 1, \dots, p-1\}$ a map of period h.*

(i) $\mathrm{Ext}^1_{\underline{\mathrm{MF}}_{\bar{k}}}(M(h; i), R^{\mathrm{DP}}) = 0$.

(ii) Via the $\mathbb{F}_{p^h}$-action on $M(h; i)$, the module $T^*_{\mathrm{crys}}(M(h; i))$ is an $\mathbb{F}_{p^h}$-vector space of dimension one.

(iii) The I_{K_0}-action on $T^*_{\mathrm{crys}}(M(h; i))$ is given by the character

$$\theta_h^{i_0+pi_1+\cdots p^{h-1}i_{h-1}} : I_{K_0} \to \mathbb{F}_{p^h}^{\times},$$

where θ_h is the fundamental character of level h.

Proof. By Lemma 4.3, we may replace R^{DP} with A to compute Hom and Ext^1 groups. First consider any extension

$$0 \longrightarrow A \longrightarrow E \longrightarrow M(h; i) \longrightarrow 0$$

in $\underline{\mathrm{MF}}_{\bar{k}}$. Let $\{e_n\}_{n \in \mathbb{Z}/h\mathbb{Z}}$ be the basis of $M(h; i)$ as before, and take a lift $\hat{e}_n$ of e_n in E^{i_n}. We have $\varphi^{i_n}(\hat{e}_n) = \hat{e}_{n-1} - d_{n-1}$ with some $d_{n-1} \in A$. Then the extension splits if and only if there exists $u_n \in A^{i_n}$ such that

$$\varphi^{i_n}(\hat{e}_n + u_n) = (\hat{e}_{n-1} + u_{n-1})$$

for any $n \in \mathbb{Z}/h\mathbb{Z}$, which is equivalent to

$$\varphi^{i_n}(u_n) - u_{n-1} = d_{n-1}.$$

Since this equation is linear, to prove that it has a solution, we may assume that $d_n = 0$ except a single index $n = n_0$. By permuting the indices, we may assume $n_0 = 0$. On the other hand, a $\bar{k}$-linear map $M(h; i) \to A$ defined by $e_n \mapsto u_n \in A^{i_n}$ gives a morphism of $\underline{\mathrm{MF}}_{\bar{k}}$ if and only if

$$\varphi^{i_n}(u_n) - u_{n-1} = 0$$

for any $n \in \mathbb{Z}/h\mathbb{Z}$. Hence we reduced ourselves to showing that the equation

$$\begin{cases} u_n \in A^{i_n}, \\ \varphi^{i_n}(u_n) - u_{n-1} = 0 \quad (n \neq 1), \\ \varphi^{i_n}(u_1) - u_0 = d \end{cases} \tag{2}$$

has a solution for any $d \in A$ in the case of the assertion (i), and has exactly p^h solutions for $d = 0$ in the case of the assertion (ii).

Put

$$u_n = a_n + b_n Y_1 + \sum_{l=2}^{p-1} c_{n,l} Y_1^l, \quad d = d_0 + d_1 Y_1 + \sum_{l=2}^{p-1} d_l Y_1^l$$

with some $a_n \in (-p)^{\frac{i_n}{p}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ and $b_n, c_{n,l}, d_l \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. We also put $\varepsilon(n) = 0$ for $n \neq p-1$ and $\varepsilon(p-1) = 1$. Then the equation (2) is equivalent to

$$\left\{\begin{array}{l} a_n \in (-p)^{\frac{i_n}{p}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}),\ b_n, c_{n,l} \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}, \\ \frac{\hat{a}_n^p}{(-p)^{i_n}}(1+Y_1)^{i_n} + \varepsilon(i_n)b_n^p - u_{n-1} = \left\{\begin{array}{ll} 0 & (n \neq 1) \\ d & (n = 1). \end{array}\right. \end{array}\right.$$

From this we see that $c_{n,l}$'s are uniquely determined by a_n's, and the solutions of the equation (2) correspond bijectively with the solutions of

$$\text{(3)} \qquad \left\{\begin{array}{l} a_n \in (-p)^{\frac{i_n}{p}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}),\ b_n \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}, \\ \frac{\hat{a}_n^p}{(-p)^{i_n}} + \varepsilon(i_n)b_n^p - a_{n-1} = \left\{\begin{array}{ll} 0 & (n \neq 1) \\ d_0 & (n = 1), \end{array}\right. \\ i_n\frac{\hat{a}_n^p}{(-p)^{i_n}} - b_{n-1} = \left\{\begin{array}{ll} 0 & (n \neq 1) \\ d_1 & (n = 1). \end{array}\right. \end{array}\right.$$

First let us assume $M(h;i) \neq M(1;p-1)$, and suppose $d = 0$. Then we can show $\varepsilon(i_n)b_n^p = 0$ for any n, by a valuation calculation [FL82, 5.11 (iii)]. Thus b_n's are also uniquely determined by a_n's in this case. For a_n's, we can show that any solution (a_n) of the equation (3) uniquely lifts to a solution in $\mathcal{O}_{\bar{K}}$ of the equation of degree p

$$\frac{X_n^p}{(-p)^{i_n}} - X_{n-1} = 0 \text{ for any } n$$

over W [FL82, Lemme 5.12].

Let K_0^{tr} be the maximal tamely ramified extension of K_0 in $\bar{K}$. For any $\rho \in \mathbb{Z}_{(p)}$, we write $\rho = a/b$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}_{>0}$ satisfying $p \nmid b$. Using [Bou56, §7, $n^\circ 4$, Proposition 5], we fix once and for all $(-p)^\rho \in K_0^{\mathrm{tr}}$ such that $((-p)^\rho)^b = (-p)^a$ and $(-p)^\rho(-p)^{\rho'} = (-p)^{\rho+\rho'}$ for any such ρ, ρ'. Then, from the above equation, we see that the module $T^*_{\mathrm{crys}}(M(h;i)) = \mathrm{Hom}_{\underline{\mathrm{MF}}_{\bar{k}}}(M(h;i), R^{\mathrm{DP}})$ is equal to

$$\{(e_n \mapsto \lambda^{p^{-n}}(-p)^{\rho_n}(1+Y_1)^{i_{n+1}}) \mid \lambda \in \mathbb{F}_{p^h}\}$$

with $\rho_n = \frac{i_{n+1}+pi_{n+2}+\cdots+p^{h-1}i_n}{p^h-1}$, which settles the assertions (ii) and (iii) for this case. The assertion (i) can be shown by constructing a solution of the equation (3) similarly from a solution in $\mathcal{O}_{\bar{K}}$ of an equation over W [FL82, 5.11 (i), (ii)].

The case of $M(1;p-1)$ needs an extra care [FL82, 5.14]. We also have an equation over W to lift solutions, while its degree is p^2 and lifts are not unique. Nevertheless, we can show for the case of $d = 0$ that the solutions of the lifted equation have exactly p distinct reductions modulo p, which yields the proposition for this case. $\square$

Now the assertions of Theorem 2.9 except the fullness follow formally. Indeed, Lemma 2.8 reduces assertions on the module structure of $T^*_{\mathrm{crys}}(M)$ to the case of $k = \bar{k}$. Moreover, for any object $N \in \underline{\mathrm{MF}}_k$ and any extension

$$0 \longrightarrow A_{\mathrm{crys},\infty} \longrightarrow E \longrightarrow N \longrightarrow 0$$

in $\underline{\mathrm{MF}}_W$, the exact sequence

$$0 \longrightarrow A_{\mathrm{crys}}/pA_{\mathrm{crys}} \longrightarrow A_{\mathrm{crys},\infty} \xrightarrow{\times p} A_{\mathrm{crys},\infty} \longrightarrow 0$$

and the snake lemma yield the extension

$$0 \longrightarrow A_{\mathrm{crys}}/pA_{\mathrm{crys}} \longrightarrow E[p] \longrightarrow N \longrightarrow 0$$

in $\underline{\mathrm{MF}}_k$ with $E[p] = \mathrm{Ker}(p : E \to E)$, and the induced map

$$\mathrm{Ext}^1_{\underline{\mathrm{MF}}_W}(N, A_{\mathrm{crys},\infty}) \to \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(N, A_{\mathrm{crys}}/pA_{\mathrm{crys}})$$

is an injection. Using this, the exactness of T^*_{crys} in Theorem 2.9 (i) follows by showing $\mathrm{Ext}^1_{\underline{\mathrm{MF}}_{W(\bar{k})}}(M, A_{\mathrm{crys},\infty}) = 0$ for any $M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W(\bar{k}),\mathrm{tor}}$ by dévissage from Proposition 4.9 (i). By combining this exactness of T^*_{crys} with Proposition 4.9 (ii), we obtain

$$\mathrm{lg}_{\mathbb{Z}_p}(T^*_{\mathrm{crys}}(M)) = \mathrm{lg}_W(M)$$

for any $M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$. Applying this to $M[p^l] = \mathrm{Ker}(p^l : M \to M)$ for any l shows Theorem 2.9 (ii). For any morphism $f : M \to N$ of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$ satisfying $T^*_{\mathrm{crys}}(f) = 0$, applying the above length equality to $\mathrm{Im}(f)$ shows $f = 0$ and settles the faithfulness of T^*_{crys} in Theorem 2.9 (i). Thus Theorem 2.9 (iv) follows from Corollary 4.8 modulo the fullness of T^*_{crys} (Theorem 2.9 (iii)).

4.3. **The fullness of T^*_{crys}.** Now we prove Theorem 2.9 (iii). First we may assume $k = \bar{k}$ by a Galois descent argument using the faithfulness of T^*_{crys} [FL82, 6.2]. By Proposition 4.7 and Proposition 4.9, the natural map

$$\mathrm{Hom}_{\underline{\mathrm{MF}}_{W(\bar{k})}}(M, N) \to \mathrm{Hom}_{\mathbb{Z}_p[I_{K_0}]}(T^*_{\mathrm{crys}}(N), T^*_{\mathrm{crys}}(M))$$

is an isomorphism if M and N are simple. In order for a dévissage argument to imply the theorem, we need to show that the natural map

$$\mathrm{Ext}^1_{\underline{\mathrm{MF}}_{W(\bar{k})}}(M, N) \to \mathrm{Ext}^1_{\mathbb{Z}_p[I_{K_0}]}(T^*_{\mathrm{crys}}(N), T^*_{\mathrm{crys}}(M))$$

is an injection for any objects $M, N \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{W(\bar{k}),\mathrm{tor}}$. To prove this injectivity and the fullness of T^*_{crys} simultaneously, by dévissage and an induction on the length we may assume that M and N are simple. By Theorem 2.9 (ii), we only have to consider the extensions killed by p. For this case, since $T^*_{\mathrm{crys}}(M)$ and $T^*_{\mathrm{crys}}(N)$ are tamely ramified by Proposition 4.9 (iii), Maschke's theorem implies that an extension of $T^*_{\mathrm{crys}}(N)$ by $T^*_{\mathrm{crys}}(M)$ splits if and only if the extension is also tamely ramified. Thus we are reduced to showing the following lemma.

Lemma 4.10 ([FL82], 6.11)**.** *Suppose $k = \bar{k}$. Let M be an object of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{\bar{k}}$. Then M is semi-simple if and only if $T^*_{\mathrm{crys}}(M)$ is tamely ramified.*

To prove the lemma, we need to describe the fixed part $T^*_{\mathrm{crys}}(M)^{G_F}$ for any algebraic extension F/K_0, which is also one of the key steps for the proof of Theorem 2.19. This is essentially done again by uniquely lifting each element of $T^*_{\mathrm{crys}}(M)$ to a solution in $\mathcal{O}_{\bar{K}}$ of an equation over W and using $(\mathcal{O}_{\bar{K}})^{G_F} = \mathcal{O}_F$, as follows.

Proposition 4.11 ([FL82], Proposition 6.7)**.** *Suppose $k = \bar{k}$. Let M be an object of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{\bar{k}}$ and F/K_0 an algebraic extension in $\bar{K}$. Then G_F acts trivially on $T^*_{\mathrm{crys}}(M)$ if and only if $T^*_{\mathrm{crys}}(M) = T^*_{\mathrm{crys},F}(M)$.*

Proof. We proceed by induction on the length of the Fontaine-Laffaille module M. When M is simple, this follows from the explicit description of $T^*_{\mathrm{crys}}(M)$ given in the proof of Proposition 4.9. Suppose that we have an exact sequence in $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{\bar{k}}$

$$0 \longrightarrow N \longrightarrow M \longrightarrow M(h;i) \longrightarrow 0$$

and that G_F acts trivially on $T^*_{\mathrm{crys}}(M)$. Then G_F also acts trivially on $T^*_{\mathrm{crys}}(N)$ and thus $T^*_{\mathrm{crys}}(N) = T^*_{\mathrm{crys},F}(N)$ by the induction hypothesis. Take the basis $(e_n)_{n\in\mathbb{Z}/h\mathbb{Z}}$ of $M(h;i)$ as before and a lift $\hat{e}_n \in M^{i_n}$ of e_n. Then

$$\varphi^{i_n}(\hat{e}_n) = \hat{e}_{n-1} + d_{n-1}$$

with some $d_{n-1} \in N$. Let $f : M \to A$ be an element of $T^*_{\mathrm{crys}}(M)$. Since $f|_N \in T^*_{\mathrm{crys}}(N) = T^*_{\mathrm{crys},F}(N)$, we can write as

$$f(d_n) = \alpha_n + \beta_n Y_1 + \sum_{l=2}^{p-1} \gamma_{n,l} Y_1^l$$

with some $\alpha_n, \beta_n, \gamma_{n,l} \in \mathcal{O}_F/p\mathcal{O}_F$. Put

$$f(\hat{e}_n) = a_n + b_n Y_1 + \sum_{l=2}^{p-1} c_{n,l} Y_1^l \in A^{i_n}$$

with $a_n \in (-p)^{\frac{i_n}{p}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})$ and $b_n, c_{n,l} \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$. From $\varphi^{i_n}(f(\hat{e}_n)) = f(\varphi^{i_n}(\hat{e}_n))$, we obtain

$$c_{n-1,l} = \binom{i_n}{l} \frac{\hat{a}_n^p}{(-p)^{i_n}} - \gamma_{n-1,l}$$

and thus $c_{n,l}$'s are uniquely determined by a_n's in a way that if $a_n \in \mathcal{O}_F/p\mathcal{O}_F$ for any n, then $c_{n,l}$'s are also in $\mathcal{O}_F/p\mathcal{O}_F$. Moreover, we see that to give $f \in T^*_{\mathrm{crys}}(M)$ with $f(d_n)$ being as above is the same as to give a solution $(a_n, b_n)_{n\in\mathbb{Z}/h\mathbb{Z}}$ of the equation

$$\begin{cases} a_n \in (-p)^{\frac{i_n}{p}}(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}),\ b_n \in \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}, \\ \frac{\hat{a}_n^p}{(-p)^{i_n}} + \varepsilon(i_n) b_n^p - a_{n-1} = \alpha_{n-1}, \\ i_n \frac{\hat{a}_n^p}{(-p)^{i_n}} - b_{n-1} = \beta_{n-1}. \end{cases}$$

Let $\hat{\alpha}_n$ and $\hat{\beta}_n$ be any lifts of α_n and β_n in $\mathcal{O}_F$, respectively. Then we can show [FL82, Lemme 6.8] that any solution $(a_n, b_n)_{n\in\mathbb{Z}/h\mathbb{Z}}$ of the above equation uniquely lifts to a solution $(x_n, y_n)_{n\in\mathbb{Z}/h\mathbb{Z}}$ in $\mathcal{O}_{\bar{K}}$ of the equation over $\mathcal{O}_F$

$$\begin{cases} \frac{X_n^p}{(-p)^{i_n}} + \varepsilon(i_n)Y_n^p - X_{n-1} = \hat{\alpha}_{n-1}, \\ i_n\frac{X_n^p}{(-p)^{i_n}} - Y_{n-1} = \hat{\beta}_{n-1}. \end{cases}$$

Since $(g(a_n), g(b_n)) = (a_n, b_n)$ for any $g \in G_F$ by assumption, the uniqueness of the lifting implies $(g(x_n), g(y_n)) = (x_n, y_n)$ for any $g \in G_F$. Thus $x_n, y_n \in (\mathcal{O}_{\bar{K}})^{G_F} = \mathcal{O}_F$ and $a_n, b_n \in \mathcal{O}_F/p\mathcal{O}_F$ for any n. Hence we obtain $f(\hat{e}_n) \in A_F$ and $f \in T^*_{\mathrm{crys},F}(M)$. □

For any $\rho \in \mathbb{Z}_{(p)} \cap [0, 1[$, there exist uniquely $h \in \mathbb{Z}_{>0}$ and $i : \mathbb{Z}/h\mathbb{Z} \to \{0, \ldots, p-1\}$ of period h satisfying

$$\rho = \frac{i_1 + pi_2 + \cdots + p^{h-2}i_{h-1} + p^{h-1}i_0}{p^h - 1}.$$

Note that any such (h, i) except $(1, p-1)$ can be obtained in this way, since $\rho < 1$. Put

$$\bar{\omega}_\rho = (-p)^\rho(1 + Y_1)^{i_1} \in A^{i_0}_{K_0^{\mathrm{tr}}},$$

where $(-p)^\rho$ is chosen as before. Then we have $\varphi^{i_0}(\bar{\omega}_\rho) = \bar{\omega}_{p\rho-i_0} \in A^{i_{h-1}}_{K_0^{\mathrm{tr}}}$ with

$$p\rho - i_0 = \frac{i_0 + pi_1 + \cdots + p^{h-2}i_{h-2} + p^{h-1}i_{h-1}}{p^h - 1} \in \mathbb{Z}_{(p)} \cap [0, 1[.$$

Repeating this, we see that the $\bar{k}$-subspace generated by

$$\bar{\omega}_\rho,\ \varphi^{i_0}(\bar{\omega}_\rho),\ \varphi^{i_{h-1}}(\varphi^{i_0}(\bar{\omega}_\rho)), \ldots$$

defines an object of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{\bar{k}}$ which is isomorphic to $M(h; i)$.

Put

$$A_{\mathrm{ss}} = \mathrm{Span}_{\bar{k}}\{\bar{\omega}_\rho \mid \rho \in \mathbb{Z}_{(p)} \cap [0, 1[\} \subseteq A_{K_0^{\mathrm{tr}}}$$

and we give this subspace the induced structure from $A_{K_0^{\mathrm{tr}}}$ as an object of $\underline{\mathrm{MF}}_{\bar{k}}$. Then we see that A_{ss} is isomorphic to the direct sum of representatives of all isomorphism classes of simple objects of $\underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{\bar{k}}$. Moreover, we can show by induction on the length that $M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]'}_{\bar{k}}$ is semi-simple if and only if the natural injection

$$\mathrm{Hom}_{\underline{\mathrm{MF}}_{\bar{k}}}(M, A_{\mathrm{ss}}) \to T^*_{\mathrm{crys}}(M)$$

is bijective.

Thus we are reduced to comparing $A_{K_0^{\mathrm{tr}}}$ and A_{ss}. Consider the quotient

$$A_{K_0^{\mathrm{tr}}}/A_{\mathrm{ss}} \simeq \bigoplus_{l=1}^{p-1}(\mathcal{O}_{K_0^{\mathrm{tr}}}/p\mathcal{O}_{K_0^{\mathrm{tr}}})Y_1^l.$$

Then we have $A_{K_0^{\mathrm{tr}}}/A_{\mathrm{ss}} = (A_{K_0^{\mathrm{tr}}}/A_{\mathrm{ss}})^{p-1}$. Moreover, a simple calculation as in [FL82, 6.10] shows that $\varphi^{p-1} = 0$ on this quotient. By Lemma 4.4, we obtain $\mathrm{Hom}_{\underline{\mathrm{MF}}_{\bar{k}}}(M, A_{K_0^{\mathrm{tr}}}/A_{\mathrm{ss}}) = 0$ and thus

$$\mathrm{Hom}_{\underline{\mathrm{MF}}_{\bar{k}}}(M, A_{\mathrm{ss}}) = \mathrm{Hom}_{\underline{\mathrm{MF}}_{\bar{k}}}(M, A_{K_0^{\mathrm{tr}}}).$$

The equality shows that M is semi-simple if and only if $\mathrm{Hom}_{\underline{\mathrm{MF}}_{\bar{k}}}(M, A_{K_0^{\mathrm{tr}}}) = T^*_{\mathrm{crys}}(M)$, which is the same as saying that $T^*_{\mathrm{crys}}(M)$ is tamely ramified by Proposition 4.11. This concludes the proofs of Lemma 4.10 and Theorem 2.9 (iii).

4.4. **Ramification bound.** To prove Theorem 2.19, we need a lemma of Fontaine (which was improved later by Yoshida) relating ramification to a ramified variant of formal smoothness, as follows. For any algebraic extension E/K in $\bar{K}$ and any non-negative real number m, put

$$\mathfrak{a}^m_{E/K} = \{x \in \mathcal{O}_E \mid v_K(x) \geq m\}.$$

For any finite Galois extension L/K, consider the property

$$(P_m)_{L/K} \quad \left\{ \begin{array}{l} \text{For any algebraic extension } E/K, \text{ if there exists} \\ \text{an } \mathcal{O}_K\text{-algebra homomorphism } \mathcal{O}_L \to \mathcal{O}_E/\mathfrak{a}^m_{E/K}, \\ \text{then there exists a } K\text{-algebra injection } L \to E. \end{array} \right.$$

Note that we do not require the map $L \to E$ to be a lift of the given map $\mathcal{O}_L \to \mathcal{O}_E/\mathfrak{a}^m_{E/K}$. The property $(P_m)_{L/K}$ is related to $u_{L/K}$ by the following lemma.

Lemma 4.12 ([Fon85], Proposition 1.5, [Yos10], Proposition 3.3)**.**

$$u_{L/K} = \inf\{m \in \mathbb{R}_{\geq 0} \mid (P_m)_{L/K} \textit{ holds}\}.$$

For this, Krasner's lemma implies that $(P_m)_{L/K}$ holds for any $m > u_{L/K}$. Conversely, by explicitly constructing counterexamples for $(P_m)_{L/K}$, we can show that if $(P_m)_{L/K}$ holds, then $m > u_{L/K} - e^{-1}_{L/K}$, where $e_{L/K}$ is the relative ramification index of the extension L/K [Fon85]. Furthermore, we can kill the error term by using arbitrarily large tamely ramified base extensions [Yos10].

Now let $r < p-1$, $M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,r]}_k$ and L/K_0 be as in Theorem 2.19. Since the theorem concerns only about the I_{K_0}-action on $T^*_{\mathrm{crys}}(M)$, by Lemma 2.8 we may assume $k = \bar{k}$. Since $r < p-1$, Lemma 4.5 gives the identification

$$T^*_{\mathrm{crys}}(M) = \mathrm{Hom}_{\underline{\mathrm{MF}}_{\bar{k}}}(M, \mathcal{O}_{\bar{K}}/\mathfrak{b}_{\bar{K}}),\ T^*_{\mathrm{crys},F}(M) = \mathrm{Hom}_{\underline{\mathrm{MF}}_{\bar{k}}}(M, \mathcal{O}_F/\mathfrak{b}_F)$$

for any algebraic extension F/K_0 in $\bar{K}$. By Lemma 4.12, it is enough to check the property $(P_m)_{L/K_0}$ for any $m > 1 + r/(p-1)$.

Let E/K_0 be an algebraic extension in $\bar{K}$, m a real number satisfying $m > 1 + r/(p-1)$ and $\eta : \mathcal{O}_L \to \mathcal{O}_E/\mathfrak{a}^m_{E/K_0}$ a W-algebra homomorphism.

Let $P(X)$ be the minimal polynomial over W of a uniformizer π_L of L. We write as

$$P(X) = X^{e_{L/K_0}} + \sum_{s=0}^{e_{L/K_0}-1} pc_s X^s$$

with some $c_s \in W$ such that $c_0 \in W^\times$. Let $\hat{x}$ be a lift of $\eta(\pi_L)$ in $\mathcal{O}_E$. Since $P(\eta(\pi_L)) = 0$ in the ring $\mathcal{O}_E/\mathfrak{a}^m_{E/K_0}$, we have $P(\hat{x}) + \delta = 0$ with some $\delta \in \mathcal{O}_E$ satisfying $v_p(\delta) \geq m > 1 + r/(p-1)$. From the Newton polygon of the polynomial $P(X) + \delta$, we obtain $v_p(\hat{x}) = e^{-1}_{L/K_0} = v_p(\pi_L)$. Hence we see that η induces an injection

$$\bar{\eta} : \mathcal{O}_L/\mathfrak{b}_L \to \mathcal{O}_E/\mathfrak{b}_E$$

and that $\bar{\eta}$ respects filtrations. We claim that $\bar{\eta}$ also respects φ^i's. Indeed, note that the i-th filtration $(\mathcal{O}_L/\mathfrak{b}_L)^i$ is spanned over $\bar{k}$ by

$$\{\pi_L^j \mid j \in \mathbb{Z},\ j \geq \frac{e_{L/K_0}}{p} i\}.$$

For such j, put $pj = e_{L/K_0} i + l$. Then

$$\varphi^i(\pi_L^j) = \frac{\pi_L^{pj}}{(-p)^i} \bmod \mathfrak{b}_L = \pi_L^l (\sum_{s=0}^{e_{L/K_0}-1} c_s \pi_L^s)^i \bmod \mathfrak{b}_L,$$

$$\varphi^i(\bar{\eta}(\pi_L^j)) = \frac{\hat{x}^{pj}}{(-p)^i} \bmod \mathfrak{b}_E = \hat{x}^l (\sum_{s=0}^{e_{L/K_0}-1} c_s \hat{x}^s + \frac{\delta}{p})^i \bmod \mathfrak{b}_E.$$

Since $\frac{\delta}{p} \in \mathfrak{b}_E$, we obtain $\bar{\eta}(\varphi^i(\pi_L^j)) = \varphi^i(\bar{\eta}(\pi_L^j))$, which proves the claim.

Thus the map $\bar{\eta}$ defines an injection of the category $\underline{\mathrm{MF}}_{\bar{k}}$ and induces an injection

$$T^*_{\mathrm{crys},L}(M) \to T^*_{\mathrm{crys},E}(M).$$

Now Proposition 4.11 shows $T^*_{\mathrm{crys}}(M) = T^*_{\mathrm{crys},L}(M)$. Thus we also have $T^*_{\mathrm{crys}}(M) = T^*_{\mathrm{crys},E}(M)$ and G_E acts trivially on $T^*_{\mathrm{crys}}(M)$. From the definition of L, we obtain $G_E \subseteq G_L$ and $L \subseteq E$, namely the property $(P_m)_{L/K_0}$ holds for any $m > 1 + r/(p-1)$. This concludes the proof of Theorem 2.19 for $r < p - 1$.

For the case of $r = p - 1$ and $M \in \underline{\mathrm{MF}}_k^{\mathrm{f},[0,p-1]'}$, it is enough to check the property $(P_m)_{L/K_0}$ for any $m > 2$. Consider a W-algebra homomorphism $\eta : \mathcal{O}_L \to \mathcal{O}_E/\mathfrak{a}^m_{E/K_0}$. We see in the same way as above that the induced injection $\bar{\eta} : \mathcal{O}_L/p\mathcal{O}_L \to \mathcal{O}_E/p\mathcal{O}_E$ and $Y_1 \mapsto Y_1$ define an injection $A_L \to A_E$ which is compatible with filtrations and Frobenius structures. Thus we have an injection $T^*_{\mathrm{crys},L}(M) \to T^*_{\mathrm{crys},E}(M)$ also in this case, which yields the property $(P_m)_{L/K_0}$ and the desired ramification bound.

Remark 4.13. As we mentioned before, [FL82] uses the subring $S_{\mathrm{FL}} = W(R)[\frac{\xi^p}{p}]$ instead of A_{crys} to construct a G_{K_0}-representation associated with

$M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$. We can show that these two constructions are naturally isomorphic: we claim that for any $M \in \underline{\mathrm{MF}}^{\mathrm{f},[0,p-1]}_{W,\mathrm{tor}}$, the natural map

$$\mathrm{Hom}_{\underline{\mathrm{MF}}_W}(M, S_{\mathrm{FL},\infty}) \to \mathrm{Hom}_{\underline{\mathrm{MF}}_W}(M, A_{\mathrm{crys},\infty}) = T^*_{\mathrm{crys}}(M)$$

is bijective and

$$\mathrm{Ext}^1_{\underline{\mathrm{MF}}_W}(M, S_{\mathrm{FL},\infty}) \to \mathrm{Ext}^1_{\underline{\mathrm{MF}}_W}(M, A_{\mathrm{crys},\infty})$$

is injective. Indeed, by dévissage we may assume $pM = 0$. By [FL82, Lemme 5.7], we have a natural identification $(\mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}})[Y_1] \simeq S_{\mathrm{FL}}/pS_{\mathrm{FL}}$ defined by $Y_1 \mapsto \delta(\xi) \bmod p$. Since $\varphi^{p-1}(Y_1^p) = 0$ in this ring [FL82, 5.9], Lemma 4.4 implies that the natural map $S_{\mathrm{FL}}/pS_{\mathrm{FL}} \to A$ induces a bijection

$$\mathrm{Hom}_{\underline{\mathrm{MF}}_k}(M, S_{\mathrm{FL}}/pS_{\mathrm{FL}}) \simeq \mathrm{Hom}_{\underline{\mathrm{MF}}_k}(M, A) = T^*_{\mathrm{crys}}(M)$$

and an injection

$$\mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, S_{\mathrm{FL}}/pS_{\mathrm{FL}}) \to \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, A) = \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, A_{\mathrm{crys}}/pA_{\mathrm{crys}}).$$

The former one settles the assertion on Hom. Moreover, as in the proof of Theorem 2.9 (i), we have a natural injection

$$\mathrm{Ext}^1_{\underline{\mathrm{MF}}_W}(M, S_{\mathrm{FL},\infty}) \to \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, S_{\mathrm{FL}}/pS_{\mathrm{FL}})$$

and the assertion on Ext^1 follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}^1_{\underline{\mathrm{MF}}_W}(M, S_{\mathrm{FL},\infty}) \arrow[r, hook] \arrow[d] & \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, S_{\mathrm{FL}}/pS_{\mathrm{FL}}) \arrow[d, hook] \\
\mathrm{Ext}^1_{\underline{\mathrm{MF}}_W}(M, A_{\mathrm{crys},\infty}) \arrow[r, hook] & \mathrm{Ext}^1_{\underline{\mathrm{MF}}_k}(M, A_{\mathrm{crys}}/pA_{\mathrm{crys}}).
\end{tikzcd}$$

References

[AM04] A. Abbes and A. Mokrane: *Sous-groupes canoniques et cycles évanescents p-adiques pour les variétés abéliennes*, Publ. Math. Inst. Hautes Études Sci. **99** (2004), 117–162.

[AS02] A. Abbes and T. Saito: *Ramification of local fields with imperfect residue fields*, Amer. J. Math. **124** (2002), no. 5, 879–920.

[AS03] A. Abbes and T. Saito: *Ramification of local fields with imperfect residue fields II*, Kazuya Kato's fiftieth birthday. Doc. Math. 2003, Extra Vol., 5–72.

[Abr87] V. Abrashkin: *Galois modules of group schemes of period p over the ring of Witt vectors*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 4, 691–736, 910; translation in Math. USSR-Izv. **31** (1988), no. 1, 1–46.

[Abr89a] V. Abrashkin: *Modification of the Fontaine-Laffaille functor*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 3, 451–497, 671; translation in Math. USSR-Izv. **34** (1990), no. 3, 467–516.

[Abr89b] V. Abrashkin: *Modular representations of the Galois group of a local field and a generalization of a conjecture of Shafarevich*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1135–1182, 1337; translation in Math. USSR-Izv. **35** (1990), no. 3, 469–518.

[Abr00] V. Abrashkin: *On a local analogue of the Grothendieck conjecture*, Internat. J. Math. **11** (2000), no. 2, 133–175.

[Abr02] V. Abrashkin: *Ramification theory for higher dimensional local fields*, Algebraic number theory and algebraic geometry, 1–16, Contemp. Math. **300**, Amer. Math. Soc., Providence, RI, 2002.

[Abr03] V. Abrashkin: *An analogue of Grothendieck's conjecture for two-dimensional local fields of finite characteristic*, Tr. Mat. Inst. Steklova **241** (2003), Teor. Chisel, Algebra i Algebr. Geom., 8–42; translation in Proc. Steklov Inst. Math. 2003, no. 2 (**241**), 2–34.

[Abr07] V. Abrashkin: *An analogue of the field-of-norms functor and of the Grothendieck conjecture*, J. Algebraic Geom. **16** (2007), no. 4, 671–730.

[Abr10] V. Abrashkin: *Modified proof of a local analogue of the Grothendieck conjecture*, J. Théor. Nombres Bordeaux **22** (2010), no. 1, 1–50.

[Abr12+] V. Abrashkin: *A semi-stable case of the Shafarevich conjecture*, to appear in Automorphic Forms and Galois Representations, London Math. Soc. Lecture Note Ser. **414**, Cambridge Univ. Press, 2014.

[Abr13] V. Abrashkin: *Projective varieties with bad semi-stable reduction at* 3 *only*, Doc. Math. **18** (2013), 547–619.

[Abr14+] V. Abrashkin: *Ramification estimate for Fontaine-Laffaille Galois modules*, arXiv:1405.3861v1 (2014).

[And06] F. Andreatta: *Generalized ring of norms and generalized (φ,Γ)-modules*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 4, 599–647.

[ABPS13+] A.-M. Aubert, P. Baum, R. Plymen and M. Solleveld: *The local Langlands correspondence for inner forms of* SL_n, arXiv:1305.2638v3 (2013).

[Ben00] D. Benois: *On Iwasawa theory of crystalline representations*, Duke Math. J. **104** (2000), no. 2, 211–267.

[BB08] D. Benois and L. Berger: *Théorie d'Iwasawa des représentations cristallines II*, Comment. Math. Helv. **83** (2008), no. 3, 603–677.

[Berg03] L. Berger: *Bloch and Kato's exponential map: three explicit formulas*, Kazuya Kato's fiftieth birthday. Doc. Math. 2003, Extra Vol., 99–129.

[Berg04] L. Berger: *Limites de représentations cristallines*, Compos. Math. **140** (2004), no. 6, 1473–1498.

[Berg12] L. Berger: *Local constancy for the reduction mod p of* 2*-dimensional crystalline representations*, Bull. Lond. Math. Soc. **44** (2012), no. 3, 451–459.

[Berg13+] L. Berger: *Multivariable Lubin-Tate (φ,Γ)-modules and filtered φ-modules*, arXiv:1211.4431v2 (2013).

[BLZ04] L. Berger, H. Li and H. Zhu: *Construction of some families of* 2*-dimensional crystalline representations*, Math. Ann. **329** (2004), no. 2, 365–377.

[BeK13] T. Berger and K. Klosin: *On deformation rings of residually reducible Galois representations and $R = T$ theorems*, Math. Ann. **355** (2013), no. 2, 481–518.

[Bert74] P. Berthelot: *Cohomologie cristalline des schémas de caractéristique $p > 0$*, Lecture Notes in Math. **407**, Springer-Verlag, Berlin-New York, 1974.

[BO78] P. Berthelot and A. Ogus: *Notes on crystalline cohomology*, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[Bor04] J. Borger: *Conductors and the moduli of residual perfection*, Math. Ann. **329** (2004), no. 1, 1–30.

[Bou56] N. Bourbaki: *Éléments de mathématique*, Fasc. XX. Livre I: Théorie des ensembles. Chapitre 3: Ensembles ordonnés. Cardinaux. Nombres entiers. Seconde édition, revue et augmentée. Actualités Sci. Ind., no. 1243. Hermann, Paris, 1956.

[Bre97] C. Breuil: *Représentations p-adiques semi-stables et transversalité de Griffiths*, Math. Ann. **307** (1997), no. 2, 191–224.

[Bre98a] C. Breuil: *Construction de représentations p-adiques semi-stables*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 3, 281–327.

[Bre98b] C. Breuil: *Cohomologie étale de p-torsion et cohomologie cristalline en réduction semi-stable*, Duke Math. J. **95** (1998), no. 3, 523–620.

[Bre98+] C. Breuil: *Schémas en groupes et corps des normes*, unpublished, available at `http://www.math.u-psud.fr/~breuil/`.

[Bre99] C. Breuil: *Une remarque sur les représentations locales p-adiques et les congruences entre formes modulaires de Hilbert*, Bull. Soc. Math. France **127** (1999), no. 3, 459–472.

[Bre00] C. Breuil: *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549.

[Bre02] C. Breuil: *Integral p-adic Hodge theory*, Algebraic geometry 2000, Azumino (Hotaka), 51–80, Adv. Stud. Pure Math. **36**, Math. Soc. Japan, Tokyo, 2002.

[BCDT01] C. Breuil, B. Conrad, F. Diamond and R. Taylor: *On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), no. 4, 843–939.

[BMs02] C. Breuil and W. Messing: *Torsion étale and crystalline cohomologies*, Cohomologies p-adiques et applications arithmétiques II, Asterisque **279** (2002), 81–124.

[BMz02] C. Breuil and A. Mézard: *Multiplicités modulaires et représentations de $GL_2(\mathbb{Z}_p)$ et de $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ en $l = p$*, Duke Math. J. **115** (2002), no. 2, 205–310.

[Bri06] O. Brinon: *Représentations cristallines dans le cas d'un corps résiduel imparfait*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 919–999.

[BC14+] O. Brinon and B. Conrad: *CMI summer school notes on p-adic Hodge theory*, available at `http://math.stanford.edu/~conrad/`.

[BT08] O. Brinon and F. Trihan: *Représentations cristallines et F-cristaux: le cas d'un corps résiduel imparfait*, Rend. Semin. Mat. Univ. Padova **119** (2008), 141–171.

[Bru99] S. Brueggeman: *The nonexistence of certain Galois extensions unramified outside* 5, J. Number Theory **75** (1999), no. 1, 47–52.

[Bru05] S. Brueggeman: *The nonexistence of certain nonsolvable Galois extensions of number fields of small degree*, Int. J. Number Theory **1** (2005), no. 1, 155–160.

[BrK01] A. Brumer and K. Kramer: *Non-existence of certain semistable abelian varieties*, Manuscripta Math. **106** (2001), no. 3, 291–304.

[BrK04] A. Brumer and K. Kramer: *Semistable abelian varieties with small division fields*, Galois theory and modular forms, 13–37, Dev. Math. **11**, Kluwer Acad. Publ., Boston, MA, 2004.

[BrK14] A. Brumer and K. Kramer: *Paramodular abelian varieties of odd conductor*, Trans. Amer. Math. Soc. **366** (2014), no. 5, 2463–2516.

[Car06] X. Caruso: *Représentations semi-stables de torsion dans le case $er < p - 1$*, J. Reine Angew. Math. **594** (2006), 35–92.

[Car08] X. Caruso: *Conjecture de l'inertie modérée de Serre*, Invent. Math. **171** (2008), no. 3, 629–699.

[Car13] X. Caruso: *Représentations galoisiennes p-adiques et (φ, τ)-modules*, Duke Math. J. **162** (2013), no. 13, 2525–2607.

[CL11] X. Caruso and T. Liu: *Some bounds for ramification of p^n-torsion semi-stable representations*, J. Algebra **325** (2011), 70–96.

[CS09] X. Caruso and D. Savitt: *Polygones de Hodge, de Newton et de l'inertie modérée des représentations semi-stables*, Math. Ann. **343** (2009), no. 4, 773–789.

[CS10] X. Caruso and D. Savitt: *Poids de l'inertie modérée de certaines représentations cristallines*, J. Théor. Nombres Bordeaux **22** (2010), no. 1, 79–96.

[CD11] S. Chang and F. Diamond: *Extensions of rank one (φ, Γ)-modules and crystalline representations*, Compos. Math. **147** (2011), no. 2, 375–427.

[CP09] B. Chiarellotto and A. Pulita: *Arithmetic and differential Swan conductors of rank one representations with finite local monodromy*, Amer. J. Math. **131** (2009), no. 6, 1743–1794.

[CHT08] L. Clozel, M. Harris and R. Taylor: *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. **108** (2008), 1–181.

[Col99] P. Colmez: *Représentations cristallines et représentations de hauteur finie*, J. Reine Angew. Math. **514** (1999), 119–143.

[CF00] P. Colmez and J.-M. Fontaine: *Construction des représentations p-adiques semi-stables*, Invent. Math. **140** (2000), no. 1, 1–43.

[Con99] B. Conrad: *Finite group schemes over bases with low ramification*, Compos. Math. **119** (1999), no. 3, 239–320.

[CDT99] B. Conrad, F. Diamond and R. Taylor: *Modularity of certain potentially Barsotti-Tate Galois representations*, J. Amer. Math. Soc. **12** (1999), no. 2, 521–567.

[Cre00] R. Crew: *Canonical extensions, irregularities, and the Swan conductor*, Math. Ann. **316** (2000), no. 1, 19–37.

[Del84] P. Deligne: *Les corps locaux de caractéristique p, limites de corps locaux de caractéristique* 0, Representations of reductive groups over a local field, 119–157, Travaux en Cours, Hermann, Paris, 1984.

[DFG04] F. Diamond, M. Flach and L. Guo: *The Tamagawa number conjecture of adjoint motives of modular forms*, Ann. Sci. École Norm. Sup. **37** (4) (2004), no. 5, 663–727.

[Dim05] M. Dimitrov: *Galois representations modulo p and cohomology of Hilbert modular varieties*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 4, 505–551.

[Dou10] G. Dousmanis: *On reductions of families of crystalline Galois representations*, Doc. Math. **15** (2010), 873–938.

[Eme11+] M. Emerton: *Local-global compatibility in the p-adic Langlands programme for $GL_{2/\mathbb{Q}}$*, available at `http://www.math.uchicago.edu/~emerton/`.

[EGH13] M. Emerton, T. Gee and F. Herzig: *Weight cycling and Serre-type conjectures for unitary groups*, Duke Math. J. **162** (2013), no. 9, 1649–1722.

[Fal83] G. Faltings: *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), no. 3, 349–366.

[Fon75] J.-M. Fontaine: *Groupes finis commutatifs sur les vecteurs de Witt*, C. R. Acad. Sci. Paris Sér. A-B **280** (1975), Ai, A1423–A1425.

[Fon77] J.-M. Fontaine: *Groupes p-divisibles sur les corps locaux*, Astérisque, **47-48**. Société Mathématique de France, Paris, 1977.

[Fon83] J.-M. Fontaine: *Cohomologie de de Rham, cohomologie cristalline et représentations p-adiques*, Algebraic geometry (Tokyo/Kyoto, 1982), 86–108, Lecture Notes in Math. **1016**, Springer, Berlin, 1983.

[Fon85] J.-M. Fontaine: *Il n'y a pas de variété abélienne sur $\mathbb{Z}$*, Invent. Math. **81** (1985), no. 3, 515–538.

[Fon90] J.-M. Fontaine: *Représentations p-adiques des corps locaux I*, The Grothendieck Festschrift, Vol. II, 249–309, Progr. Math. **87**, Birkhauser Boston, Boston, MA, 1990.

[Fon93] J.-M. Fontaine: *Schémas propres et lisses sur $\mathbb{Z}$*, Proceedings of the Indo-French Conference on Geometry (Bombay, 1989), 43–56, Hindustan Book Agency, Delhi, 1993.

[FL82] J.-M. Fontaine and G. Laffaille: *Construction de représentations p-adiques*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 4, 547–608.

[FMa95] J.-M. Fontaine and B. Mazur: *Geometric Galois representations*, Elliptic curves, modular forms & Fermat's last theorem (Hong Kong, 1993), 41–78, Ser. Number Theory, I, Int. Press, Cambridge, MA, 1995.

[FMe87] J.-M. Fontaine and W. Messing: *p-adic periods and p-adic etale cohomology*, Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), 179–207, Contemp. Math. **67**, Amer. Math. Soc., Providence, RI, 1987.

[FX13] L. Fourquaux and B. Xie: *Triangulable $\mathcal{O}_F$-analytic (φ_q,Γ)-modules of rank* 2, Algebra Number Theory **7** (2013), no. 10, 2545–2592.

[Fuj06+] K. Fujiwara: *Deformation rings and Hecke algebras in the totally real case*, arXiv:math/0602606v2 (2006).

[Gan13+] R. Ganapathy: *The local Langlands correspondence for* GSp_4 *over local function fields*, arXiv:1305.6088 (2013).

[Gao14+] H. Gao: *Galois lattices and strongly divisible lattices in the unipotent case*, arXiv:1305.2664v2 (2014).

[Gee11] T. Gee: *On the weights of mod p Hilbert modular forms*, Invent. Math. **184** (2011), no. 1, 1–46.

[GLS12] T. Gee, T. Liu and D. Savitt: *Crystalline extensions and the weight part of Serre's conjecture*, Algebra Number Theory **6** (2012), no. 7, 1537–1559.

[GLS14] T. Gee, T. Liu and D. Savitt: *The Buzzard-Diamond-Jarvis conjecture for unitary groups*, J. Amer. Math. Soc. **27** (2014), no. 2, 389–435.

[Gro74] A. Grothendieck: *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Séminaire de Mathématiques Supérieures **45** (Été, 1970). Les Presses de l'Université de Montréal, Montréal, Que., 1974.

[Gro77] A. Grothendieck: *Formule d'Euler-Poincaré en cohomologie étale*, Cohomologie l-adique et fonctions L, Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Exp. X, Lecture Notes in Math. **589**, Springer-Verlag, Berlin-New York, 1977.

[Hat06] S. Hattori: *Ramification of a finite flat group scheme over a local field*, J. Number Theory **118** (2006), no. 2, 145–154.

[Hat09] S. Hattori: *On a ramification bound of torsion semi-stable representations over a local field*, J. Number Theory **129** (2009), no. 10, 2474–2503.

[Hat13] S. Hattori: *Canonical subgroups via Breuil-Kisin modules*, Math. Z. **274** (2013), no. 3-4, 933–953.

[Hat14a] S. Hattori: *Canonical subgroups via Breuil-Kisin modules for* $p = 2$, J. Number Theory **137** (2014), 142–159.

[Hat14b] S. Hattori: *Ramification theory and perfectoid spaces*, Compos. Math. **150** (2014), no. 5, 798–834.

[Hon70] T. Honda: *On the theory of commutative formal groups*, J. Math. Soc. Japan **22** (1970), 213–246.

[Jos99] K. Joshi: *Remarks on methods of Fontaine and Faltings*, Internat. Math. Res. Notices **22** (1999), 1199–1209.

[Kas13] P. Kassaei: *Modularity lifting in parallel weight one*, J. Amer. Math. Soc. **26** (2013), no. 1, 199–225.

[Kat89] K. Kato: *Swan conductors for characters of degree one in the imperfect residue field case*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), 101–131, Contemp. Math. **83**, Amer. Math. Soc., Providence, RI, 1989.

[Kat94] K. Kato: *Semi-stable reduction and p-adic étale cohomology*, Périodes p-adiques (Bures-sur-Yvette, 1988), Astérisque **223** (1994), 269–293.

[Kaz86] D. Kazhdan: *Representations of groups over close local fields*, J. Analyse Math. **47** (1986), 175–179.

[Ked05] K. Kedlaya: *Local monodromy of p-adic differential equations: an overview*, Int. J. Number Theory **1** (2005), no. 1, 109–154.

[Ked07a] K. Kedlaya: *Swan conductors for p-adic differential modules, I: A local construction*, Algebra Number Theory **1** (2007), no. 3, 269–300.

[Ked07b] K. Kedlaya: *Semistable reduction for overconvergent F-isocrystals, I: Unipotence and logarithmic extensions*, Compos. Math. **143** (2007), no. 5, 1164–1212.

[Ked08] K. Kedlaya: *Semistable reduction for overconvergent F-isocrystals, II: A valuation-theoretic approach*, Compos. Math. **144** (2008), no. 3, 657–672.

[Ked09] K. Kedlaya: *Semistable reduction for overconvergent F-isocrystals, III: Local semistable reduction at monomial valuations*, Compos. Math. **145** (2009), no. 1, 143–172.

[Ked11] K. Kedlaya: *Semistable reduction for overconvergent F-isocrystals, IV: local semistable reduction at nonmonomial valuations*, Compos. Math. **147** (2011), no. 2, 467–523.

[Kim12] W. Kim: *The classification of p-divisible groups over 2-adic discrete valuation rings*, Math. Res. Lett. **19** (2012), no. 1, 121–141.

[Kim13+] W. Kim: *The relative Breuil-Kisin classification of p-divisible groups and finite flat group schemes*, arXiv:1201.0121v4 (2013).

[Kis06] M. Kisin: *Crystalline representations and F-crystals*, Algebraic geometry and number theory, 459–496, Progr. Math. **253**, Birkhauser Boston, Boston, MA, 2006.

[Kis08] M. Kisin: *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546.

[Kis09a] M. Kisin: *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180.

[Kis09b] M. Kisin: *Modularity of 2-adic Barsotti-Tate representations*, Invent. Math. **178** (2009), no. 3, 587–634.

[Kis09c] M. Kisin: *The Fontaine-Mazur conjecture for GL_2*, J. Amer. Math. Soc. **22** (2009), no. 3, 641–690.

[Kis10] M. Kisin: *Deformations of $G_{\mathbb{Q}_p}$ and $GL_2(\mathbb{Q}_p)$ representations*, Astérisque **330** (2010), 511–528.

[KR09] M. Kisin and W. Ren: *Galois representations and Lubin-Tate groups*, Doc. Math. **14** (2009), 441–461.

[Laf80] G. Laffaille: *Groupes p-divisibles et modules filtrés: le cas peu ramifié*, Bull. Soc. Math. France **108** (1980), 187–206.

[Lau10] E. Lau: *Frames and finite group schemes over complete regular local rings*, Doc. Math. **15** (2010), 545–569.

[Lau10+] E. Lau: *A relation between Dieudonné displays and crystalline Dieudonné theory*, arXiv:1006.2720v2 (2010).

[LLZ10] A. Lei, D. Loeffler and S. Zerbes: *Wach modules and Iwasawa theory for modular forms*, Asian J. Math. **14** (2010), no. 4, 475–528.

[LLZ11] A. Lei, D. Loeffler and S. Zerbes: *Coleman maps and the p-adic regulator*, Algebra Number Theory **5** (2011), no. 8, 1095–1131.

[LeZ12] A. Lei and S. Zerbes: *Signed Selmer groups over p-adic Lie extensions*, J. Théor. Nombres Bordeaux **24** (2012), no. 2, 377–403.

[Liu07] T. Liu: *Torsion p-adic Galois representations and a conjecture of Fontaine*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 4, 633–674.

[Liu08] T. Liu: *On lattices in semi-stable representations: a proof of a conjecture of Breuil*, Compos. Math. **144** (2008), no. 1, 61–88.

[Liu10] T. Liu: *A note on lattices in semi-stable representations*, Math. Ann. **346** (2010), no. 1, 117–138.

[Liu13] T. Liu: *The correspondence between Barsotti-Tate groups and Kisin modules when $p = 2$*, J. Théor. Nombres Bordeaux **25** (2013), no. 3, 661–676.

[Liu13+] T. Liu: *Compatibility of Kisin modules for different uniformizers*, arXiv:1302.1888v1 (2013).

[LoZ13] D. Loeffler and S. Zerbes: *Wach modules and critical slope p-adic L-functions*, J. Reine Angew. Math. **679** (2013), 181–206.

[Mat02] S. Matsuda: *Katz correspondence for quasi-unipotent overconvergent isocrystals*, Compos. Math. **134** (2002), no. 1, 1–34.

[Mat04] S. Matsuda: *Conjecture on Abbes-Saito filtration and Christol-Mebkhout filtration*, Geometric aspects of Dwork theory. Vol. I, II, 845–856, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[Mes72] W. Messing: *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Lecture Notes in Math. **264**, Springer-Verlag, Berlin-New York, 1972.

[Moc97] S. Mochizuki: *A version of the Grothendieck conjecture for p-adic local fields*, Internat. J. Math. **8** (1997), no. 4, 499–506.

[Moo00] H. Moon: *Finiteness results on certain mod p Galois representations*, J. Number Theory **84** (2000), no. 1, 156–165.

[Moo03] H. Moon: *The non-existence of certain mod p Galois representations*, Bull. Korean Math. Soc. **40** (2003), no. 3, 537–544.

[MT03] H. Moon and Y. Taguchi: *Refinement of Tate's discriminant bound and non-existence theorems for mod p Galois representations*, Kazuya Kato's fiftieth birthday. Doc. Math. 2003, Extra Vol., 641–654.

[Oze11] Y. Ozeki: *Non-existence of certain Galois representations with a uniform tame inertia weight*, Int. Math. Res. Not. IMRN 2011 **11**, 2377–2395.

[Oze13a] Y. Ozeki: *Torsion representations arising from $(\varphi, \hat{G})$-modules*, J. Number Theory **133** (2013), no. 11, 3810–3861.

[Oze13b] Y. Ozeki: *Non-existence of certain CM abelian varieties with prime power torsion*, Tohoku Math. J. (2) **65** (2013), no. 3, 357–371.

[OT14] Y. Ozeki and Y. Taguchi: *On congruences of Galois representations of number fields*, Publ. Res. Inst. Math. Sci. **50** (2014), no. 2, 287–306.

[Qui73] D. Quillen: *Higher algebraic K-theory: I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math. **341**, Springer, Berlin 1973.

[Ram93] R. Ramakrishna: *On a variation of Mazur's deformation functor*, Compos. Math. **87** (1993), no. 3, 269–286.

[Sai09] T. Saito: *Wild ramification and the characteristic cycle of an l-adic sheaf*, J. Inst. Math. Jussieu **8** (2009), no. 4, 769–829.

[Sai14+] T. Saito: *Characteristic cycle and the Euler number of a constructible sheaf on a surface*, arXiv:1402.5720v1 (2014).

[Sav04] D. Savitt: *Modularity of some potentially Barsotti-Tate Galois representations*, Compos. Math. **140** (2004), no. 1, 31–63.

[Sav05] D. Savitt: *On a conjecture of Conrad, Diamond, and Taylor*, Duke Math. J. **128** (2005), no. 1, 141–197.

[Schl06] A. Scholl: *Higher fields of norms and (φ, Γ)-modules*, Doc. Math. 2006, Extra Vol., 685–709.

[Schz12] P. Scholze: *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. **116** (2012), 245–313.

[Schf03] R. Schoof: *Abelian varieties over cyclotomic fields with good reduction everywhere*, Math. Ann. **325** (2003), no. 3, 413–448.

[Schf05] R. Schoof: *Abelian varieties over $\mathbb{Q}$ with bad reduction in one prime only*, Compos. Math. **141** (2005), no. 4, 847–868.

[Schf12] R. Schoof: *Semistable abelian varieties with good reduction outside* 15, Manuscripta Math. **139** (2012), no. 1-2, 49–70.

[Sen72] S. Sen: *Ramification in p-adic Lie extensions*, Invent. Math. **17** (1972), 44–50.

[Ser68] J.-P. Serre: *Corps locaux*, Deuxième édition. Publications de l'Université de Nancago, No. VIII. Hermann, Paris, 1968.

[Ser72] J.-P. Serre: *Propriétés galoisiennes des points d'ordre fini des courbes elliptiques*, Invent. Math. **15** (1972), no. 4, 259–331.

[Ser87] J.-P. Serre: *Sur les représentations modulaires de degré* 2 *de* $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230.

[Shi02] A. Shiho: *Crystalline fundamental groups II—Log convergent cohomology and rigid cohomology*, J. Math. Sci. Univ. Tokyo **9** (2002), no. 1, 1–163.

[Tat94] J. Tate: *The non-existence of certain Galois extensions of* $\mathbb{Q}$ *unramified outside* 2, Arithmetic geometry (Tempe, AZ, 1993), 153–156, Contemp. Math. **174**, Amer. Math. Soc., Providence, RI, 1994.

[Tav11] F. Tavares Ribeiro: *An explicit formula for the Hilbert symbol of a formal group*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 1, 261–318.

[Tay03] R. Taylor: *On icosahedral Artin representations II*, Amer. J. Math. **125** (2003), no. 3, 549–566.

[Tay06] R. Taylor: *On the meromorphic continuation of degree two* L*-functions*, Doc. Math. 2006, Extra Vol., 729–779.

[Tay08] R. Taylor: *Automorphy for some* l*-adic lifts of automorphic mod* l *Galois representations II*, Publ. Math. Inst. Hautes Études Sci. **108** (2008), 183–239.

[TW95] R. Taylor and A. Wiles: *Ring-theoretic properties of certain Hecke algebras*, Ann. of Math. (2) **141** (1995), no. 3, 553–572.

[Tia10] Y. Tian: *Canonical subgroups of Barsotti-Tate groups*, Ann. of Math. (2) **172** (2010), no. 2, 955–988.

[Tia11+] Y. Tian: *Classicality of overconvergent Hilbert eigenforms: case of quadratic residue degrees*, to appear in Rend. Sem. Mat. Univ. Padova.

[Tia12] Y. Tian: *An upper bound on the Abbes-Saito filtration for finite flat group schemes and applications*, Algebra Number Theory **6** (2012), no. 2, 231–242.

[Tsu98] N. Tsuzuki: *The local index and the Swan conductor*, Compos. Math. **111** (1998), no. 3, 245–288.

[VZ10] A. Vasiu and T. Zink: *Breuil's classification of* p*-divisible groups over regular local rings of arbitrary dimension*, Algebraic and arithmetic structures of moduli spaces (Sapporo 2007), 461–479, Adv. Stud. Pure Math. **58**, Math. Soc. Japan, Tokyo, 2010.

[Ver13] H. Verhoek: *Modularity of abelian varieties over* $\mathbb{Q}$ *with bad reduction in one prime only*, J. Number Theory **133** (2013), no. 9, 3065–3098.

[Wac96] N. Wach: *Représentations* p*-adiques potentiellement cristallines*, Bull. Soc. Math. France **124** (1996), no. 3, 375–400.

[Wac97] N. Wach: *Représentations cristallines de torsion*, Compos. Math. **108** (1997), no. 2, 185–240.

[Wes04] T. Weston: *Unobstructed modular deformation problems*, Amer. J. Math. **126** (2004), no. 6, 1237–1252.

[Wil95] A. Wiles: *Modular elliptic curves and Fermat's last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551.

[Win83] J.-P. Wintenberger: *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Ann. Sci. École Norm. Sup. (4) **16** (1983), no. 1, 59–89.

[Win84] J.-P. Wintenberger: *Un scindage de la filtration de Hodge pour certaines variétés algébriques sur les corps locaux*, Ann. of Math. (2) **119** (1984), no. 3, 511–548.

[Xia10] L. Xiao: *On ramification filtrations and* p*-adic differential modules, I: the equal characteristic case*, Algebra Number Theory **4** (2010), no. 8, 969–1027.

[Xia12] L. Xiao: *On ramification filtrations and* p*-adic differential equations, II: mixed characteristic case*, Compos. Math. **148** (2012), no. 2, 415–463.

[Yos10] M. Yoshida: *Ramification of local fields and Fontaine's property* (P_m), J. Math. Sci. Univ. Tokyo **17** (2010), no. 3, 247–265.

[Zhu03] I. Zhukov: *On ramification theory in the case of an imperfect residue field*, Mat. Sb. **194** (2003), no. 12, 3–30; translation in Sb. Math. **194** (2003), no. 11-12, 1747–1774.

[Zin02] T. Zink: *The display of a formal* p*-divisible group*, Cohomologies p-adiques et applications arithmétiques I, Astérisque **278** (2002), 127–248.

E-mail address: `shin-h@math.kyushu-u.ac.jp`

Faculty of Mathematics, Kyushu University