ON THE COMPACTIFICATION OF THE DRINFELD MODULAR CURVE OF LEVEL $\Gamma_1^{\Delta}(\mathfrak{n})$

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ABSTRACT. Let p be a rational prime and q a power of p. Let $\mathfrak{n}$ be a non-constant monic polynomial in $\mathbb{F}_q[t]$ which has a prime factor of degree prime to $q-1$. In this paper, we define a Drinfeld modular curve $Y_1^{\Delta}(\mathfrak{n})$ over $A[1/\mathfrak{n}]$ and study the structure around cusps of its compactification $X_1^{\Delta}(\mathfrak{n})$, in a parallel way to Katz-Mazur's work on classical modular curves. Using them, we also define a Hodge bundle over $X_1^{\Delta}(\mathfrak{n})$ such that Drinfeld modular forms of level $\Gamma_1(\mathfrak{n})$, weight k and some type are identified with global sections of its k-th tensor power.

1. INTRODUCTION

Let p be a rational prime and q a power of p. Put $A = \mathbb{F}_q[t]$, $K_\infty = \mathbb{F}_q((1/t))$ and let $\mathbb{C}_\infty$ be the $(1/t)$-adic completion of an algebraic closure of K_∞. We denote by Ω the Drinfeld upper half plane $\mathbb{C}_\infty \backslash K_\infty$, which has a natural structure of a rigid analytic variety over K_∞. Let $\mathfrak{n}$ and $\wp$ be monic polynomials in A such that $\wp$ is irreducible of degree $d > 0$ and prime to $\mathfrak{n}$. We put

$$\Gamma_1(\mathfrak{n}) = \left\{ \gamma \in GL_2(A) \,\middle|\, \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{n} \right\}$$

and $\Gamma_{11}(\mathfrak{n}) = \Gamma_1(\mathfrak{n}) \cap SL_2(A)$. Let K be the $\wp$-adic completion of $\mathbb{F}_q(t)$, which is a complete discrete valuation field with uniformizer $\wp$.

For any $k \in \mathbb{Z}$ and $l \in \mathbb{Z}/(q-1)$, a Drinfeld modular form of level $\Gamma_1(\mathfrak{n})$, weight k and type l is a rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (ad-bc)^{-l}(cz+d)^k f(z) \text{ for any } z \in \Omega,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(\mathfrak{n})$$

and a certain holomorphy condition at cusps. It is a function field analogue of the notion of elliptic modular form of level $\Gamma_1(N)$ and weight k. As in the latter case, for any non-constant $\mathfrak{n}$, Drinfeld modular forms of level $\Gamma_1(\mathfrak{n})$ and weight k are identified with global sections of the

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k-th tensor power of a natural line bundle $\bar{\omega}_{\mathbb{C}_\infty}$ on an algebraic curve $X_{11}(\mathfrak{n})_{\mathbb{C}_\infty}$ over $\mathbb{C}_\infty$ called Drinfeld modular curve of level $\Gamma_{11}(\mathfrak{n})$. The curve $X_{11}(\mathfrak{n})_{\mathbb{C}_\infty}$ is the compactification of a moduli space $Y_{11}(\mathfrak{n})_{\mathbb{C}_\infty}$ of Drinfeld modules of rank two endowed with some level structures such that $Y_{11}(\mathfrak{n})_{\mathbb{C}_\infty}(\mathbb{C}_\infty)$ is identified with $\Gamma_{11}(\mathfrak{n})\backslash\Omega$.

We also have a $\wp$-adic version of the notion of Drinfeld modular form—$\wp$-adic Drinfeld modular form [Vin, Gos2]. The latter is defined as the $\wp$-adic limit in $K[[x]]$ of Fourier expansions at ∞ of Drinfeld modular forms with expansion coefficients in $\mathbb{F}_q(t)$. It is expected that Drinfeld modular forms have deep $\wp$-adic properties which are comparable to p-adic properties of elliptic modular forms.

To investigate $\wp$-adic properties of Drinfeld modular forms, we need to define models X and $\bar{\omega}$ of $X_{11}(\mathfrak{n})_{\mathbb{C}_\infty}$ and $\bar{\omega}_{\mathbb{C}_\infty}$ over $A[1/\mathfrak{n}]$ in order to pass to $\mathcal{O}_K$. The problem is that, the study around cusps of Drinfeld modular curves in the literature [Dri, Gos1, Gek1, Gek2, Gek3, vdPT, vdH, Böc] is carried out by, first describing the formal completion for the case of the Drinfeld modular curve $X(\mathfrak{n})$ of full level over $A[1/\mathfrak{n}]$ and then taking the quotient by an appropriate group acting on $X(\mathfrak{n})$. Since this group action is not necessarily free at cusps (in fact, the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_{11}(\mathfrak{n})/\Gamma(\mathfrak{n})$ stabilizes ∞), it is unclear if the Hodge bundle on $X(\mathfrak{n})$ descends to a model X over $A[1/\mathfrak{n}]$ and we need a more precise study of the formal completion along cusps.

In this paper, we resolve it by following the method of Katz-Mazur [KM] in the case of classical modular curves. For this, we need to assume that the level $\mathfrak{n}$ has a prime factor of degree prime to $q-1$. This ensures the existence of a subgroup $\Delta \subseteq (A/(\mathfrak{n}))^\times$ which is a direct summand of $\mathbb{F}_q^\times$. Under this mild assumption, a $\Gamma_1^\Delta(\mathfrak{n})$-structure is defined as a pair of a usual $\Gamma_1(\mathfrak{n})$-structure and an additional structure admitting an $\mathbb{F}_q^\times$-action. In particular, for any $A[1/\mathfrak{n}]$-algebra R_0 which is an excellent regular ring, we have a fine moduli scheme $Y_1^\Delta(\mathfrak{n})_{R_0}$ classifying Drinfeld modules with $\Gamma_1^\Delta(\mathfrak{n})$-structures and also its compactification $X_1^\Delta(\mathfrak{n})_{R_0}$. Then we can show that $X_1^\Delta(\mathfrak{n})_{A[1/\mathfrak{n}]}$ is a model of $X_{11}(\mathfrak{n})_{\mathbb{C}_\infty}$. It also enables us to control types of Drinfeld modular forms by a diamond operator [Hat].

Let $\widehat{\mathrm{Cusps}}^\Delta_{R_0}$ be the formal completion of $X_1^\Delta(\mathfrak{n})_{R_0}$ along the cusps and $\mathrm{Cusps}^\Delta_{R_0}$ its reduction. Then we will prove the following theorems.

Theorem 1.1 (Theorem 5.3)**.** *Let R_0 be a flat $A[1/\mathfrak{n}]$-algebra which is an excellent regular domain.*

(1) *Let* P^{Δ}_{∞} *be the* ∞*-cusp of* $X^{\Delta}_1(\mathfrak{n})_{R_0}$. *Then there exists a natural isomorphism of complete local rings*

$$(x^{\Delta}_{\infty})^* : \hat{\mathcal{O}}_{X^{\Delta}_1(\mathfrak{n})_{R_0}, P^{\Delta}_{\infty}} \to R_0[[x]].$$

(2) *The Hodge bundle on* $Y^{\Delta}_1(\mathfrak{n})_{R_0}$ *extends to an invertible sheaf* $\bar{\omega}^{\Delta}_{\mathrm{un}}$ *on* $X^{\Delta}_1(\mathfrak{n})_{R_0}$ *satisfying*

$$(x^{\Delta}_{\infty})^*(\bar{\omega}^{\Delta}_{\mathrm{un}}) = R_0[[x]]dX,$$

where dX *denotes an invariant differential form of a Tate-Drinfeld module* $\mathrm{TD}^{\nabla}(\Lambda)$.

(3) *The formation of* $\bar{\omega}^{\Delta}_{\mathrm{un}}$ *is compatible with any base change* $R_0 \to R'_0$ *of flat* $A[1/\mathfrak{n}]$*-algebras which are excellent regular domains.*

(4) *There exist natural actions of* $\mathbb{F}_q^{\times}$ *on* $X^{\Delta}_1(\mathfrak{n})_{R_0}$ *and on* $\bar{\omega}^{\Delta}_{\mathrm{un}}$ *covering the former action.*

Theorem 1.2 (Theorem 6.3)**.** *Let* R_0 *be a flat* $A[1/\mathfrak{n}]$*-algebra which is an excellent regular domain. Let* $W_{\mathfrak{n}}(X)$ *be the* $\mathfrak{n}$*-th Carlitz cyclotomic polynomial* [Car] *and* $R_{\mathfrak{n}}$ *the affine ring of a connected component of* $\mathrm{Spec}(R_0[X]/(W_{\mathfrak{n}}(X)))$. *We also put*

$$\bar{\Gamma}^1_1 = \left\{ \gamma \in SL_2(A/(\mathfrak{n})) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \bmod \mathfrak{n} \right\}.$$

(1) *We have a natural isomorphism*

$$\widehat{\mathrm{Cusps}}^{\Delta}_{R_0} \times_{R_0} R_{\mathfrak{n}} \simeq \coprod_{(a,b)} \mathrm{Spec}(R_{\mathfrak{n}}[[w]]),$$

where the direct sum is taken over a complete representative of the set

$$\mathbb{F}_q^{\times} \backslash \{(a,b) \in (A/(\mathfrak{n}))^2 \mid (a,b) = (1)\} / \bar{\Gamma}^1_1.$$

(2) $\mathrm{Cusps}^{\Delta}_{R_0}$ *is finite etale over* R_0. *In particular, it defines an effective Cartier divisor of* $X^{\Delta}_1(\mathfrak{n})_{R_0}$ *over* R_0.

(3) *For any* $(a,b) \in (A/(\mathfrak{n}))^2$ *satisfying* $(a,b) = (1)$, *we denote by* f_b *the monic generator of the ideal* $\mathrm{Ann}_A(b(A/(\mathfrak{n})))$ *and by* $\Phi^C_{f_b}$ *the* f_b*-multiplication map of the Carlitz module* C. *Then, at each point of* $\mathrm{Cusps}^{\Delta}_{R_0}$ *in the component labeled by* (a,b), *the invertible sheaf*

$$\Omega^1_{X^{\Delta}_1(\mathfrak{n})_{R_0}/R_0}(2\mathrm{Cusps}^{\Delta}_{R_0})$$

is locally generated by the section dx/x^2, *where* x *is defined by* $1/x = \Phi^C_{f_b}(1/w)$.

We also have similar results for the case of level $\Gamma_1^\Delta(\mathfrak{n}) \cap \Gamma_0(\wp)$ (§7).

For the proof of the above theorems, the main differences from [KM] are twofold: First, the j-invariant j_t of the usual Tate-Drinfeld module does not give (the inverse of) a uniformizer of the j-line at the infinity, contrary to the case of the Tate curve. For this, we use a descent $\mathrm{TD}^\triangledown(\Lambda)$ of the Tate-Drinfeld module by an $\mathbb{F}_q^\times$-action on the coefficients to obtain a right j-invariant (see (5.1)). This enables us to study Drinfeld modular curves directly in §5, not via taking quotients of $X(\mathfrak{n})$. As a trade-off, we need to consider $\Gamma_1^\Delta(\mathfrak{n})$-structures, not just $\Gamma_1(\mathfrak{n})$-structures, in order to kill an effect of the descent. The author learned the idea of the use of the descent from a work of Armana [Arm].

Second, since we are in the positive characteristic situation with wild ramification along cusps, we cannot use Abhyankar's lemma to study the structure of Drinfeld modular curves around cusps. This is bypassed by a direct computation of the formal completion along each cusp over $R_\mathfrak{n}$ (§6).

In the paper [Hat], the above theorems are combined with a duality theory of Taguchi [Tag] for Drinfeld modules of rank two, which compensates the lack of autoduality for Drinfeld modules, to develop a geometric theory of $\wp$-adic Drinfeld modular forms in a similar way to [Kat].

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2. Drinfeld modules

For any scheme S over $\mathbb{F}_q$, we denote the q-th power Frobenius map on S by $F_S : S \to S$. For any S-scheme T and $\mathcal{O}_S$-module $\mathcal{L}$, we put $T^{(q)} = T \times_{S,F_S} S$ and $\mathcal{L}^{(q)} = F_S^*(\mathcal{L})$. For any A-scheme S, the image of $t \in A$ by the structure map $A \to \mathcal{O}_S(S)$ is denoted by θ.

For any scheme S over $\mathbb{F}_q$ and any invertible $\mathcal{O}_S$-module $\mathcal{L}$, we write the associated covariant line bundle to $\mathcal{L}$ as

$$\mathbb{V}_*(\mathcal{L}) = \mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{L}^{\otimes -1}))$$

with $\mathcal{L}^{\otimes -1} := \mathcal{L}^\vee = \mathscr{H}om_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_S)$. It represents the functor over S defined by $T \mapsto \mathcal{L}|_T(T)$, where $\mathcal{L}|_T$ denotes the pull-back to T, and thus we identify $\mathcal{L}$ with $\mathbb{V}_*(\mathcal{L})$. We have the q-th power Frobenius map

$$\tau : \mathcal{L} \to \mathcal{L}^{\otimes q}, \quad l \mapsto l^{\otimes q},$$

by which we identify $\mathcal{L}^{(q)}$ with $\mathcal{L}^{\otimes q}$. This map induces a homomorphism of group schemes over S

$$\tau : \mathbb{V}_*(\mathcal{L}) \to \mathbb{V}_*(\mathcal{L}^{\otimes q}).$$

Definition 2.1 ([Lau], Remark (1.2.2))**.** Let S be a scheme over A and r a positive integer. A (standard) Drinfeld (A-)module of rank r over S is a pair $E = (\mathcal{L}, \Phi^E)$ of an invertible sheaf $\mathcal{L}$ on S and an $\mathbb{F}_q$-algebra homomorphism

$$\Phi^E : A \to \mathrm{End}_S(\mathbb{V}_*(\mathcal{L}))$$

satisfying the following conditions for any $a \in A \backslash \{0\}$:

- the image Φ^E_a of a by Φ^E is written as

$$\Phi^E_a = \sum_{i=0}^{r \deg(a)} \alpha_i(a)\tau^i, \quad \alpha_i(a) \in \mathcal{L}^{\otimes 1-q^i}(S)$$

  with $\alpha_{r\deg(a)}(a)$ nowhere vanishing.
- $\alpha_0(a)$ is equal to the image of a by the structure map $A \to \mathcal{O}_S(S)$.

We often refer to the underlying A-module scheme $\mathbb{V}_*(\mathcal{L})$ as E. A morphism $(\mathcal{L}, \Phi) \to (\mathcal{L}', \Phi')$ of Drinfeld modules over S is defined to be a morphism of A-module schemes $\mathbb{V}_*(\mathcal{L}) \to \mathbb{V}_*(\mathcal{L}')$ over S.

We denote the Carlitz module over S by C: it is the Drinfeld module $(\mathcal{O}_S, \Phi^C)$ of rank one over S defined by $\Phi^C_t = \theta + \tau$. We identify the underlying group scheme of C with $\mathbb{G}_{\mathrm{a}} = \mathrm{Spec}_S(\mathcal{O}_S[Z])$ using $1 \in \mathcal{O}_S(S)$.

Lemma 2.2. *(1) Let E be a line bundle over S. Let $\mathcal{H}$ be a finite locally free closed $\mathbb{F}_q$-submodule scheme of E over S. Suppose that the rank of $\mathcal{H}$ is a constant q-power. Then $E/\mathcal{H}$ is a line bundle over S.*

(2) Let E be a Drinfeld module of rank r. Let $\mathcal{H}$ be a finite locally free closed A-submodule scheme of E of constant q-power rank over S. Suppose either
- *$\mathcal{H}$ is etale over S, or*
- *S is reduced and for any maximal point η of S, the fiber $\mathcal{H}_\eta$ of $\mathcal{H}$ over η is etale.*

Then $E/\mathcal{H}$ is a Drinfeld module of rank r with the induced A-action.

Proof. The assertion (1) follows in the same way as [Leh, Ch. 1, Proposition 3.2]. For (2), we may assume that $S = \mathrm{Spec}(B)$ is affine, the underlying invertible sheaves of E and $E/\mathcal{H}$ are trivial and $\mathcal{H}$ is free of rank q^n over S. We write the t-multiplication maps of E and $E/\mathcal{H}$ as

$$\Phi^E_t(X) = \theta X + a_1 X^q + \cdots + a_r X^{q^r}, \quad \Phi^{E/\mathcal{H}}_t(X) = b_0 X + b_1 X^q + \cdots + b_s X^{q^s}$$

with $b_s \neq 0$. From the proof of [Leh, Ch. 1, Proposition 3.2], we may also assume that the map $E \to E/\mathcal{H}$ is defined by an $\mathbb{F}_q$-linear monic additive polynomial

$$X \mapsto P(X) = p_1 X + \cdots + p_{n-1} X^{q^{n-1}} + X^{q^n}.$$

From the equality $\Phi_t^{E/\mathcal{H}}(P(X)) = P(\Phi_t^E(X))$, we obtain $r = s$, $b_r = a_r^{q^n}$ and $p_1(b_0 - \theta) = 0$. If $\mathcal{H}$ is etale over B, then we have $p_1 \in B^\times$ and thus $b_0 = \theta$. If the latter assumption in the lemma holds, then $p_1 \in B$ is a non-zero divisor in the ring $B/\mathfrak{p}$ for any minimal prime ideal $\mathfrak{p}$. Since B is reduced, it is a subring of $\prod B/\mathfrak{p}$, where the product is taken over the set of minimal prime ideals $\mathfrak{p}$ of B. This also yields $b_0 = \theta$, and thus $E/\mathcal{H}$ is a Drinfeld module of rank r in both cases. □

Next let $\wp$ be a monic irreducible polynomial of degree $d > 0$ in $A = \mathbb{F}_q[t]$, as before. Let $\bar{S}$ be an A-scheme of characteristic $\wp$ and $\bar{E} = (\bar{\mathcal{L}}, \Phi^{\bar{E}})$ a Drinfeld module of rank two over $\bar{S}$. By [Sha, Proposition 2.7], we can write as

$$\Phi_\wp^{\bar{E}} = (\alpha_d(\bar{E}) + \cdots + \alpha_{2d}(\bar{E})\tau^d)\tau^d, \quad \alpha_i(\bar{E}) \in \bar{\mathcal{L}}^{\otimes 1-q^i}(\bar{S}).$$

We put

$$F_{d,\bar{E}} = \tau^d : \bar{E} \to \bar{E}^{(q^d)}, \quad V_{d,\bar{E}} = \alpha_d(\bar{E}) + \cdots + \alpha_{2d}(\bar{E})\tau^d : \bar{E}^{(q^d)} \to \bar{E}.$$

We also denote them by F_d and V_d if no confusion may occur. They are isogenies of Drinfeld modules satisfying $V_d \circ F_d = \Phi_\wp^{\bar{E}}$ and $F_d \circ V_d = \Phi_\wp^{\bar{E}^{(q^d)}}$ [Sha, §2.8].

Definition 2.3. We say $\bar{E}$ is ordinary if $\alpha_d(\bar{E}) \in \bar{\mathcal{L}}^{\otimes 1-q^d}(\bar{S})$ is nowhere vanishing, and supersingular if $\alpha_d(\bar{E}) = 0$.

By [Sha, Proposition 2.14], $\bar{E}$ is ordinary if and only if $\mathrm{Ker}(V_d)$ is etale.

3. Drinfeld modular curves

Let $\mathfrak{n}$ be a non-constant monic polynomial in $A = \mathbb{F}_q[t]$ which is prime to $\wp$. Put $A_\mathfrak{n} = A[1/\mathfrak{n}]$. For any Drinfeld module E of rank two over an A-scheme S and a non-constant monic polynomial $\mathfrak{m} \in A$, a $\Gamma(\mathfrak{m})$-structure on E is an A-linear homomorphism $\alpha : (A/(\mathfrak{m}))^2 \to E(S)$ inducing the equality of effective Cartier divisors of E

$$\sum_{a \in (A/(\mathfrak{m}))^2} [\alpha(a)] = E[\mathfrak{m}].$$

If $\mathfrak{m}$ is invertible in S, then it is the same as an isomorphism of A-module schemes $\alpha : \underline{(A/(\mathfrak{m}))^2} \to E[\mathfrak{m}]$ over S, where the underline means the constant A-module scheme. If $\mathfrak{m}$ has at least two different prime factors, then the functor over A sending S to the set of isomorphism classes of such pairs (E,α) over S is represented by a regular affine scheme $Y(\mathfrak{m})$ of dimension two which is flat and of finite type over A. Over $A[1/\mathfrak{m}]$, for any non-constant $\mathfrak{m}$ this functor is representable by an affine scheme $Y(\mathfrak{m})$ which is smooth of relative dimension one over $A[1/\mathfrak{m}]$. The natural left action of $GL_2(A/(\mathfrak{m}))$ on $(A/(\mathfrak{m}))^2$ induces a right action of this group on $Y(\mathfrak{m})$.

For any Drinfeld module E of rank two over an $A_{\mathfrak{n}}$-scheme S, we define a $\Gamma_1(\mathfrak{n})$-structure on E as a closed immersion of A-module schemes $\lambda : C[\mathfrak{n}] \to E$ over S. Since $C[\mathfrak{n}]$ is etale over S, we see that over a finite etale cover of $A_{\mathfrak{n}}$ a $\Gamma_1(\mathfrak{n})$-structure on E is identified with a closed immersion of A-module schemes $\underline{A/(\mathfrak{n})} \to E$. Then [Fli, Proposition 4.2 (2)] implies that E has no non-trivial automorphism fixing λ. Note that the quotient $E[\mathfrak{n}]/\mathrm{Im}(\lambda)$ is a finite etale A-module scheme over S which is etale locally isomorphic to $\underline{A/(\mathfrak{n})}$, and thus the functor

$$\mathscr{I}\!som_{A,S}(\underline{A/(\mathfrak{n})}, E[\mathfrak{n}]/\mathrm{Im}(\lambda))$$

is represented by a finite etale $(A/(\mathfrak{n}))^\times$-torsor $I_{(E,\lambda)}$ over S.

Consider the functor over $A_{\mathfrak{n}}$ sending an $A_{\mathfrak{n}}$-scheme S to the set of isomorphism classes $[(E,\lambda)]$ of pairs (E,λ) consisting of a Drinfeld module E of rank two over S and a $\Gamma_1(\mathfrak{n})$-structure λ on E. Then we can show that this functor is representable by an affine scheme $Y_1(\mathfrak{n})$ which is smooth over $A_{\mathfrak{n}}$ of relative dimension one.

Suppose that there exists a prime factor $\mathfrak{q}$ of $\mathfrak{n}$ such that its residue extension $k(\mathfrak{q})/\mathbb{F}_q$ is of degree prime to $q-1$. In this case, the inclusion $\mathbb{F}_q^\times \to k(\mathfrak{q})^\times$ splits and we can choose a subgroup $\Delta \subseteq (A/(\mathfrak{n}))^\times$ such that the natural map $\Delta \to (A/(\mathfrak{n}))^\times/\mathbb{F}_q^\times$ is an isomorphism. For such Δ, we define a $\Gamma_1^\Delta(\mathfrak{n})$-structure on E as a pair $(\lambda,[\mu])$ of a $\Gamma_1(\mathfrak{n})$-structure λ on E and an element $[\mu] \in (I_{(E,\lambda)}/\Delta)(S)$. We have a fine moduli scheme $Y_1^\Delta(\mathfrak{n})$ of the isomorphism classes of triples $(E,\lambda,[\mu])$, which is finite etale over $Y_1(\mathfrak{n})$. The universal Drinfeld module over $Y_1^\Delta(\mathfrak{n})$ is denoted by $E^\Delta_{\mathrm{un}} = \mathbb{V}_*(\mathcal{L}^\Delta_{\mathrm{un}})$ and put

$$\omega^\Delta_{\mathrm{un}} := \omega_{E^\Delta_{\mathrm{un}}} = (\mathcal{L}^\Delta_{\mathrm{un}})^\vee,$$

where $\omega_{E^\Delta_{\mathrm{un}}}$ denotes the sheaf of invariant differential forms on E^Δ_{un}.

For any Drinfeld module E over an $A_{\mathfrak{n}}$-scheme S, a $\Gamma_0(\wp)$-structure on E is a finite locally free closed A-submodule scheme $\mathcal{G}$ of $E[\wp]$ of rank q^d over S. Then we have a fine moduli scheme $Y_1^\Delta(\mathfrak{n},\wp)$ classifying tuples $(E,\lambda,[\mu],\mathcal{G})$ consisting of a Drinfeld module E of rank two over an $A_{\mathfrak{n}}$-scheme S, a $\Gamma_1^\Delta(\mathfrak{n})$-structure $(\lambda,[\mu])$ and a $\Gamma_0(\wp)$-structure $\mathcal{G}$ on

E. From the theory of Hilbert schemes, we see that the natural map $Y_1^\Delta(\mathfrak{n},\wp) \to Y_1^\Delta(\mathfrak{n})$ is finite, and it is also etale over $A_\mathfrak{n}[1/\wp]$. For any $A_\mathfrak{n}$-algebra R, we write as $Y_1^\Delta(\mathfrak{n})_R = Y_1^\Delta(\mathfrak{n}) \times_{A_\mathfrak{n}} \mathrm{Spec}(R)$ and similarly for other Drinfeld modular curves.

Lemma 3.1. $Y_1^\Delta(\mathfrak{n},\wp)$ *is smooth over* $A_\mathfrak{n}$ *outside finitely many supersingular points on the fiber over* $(\wp)$*.*

Proof. Let B be an Artinian local $A_\mathfrak{n}$-algebra of characteristic $\wp$ and J an ideal of B satisfying $J^2 = 0$. Let E be an ordinary Drinfeld module of rank two over B/J and $\mathcal{G}$ a $\Gamma_0(\wp)$-structure on E. Since B is local, the underlying invertible sheaf of E is trivial. It is enough to show that the isomorphism class of the pair $(E,\mathcal{G})$ lifts to B.

Since E is ordinary and B/J is Artinian local, we have either $\mathcal{G} = \mathrm{Ker}(F_{d,E})$ or the composite $\mathcal{G} \to E[\wp] \to \mathrm{Ker}(V_{d,E})$ is an isomorphism. In the former case, write as $\Phi_t^E = \theta + a_1\tau + a_2\tau^2$. For any lift $\hat{a}_i \in B$ of a_i, we can define a structure of a Drinfeld module of rank two over B on $\hat{E} = \mathrm{Spec}(B[X])$ by putting $\Phi_t^{\hat{E}} = \theta + \hat{a}_1\tau + \hat{a}_2\tau^2$, which is also ordinary. Then $\mathcal{G}$ lifts to $\mathrm{Ker}(F_{d,\hat{E}})$. In the latter case $\mathcal{G}$ is etale and, by Lemma 2.2 (2), $E/\mathcal{G}$ has a structure of a Drinfeld module of rank two. Moreover, it is also ordinary since $(E/\mathcal{G})[\wp]$ has the etale quotient $\mathcal{G}$. Thus we have isomorphisms

$$(E/\mathcal{G})^{(q^d)} \xleftarrow[\sim]{F_{d,E/\mathcal{G}}} (E/\mathcal{G})/\mathrm{Ker}(F_{d,E}) \xrightarrow[\sim]{\wp} E$$

sending $\mathrm{Ker}(V_{d,E/\mathcal{G}})$ to $\mathcal{G}$. Since the above argument shows that $E/\mathcal{G}$ also lifts to an ordinary Drinfeld module $\hat{F}$ of rank two over B, the pair $(E,\mathcal{G})$ lifts to the pair $(\hat{F}^{(q^d)}, \mathrm{Ker}(V_{d,\hat{F}}))$ over B. $\square$

Put $K_\infty = \mathbb{F}_q((1/t))$ and let $\mathbb{C}_\infty$ be the $(1/t)$-adic completion of an algebraic closure of K_∞. Let $\mathbb{A}_f$ be the ring of finite adeles (namely, the restricted direct product over the set of places of $\mathbb{F}_q(t)$ other than the $(1/t)$-adic one) and $\hat{A}$ its subring of elements which are integral at all finite places. Let Ω be the Drinfeld upper half plane over $\mathbb{C}_\infty$. Put

$$K_1^\Delta(\mathfrak{n}) = \left\{ g \in GL_2(\hat{A}) \;\middle|\; g \bmod \mathfrak{n}\hat{A} \in \begin{pmatrix} \Delta & A/(\mathfrak{n}) \\ 0 & 1 \end{pmatrix} \right\},$$

$$\Gamma(\mathfrak{n}) = \left\{ g \in GL_2(A) \;\middle|\; g \bmod (\mathfrak{n}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and $\Gamma_1^\Delta(\mathfrak{n}) = GL_2(A) \cap K_1^\Delta(\mathfrak{n})$. Since $A^\times = \mathbb{F}_q^\times$, we have $\Gamma_1^\Delta(\mathfrak{n}) \subseteq SL_2(A)$. This yields

$$\Gamma_1^\Delta(\mathfrak{n}) = \left\{ g \in SL_2(A) \;\middle|\; g \bmod (\mathfrak{n}) \in \begin{pmatrix} 1 & A/(\mathfrak{n}) \\ 0 & 1 \end{pmatrix} \right\}.$$

In particular, the group $\Gamma_1^{\Delta}(\mathfrak{n})$ is independent of the choice of Δ. Note that the natural right action of $g \in GL_2(A/(\mathfrak{n}))$ on $Y(\mathfrak{n})_{\mathbb{C}_\infty}$ corresponds to the left action of ${}^t g$ on $\Gamma(\mathfrak{n})\backslash\Omega$ via the Möbius transformation. Since $\mathbb{F}_q^\times \det(K_1^{\Delta}(\mathfrak{n})) = \hat{A}^\times$, [Dri, Proposition 6.6] implies that the analytification of $Y_1^{\Delta}(\mathfrak{n})_{\mathbb{C}_\infty}$ is identified with

$$GL_2(\mathbb{F}_q(t))\backslash\Omega \times GL_2(\mathbb{A}_f)/K_1^{\Delta}(\mathfrak{n}) = \Gamma_1^{\Delta}(\mathfrak{n})\backslash\Omega,$$

and thus the fiber $Y_1^{\Delta}(\mathfrak{n})_{K_\infty}$ is geometrically connected. Similarly, we see that $Y_1^{\Delta}(\mathfrak{n},\wp)_{K_\infty}$ is also geometrically connected.

For any Drinfeld module E of rank two over S, we write the t-multiplication map of E as $\Phi_t^E = \theta + a_1\tau + a_2\tau^2$ and put

$$j_t(E) = a_1^{\otimes q+1} \otimes a_2^{\otimes -1} \in \mathcal{O}_S(S).$$

Consider the finite flat map

$$j_t : Y_1^{\Delta}(\mathfrak{n}) \to \mathbb{A}^1_{A_\mathfrak{n}} = \mathrm{Spec}(A_\mathfrak{n}[j]), \quad j \mapsto j_t(E^{\Delta}_{\mathrm{un}})$$

and a similar finite map for $Y_1^{\Delta}(\mathfrak{n},\wp)$. We define the compactifications $X_1^{\Delta}(\mathfrak{n})$ and $X_1^{\Delta}(\mathfrak{n},\wp)$ of $Y_1^{\Delta}(\mathfrak{n})$ and $Y_1^{\Delta}(\mathfrak{n},\wp)$ as the normalizations of $\mathbb{P}^1_{A_\mathfrak{n}}$ in $Y_1^{\Delta}(\mathfrak{n})$ and $Y_1^{\Delta}(\mathfrak{n},\wp)$ via this map, respectively. As in [Sha, §7.2], we see that $X_1^{\Delta}(\mathfrak{n})$ is smooth over $A_\mathfrak{n}$ and $X_1^{\Delta}(\mathfrak{n},\wp)$ is smooth over $A_\mathfrak{n}[1/\wp]$. By a similar argument to the proof of [KM, Corollary 10.9.2], Zariski's connectedness theorem implies that each fiber of the map $X_1^{\Delta}(\mathfrak{n}) \to \mathrm{Spec}(A_\mathfrak{n})$ is geometrically connected, and so is $X_1^{\Delta}(\mathfrak{n},\wp) \to \mathrm{Spec}(A_\mathfrak{n}[1/\wp])$. For any $A_\mathfrak{n}$-algebra R which is Noetherian, excellent and regular, we also have the compactifications $X_1^{\Delta}(\mathfrak{n})_R$ and $X_1^{\Delta}(\mathfrak{n},\wp)_R$ of $Y_1^{\Delta}(\mathfrak{n})_R$ and $Y_1^{\Delta}(\mathfrak{n},\wp)_R$. From the smoothness of $X_1^{\Delta}(\mathfrak{n})$, we have $X_1^{\Delta}(\mathfrak{n})_R = X_1^{\Delta}(\mathfrak{n}) \times_{A_\mathfrak{n}} \mathrm{Spec}(R)$. The base change compatibility also holds for $X_1^{\Delta}(\mathfrak{n},\wp)_R$ if $\wp$ is invertible in R.

On the other hand, the maps

$$[(E,\lambda,[\mu])] \mapsto [(E,a\lambda,[\mu])], \quad [(E,\lambda,[\mu])] \mapsto [(E,\lambda,c[\mu])]$$

induce actions of the groups $(A/(\mathfrak{n}))^\times$ and $(A/(\mathfrak{n}))^\times/\Delta = \mathbb{F}_q^\times$ on $X_1^{\Delta}(\mathfrak{n})_R$. We denote them by $\langle a\rangle_\mathfrak{n}$ and $\langle c\rangle_\Delta$, respectively.

Lemma 3.2. *Let S be a scheme over A and E a Drinfeld module of rank two over S. If $j_t(E) \in \mathcal{O}_S(S)$ is invertible, then for the big fppf sheaf $\mathscr{A}ut_{A,S}(E)$ defined by*

$$T \mapsto \mathrm{Aut}_{A,T}(E|_T),$$

the natural map $\underline{\mathbb{F}_q^\times} \to \mathscr{A}ut_{A,S}(E)$ is an isomorphism.

Proof. We may assume that $S = \mathrm{Spec}(B)$ is affine and the underlying invertible sheaf of E is trivial. By [Fli, Proposition 4.2 (2)], any automorphism of $E = \mathrm{Spec}(B[X])$ is linear, namely it is given by $X \mapsto bX$ for some $b \in B^\times$. Write as $\Phi_t^E = \theta + a_1\tau + a_2\tau^2$. From the assumption, we have $a_1 \in B^\times$ and the equality $\Phi_t^E(bX) = b\Phi_t^E(X)$ yields $b^{q-1} = 1$. Since the group scheme μ_{q-1} over $\mathbb{F}_q$ is isomorphic to the constant group scheme $\underline{\mathbb{F}_q^\times}$, so is $\mu_{q-1}|_B$ over the $\mathbb{F}_q$-algebra B. This concludes the proof. □

Lemma 3.3. *Let S be a scheme over A. Let E and E' be Drinfeld modules of rank two over S satisfying $j_t(E) = j_t(E') \in \mathcal{O}_S(S)^\times$. Then the big fppf sheaf $\mathscr{I}som_{A,S}(E, E')$ over S defined by*

$$T \mapsto \mathrm{Isom}_{A,T}(E|_T, E'|_T)$$

is represented by a Galois covering of S with Galois group $\mathbb{F}_q^\times$.

Proof. By gluing, we reduce ourselves to the case where $S = \mathrm{Spec}(B)$ is affine and the underlying line bundles of E and E' are trivial. We write the t-multiplication maps of E and E' as

$$\Phi_t^E = \theta + a_1\tau + a_2\tau^2, \quad \Phi_t^{E'} = \theta + a_1'\tau + a_2'\tau^2$$

with some $a_1, a_1' \in B$ and $a_2, a_2' \in B^\times$. By assumption, we have $a_1^{q+1}/a_2 = (a_1')^{q+1}/a_2' \in B^\times$ and thus $a_1, a_1' \in B^\times$. Hence the scheme

$$J = \mathrm{Spec}(B[Y]/(Y^{q-1} - a_1/a_1'))$$

is a finite etale $\mathbb{F}_q^\times$-torsor over B. By $Y \mapsto (X \mapsto YX)$, we obtain a map of functors $J \to \mathscr{I}som_{A,S}(E, E')$. To show that it is an isomorphism, we may prove it over J. In this case, it follows from Lemma 3.2. □

4. Tate-Drinfeld modules

To investigate the structure around cusps of Drinfeld modular curves and extend the sheaf $\omega_{\mathrm{un}}^{\Delta}$, we need to introduce Tate-Drinfeld modules. Let R_0 be a flat $A_{\mathfrak{n}}$-algebra which is an excellent Noetherian domain with fraction field K_0. Let $R_0((x))$ and $K_0((x))$ be the Laurent power series rings over R_0 and K_0, respectively. Put $T_0 = \mathrm{Spec}(R_0((x)))$. We denote the normalized x-adic valuation on $K_0((x))$ by v_x. We also denote the ring of entire series over $K_0((x))$ by $K_0((x))\{\{X\}\}$; it is the subring of $K_0((x))[[X]]$ consisting of elements $\sum_{i\geqslant 0} a_i X^i$ satisfying

$$\lim_{i\to\infty}(v_x(a_i) + i\rho) = +\infty \text{ for any } \rho \in \mathbb{R}.$$

We put $R_0[[x]]\{\{X\}\} = K_0((x))\{\{X\}\} \cap R_0[[x]][[X]]$.

Let (C, Φ^C) be the Carlitz module over R_0. For any non-zero element $f \in A$, put

$$f\Lambda = \left\{ \Phi^C_{fa}\left(\frac{1}{x}\right) \;\middle|\; a \in A \right\} \subseteq R_0((x)), \tag{4.1}$$

$$e_{f\Lambda}(X) = X \prod_{\alpha \neq 0 \in f\Lambda} \left(1 - \frac{X}{\alpha}\right) \in X + xX^2 R_0[[x]][[X]] \tag{4.2}$$

as in [Leh, Ch. 5, §2]. Note that any non-zero element of $f\Lambda$ is invertible in $R_0((x))$. We consider $f\Lambda$ as an A-module via Φ^C. Then it is a free A-module of rank one, and it is also discrete inside $K_0((x))$. Hence the power series $e_{f\Lambda}(X)$ is entire, and it is an element of $R_0[[x]]\{\{X\}\}$.

Put

$$F_f(x) = \frac{1}{\Phi^C_f\left(\frac{1}{x}\right)} \in x^{q^{\deg(f)}} \mathbb{F}_q^\times (1 + xR_0[[x]]).$$

Then $x \mapsto F_f(x)$ defines an R_0-algebra homomorphism $\nu_f^\sharp : R_0((x)) \to R_0((x))$ and a map $\nu_f : T_0 \to T_0$. For any element $h(X) = \sum_i a_i X^i \in R_0((x))[[X]]$, we put $\nu_f^*(h)(X) = \sum_i \nu_f^\sharp(a_i) X^i$. Then we have $\nu_f^\sharp(\Lambda) = f\Lambda$ and $\nu_f^*(e_\Lambda)(X) = e_{f\Lambda}(X)$.

For any element $a \in A$, consider the power series

$$\Phi^{f\Lambda}_a(X) = e_{f\Lambda}(\Phi^C_a(e^{-1}_{f\Lambda}(X))) \in R_0[[x]][[X]]. \tag{4.3}$$

Note that (4.2) yields

$$\Phi^{f\Lambda}_a(X) \equiv \Phi^C_a(X) \bmod xR_0[[x]] \text{ for any } a \in A. \tag{4.4}$$

Let $K_0((x))^{\mathrm{alg}}$ be an algebraic closure of $K_0((x))$. For any $a \in A$, put

$$(\Phi^C_a)^{-1}(f\Lambda) = \{y \in K_0((x))^{\mathrm{alg}} \mid \Phi^C_a(y) \in f\Lambda\},$$

which is an A-module, and let $\Sigma_a \subseteq (\Phi^C_a)^{-1}(f\Lambda)$ be a representative of the set

$$((\Phi^C_a)^{-1}(f\Lambda)/f\Lambda)\backslash\{0\}.$$

Since R_0 is flat over A, we have

$$\Phi^{f\Lambda}_a(X) = aX \prod_{\beta \in \Sigma_a} \left(1 - \frac{X}{e_{f\Lambda}(\beta)}\right) \tag{4.5}$$

(see for example the proof of [Böc, Proposition 2.9]). In particular, it is an $\mathbb{F}_q$-linear additive polynomial of degree $q^{2\deg(a)}$.

Lemma 4.1. *If we write as* $\Phi^\Lambda_t = \theta + a_1\tau + a_2\tau^2$ *for some* $a_i \in R_0[[x]]$, *then we have*

$$a_1 \in 1 + xR_0[[x]], \quad a_2 \in x^{q-1} R_0[[x]]^\times.$$

Proof. The assertion on a_1 follows from (4.4). That on a_2 is proved by the computation in the proof of [Böc, Lemma 2.10]. Indeed, we choose a root $\eta \in K_0((x))^{\mathrm{alg}}$ of the equation

$$\Phi_t^C(X) = \theta X + X^q = \frac{1}{x}.$$

Put $\tilde{\Sigma} = \{c\eta \mid c \in \mathbb{F}_q^\times\}$, $\Sigma_0 = \{\zeta \in K_0((x))^{\mathrm{alg}} \mid \Phi_t^C(\zeta) = 0\}$ and $\Sigma_t = (\tilde{\Sigma} + \Sigma_0) \cup (\Sigma_0 \backslash \{0\})$. By (4.5), we have $a_2 = \theta/(\prod_{\beta\in\Sigma_t} e_\Lambda(\beta))$. The denominator $\prod_{\beta\in\Sigma_t} e_\Lambda(\beta)$ is equal to

$$\prod_{\beta\in\tilde{\Sigma}}\prod_{\zeta\in\Sigma_0}(\beta+\zeta)\prod_{\alpha\neq 0\in\Lambda}\left(\frac{\alpha-(\beta+\zeta)}{\alpha}\right)\cdot\prod_{\zeta\in\Sigma_0\backslash\{0\}}\zeta\prod_{\alpha\neq 0\in\Lambda}\left(\frac{\alpha-\zeta}{\alpha}\right).$$

The first term is equal to

$$\prod_{\beta\in\tilde{\Sigma}}\Phi_t^C(\beta)\prod_{\alpha\neq 0\in\Lambda}\left(\frac{\Phi_t^C(\alpha-\beta)}{\alpha^q}\right) = \frac{\left(\prod_{c\in\mathbb{F}_q^\times} c\right)}{x^{q-1}}\cdot\prod_{c\in\mathbb{F}_q^\times}\prod_{\alpha\neq 0\in\Lambda}\frac{\theta\alpha+\alpha^q-\frac{c}{x}}{\alpha^q}.$$

By the definition (4.1) of Λ, any $\alpha \neq 0 \in \Lambda$ can be written as $\alpha = \Phi_a^C(1/x)$ for some $a \neq 0 \in A$. Thus we have $\alpha = x^{-q^r}h$ with $r = \deg(a)$ and $h \in R_0[[x]]^\times$, which yields $(\theta\alpha + \alpha^q - c/x)/\alpha^q \in 1 + xR_0[[x]]$. By a similar computation, the second term is equal to

$$\theta\prod_{\alpha\neq 0\in\Lambda}\frac{\theta+\alpha^{q-1}}{\alpha^{q-1}} \in \theta(1 + xR_0[[x]]).$$

Hence we obtain the assertion on a_2. □

Using Lemma 4.1 and the map ν_f, we see that the polynomials $\Phi_a^{f\Lambda}$ define a structure of a Drinfeld module of rank two over T_0. We refer to it as the Tate-Drinfeld module $\mathrm{TD}(f\Lambda)$ over T_0.

Lemma 4.2. *For any monic polynomial $\mathfrak{m} \in A$, there exists a natural A-linear closed immersion $\lambda_{\infty,\mathfrak{m}}^{f\Lambda} : C[\mathfrak{m}] \to \mathrm{TD}(f\Lambda)$ over T_0 satisfying $\nu_f^*(\lambda_{\infty,\mathfrak{m}}^{\Lambda}) = \lambda_{\infty,\mathfrak{m}}^{f\Lambda}$. In particular, the Tate-Drinfeld module $\mathrm{TD}(f\Lambda)$ is endowed with a natural $\Gamma_1(\mathfrak{n})$-structure $\lambda_{\infty,\mathfrak{n}}^{f\Lambda}$ over T_0.*

Proof. Let $R_0[[x]]\langle Z\rangle$ be the x-adic completion of the ring $R_0[[x]][Z]$. We have a natural map

$$i : R_0[[x]][Z]/(\Phi_{\mathfrak{m}}^C(Z)) \to R_0[[x]]\langle Z\rangle/(\Phi_{\mathfrak{m}}^C(Z)).$$

Since $\Phi_{\mathfrak{m}}^C(Z) \in R_0[Z]$ is monic, the ring on the left-hand side is finite over the x-adically complete Noetherian ring $R_0[[x]]$. Hence this ring is also x-adically complete and the map i is an isomorphism. Since $R_0[[x]]\{\{Z\}\} \subseteq R_0[[x]]\langle Z\rangle$, the map

$$R_0[[x]][X] \to R_0[[x]]\{\{Z\}\}, \quad X \mapsto e_{f\Lambda}(Z)$$

induces a homomorphism of Hopf algebras

$$R_0((x))[X] \to R_0[[x]]\langle Z\rangle[1/x]/(\Phi^C_{\mathfrak{m}}(Z)) \xrightarrow{i^{-1}} R_0((x))[Z]/(\Phi^C_{\mathfrak{m}}(Z)),$$

which we denote by $(\lambda^{f\Lambda}_{\infty,\mathfrak{m}})^*$. In the ring $R[[x]]\langle Z\rangle$, we have $\Phi^{f\Lambda}_a(e_{f\Lambda}(Z)) = e_{f\Lambda}(\Phi^C_a(Z))$ for any $a \in A$ and this implies that the map $(\lambda^{f\Lambda}_{\infty,\mathfrak{m}})^*$ is compatible with A-actions. Thus we obtain a homomorphism of finite locally free A-module schemes over T_0

$$\lambda^{f\Lambda}_{\infty,\mathfrak{m}} : C[\mathfrak{m}] \to \mathrm{TD}(f\Lambda)[\mathfrak{m}]$$

which is compatible with the map ν_f.

To prove that it is a closed immersion, it is enough to show that the map $R_0[[x]][X] \to R_0[[x]]\langle Z\rangle/(\Phi^C_{\mathfrak{m}}(Z))$ defined by $X \mapsto e_{f\Lambda}(Z)$ is surjective. Since the right-hand side is x-adically complete, it suffices to show the surjectivity modulo x, which follows from (4.2). □

Lemma 4.3. *Let D be any finite flat $R_0((x))$-algebra whose restriction to* $\mathrm{Frac}(R_0((x)))$ *is etale, and δ any element of D. Let $\mathcal{D}$ be the integral closure of $R_0[[x]]$ in D. We consider D as a topological ring by taking $\{x^l\mathcal{D}\}_{l\in\mathbb{Z}_{\geqslant 0}}$ as a fundamental system of neighborhoods of $0 \in D$. Then, for any $F(X) \in R_0((x))\{\{X\}\}$, the evaluation $F(\delta)$ converges for any $\delta \in D$. In particular, we have an $\mathbb{F}_q$-linear map $e_{f\Lambda} : D \to D$ which is functorial on D.*

Proof. We have $\mathcal{D}[1/x] = D$. Since R_0 is excellent, so is the power series ring $R_0[[x]]$. Thus $\mathcal{D}$ is finite over $R_0[[x]]$ and x-adically complete. (Here the fact that $R_0[[x]]$ is excellent follows from an unpublished work of Gabber [KS, Main Theorem 2]. If we assume that R_0 is regular, then the finiteness of $\mathcal{D}$ follows from [Mat, Proposition (31.B)]. This is the only case we need.) This implies that the evaluation $F(\delta)$ converges and defines an element of D. □

Put $\mathcal{H}^{f\Lambda}_{\infty,\mathfrak{m}} = \mathrm{TD}(f\Lambda)[\mathfrak{m}]/\mathrm{Im}(\lambda^{f\Lambda}_{\infty,\mathfrak{m}})$ and

$$B^{f\Lambda}_{0,\mathfrak{m}} = R_0((x))[\eta]/(\Phi^C_{\mathfrak{m}}(\eta) - \Phi^C_f(1/x)). \tag{4.6}$$

Then $\mathrm{Spec}(B^{f\Lambda}_{0,\mathfrak{m}})$ is a finite flat $C[\mathfrak{m}]$-torsor over T_0. Since $\mathfrak{m}$ is invertible in K_0, it is etale over $\mathrm{Frac}(R_0((x)))$.

Lemma 4.4. *For any monic polynomial $\mathfrak{m} \in A$, there exists an A-linear isomorphism $\mu^{f\Lambda}_{\infty,\mathfrak{m}} : \underline{A/(\mathfrak{m})} \to \mathcal{H}^{f\Lambda}_{\infty,\mathfrak{m}}$ which is compatible with the map ν_f such that the image of $\mu^{f\Lambda}_{\infty}(1) \in \mathcal{H}^{f\Lambda}_{\infty,\mathfrak{m}}(T_0)$ in $\mathcal{H}^{f\Lambda}_{\infty,\mathfrak{m}}(B^{f\Lambda}_{0,\mathfrak{m}})$ is* *equal to the image* $\overline{e_{f\Lambda}(\eta)}$ *of the element* $e_{f\Lambda}(\eta) \in \mathrm{TD}(f\Lambda)[\mathfrak{m}](B^{f\Lambda}_{0,\mathfrak{m}})$. *In particular, we have an exact sequence of A-module schemes over* T_0

$$0 \longrightarrow C[\mathfrak{m}] \xrightarrow{\lambda^{f\Lambda}_{\infty,\mathfrak{m}}} \mathrm{TD}(f\Lambda)[\mathfrak{m}] \xrightarrow{\pi^{f\Lambda}_{\infty,\mathfrak{m}}} \underline{A/(\mathfrak{m})} \longrightarrow 0. \tag{4.7}$$

Proof. By Lemma 4.3, we have an element $e_{f\Lambda}(\eta) \in \mathrm{TD}(f\Lambda)[\mathfrak{m}](B^{f\Lambda}_{0,\mathfrak{m}})$. Since its image $\overline{e_{f\Lambda}(\eta)}$ in $\mathcal{H}^{f\Lambda}_{\infty,\mathfrak{m}}(B^{f\Lambda}_{0,\mathfrak{m}})$ is invariant under the action of $C[\mathfrak{m}]$ on $B^{f\Lambda}_{0,\mathfrak{m}}$, we obtain $\overline{e_{f\Lambda}(\eta)} \in \mathcal{H}^{f\Lambda}_{\infty,\mathfrak{m}}(T_0)$. This yields an A-linear homomorphism $\underline{A/(\mathfrak{m})} \to \mathcal{H}^{f\Lambda}_{\infty,\mathfrak{m}}$ over T_0 which is compatible with the map ν_f.

To see that it is an isomorphism, using the map ν_f we reduce ourselves to the case of $f = 1$. Since the element $\mathfrak{m}$ is invertible in K_0, using co-Lie complexes we obtain the exact sequence

$$0 \longrightarrow \omega_{\mathcal{H}^{\Lambda}_{\infty,\mathfrak{m}}} \longrightarrow \omega_{\mathrm{TD}(\Lambda)[\mathfrak{m}]} \xrightarrow{(\lambda^{\Lambda}_{\infty,\mathfrak{m}})^*} \omega_{C[\mathfrak{m}]} \longrightarrow 0.$$

We also see that the natural sequence

$$0 \longrightarrow \omega_{\mathrm{TD}(\Lambda)} \xrightarrow{\mathfrak{m}} \omega_{\mathrm{TD}(\Lambda)} \longrightarrow \omega_{\mathrm{TD}(\Lambda)[\mathfrak{m}]} \longrightarrow 0$$

is exact and similarly for $C[\mathfrak{m}]$. Since we have $d(e_\Lambda(Z)) = dZ$, the map $(\lambda^{\Lambda}_{\infty,\mathfrak{m}})^*$ is an isomorphism. Hence $\omega_{\mathcal{H}^{\Lambda}_{\infty,\mathfrak{m}}} = 0$ and $\mathcal{H}^{\Lambda}_{\infty,\mathfrak{m}}$ is etale.

Now it is enough to show $\overline{ae_\Lambda(\eta)} \neq 0$ in $\mathcal{H}^{\Lambda}_{\infty,\mathfrak{m}}(B^{f\Lambda}_{0,\mathfrak{m}})$ for any non-zero element $a \in A/(\mathfrak{m})$. For this, we may assume $R_0 = K_0$. In this case, note that the polynomial $\Phi^C_{\mathfrak{m}}(X) - 1/x$ is irreducible over $K_0((x))$, since the equation $\Phi^C_{\mathfrak{m}}(1/X) = 1/x$ gives an Eisenstein extension over $K_0[[x]]$. Hence we may consider the ring $B^{\Lambda}_{0,\mathfrak{m}}$ as a subfield of $K_0((x))^{\mathrm{alg}}$. Let $\hat{a} \in A$ be a lift of a satisfying $\deg(\hat{a}) < \deg(\mathfrak{m})$. The condition $\overline{ae_\Lambda(\eta)} = 0$ implies $\Phi^C_{\hat{a}}(\eta) \equiv \zeta \bmod \Lambda$ for some root ζ of $\Phi^C_{\mathfrak{m}}(X)$ in $K_0((x))^{\mathrm{alg}}$. By inspecting x-adic valuations it forces $\zeta = 0$, and the irreducibility of $\Phi^C_{\mathfrak{m}}(X) - 1/x$ implies $\hat{a} = 0$. This concludes the proof. □

We often write $\lambda^{\Lambda}_{\infty,\mathfrak{n}}$, $\mathcal{H}^{\Lambda}_{\infty,\mathfrak{n}}$, $B^{\Lambda}_{0,\mathfrak{n}}$, $\mu^{\Lambda}_{\infty,\mathfrak{n}}$ and $\pi^{\Lambda}_{\infty,\mathfrak{n}}$ as λ_∞, $\mathcal{H}_\infty$, B_0, μ_∞ and π_∞, respectively.

Put $S_0 = \mathrm{Spec}(R_0((y)))$ and consider the morphism

$$\sigma_{q-1} : T_0 \to S_0$$

defined by $y \mapsto x^{q-1}$. The S_0-scheme T_0 is a finite etale $\mathbb{F}_q^\times$-torsor, where $c \in \mathbb{F}_q^\times$ acts on it by the R_0-linear map

$$g_c : R_0((x)) \to R_0((x)), \quad x \mapsto c^{-1}x.$$

Since Λ is stable under this $\mathbb{F}_q^\times$-action, we see that the coefficients of $e_\Lambda(X)$ and $\Phi_a^\Lambda(X)$ are in $R_0[[x^{q-1}]]$ for any $a \in A$ [Arm, §5C1]. This means that there exists a unique pair of a Drinfeld module and its $\Gamma_1(\mathfrak{n})$-structure over S_0

$$(\mathrm{TD}^\triangledown(\Lambda), \lambda_\infty^\triangledown)$$

satisfying $\sigma_{q-1}^*(\mathrm{TD}^\triangledown(\Lambda), \lambda_\infty^\triangledown) = (\mathrm{TD}(\Lambda), \lambda_\infty)$. Over T_0, the Tate-Drinfeld module $\mathrm{TD}^\triangledown(\Lambda)|_{T_0} = \mathrm{TD}(\Lambda)$ has a $\Gamma_1^\Delta(\mathfrak{n})$-structure

$$(\mathrm{TD}(\Lambda), \lambda_\infty, [\mu_\infty])$$

with the element $[\mu_\infty] \in (I_{(\mathrm{TD}(\Lambda),\lambda_\infty)}/\Delta)(T_0)$ defined by μ_∞. We also put

$$\mathcal{H}_\infty^\triangledown = \mathrm{TD}^\triangledown(\Lambda)[\mathfrak{n}]/\mathrm{Im}(\lambda_\infty^\triangledown), \quad I_\infty^\triangledown = \mathscr{I}som_{A,S_0}(\underline{A/(\mathfrak{n})}, \mathcal{H}_\infty^\triangledown).$$

Lemma 4.5. *There exists an isomorphism of finite etale $\mathbb{F}_q^\times$-torsors over S_0*

$$T_0 \to I_\infty^\triangledown/\Delta.$$

Proof. It is enough to give an $\mathbb{F}_q^\times$-equivariant morphism $T_0 \to I_\infty^\triangledown$ over S_0, which amounts to giving an A-linear isomorphism $\mu : \underline{A/(\mathfrak{n})} \to \mathcal{H}_\infty$ over T_0 satisfying $c\mu = g_c^*(\mu)$ for any $c \in \mathbb{F}_q^\times$. The map g_c extends to a similar $R_0((x))$-linear isomorphism on B_0 via $\eta \mapsto c\eta$, which we denote by $\tilde{g}_c$. Then the inclusion $\mathcal{H}_\infty(R_0((x))) \to \mathcal{H}_\infty(B_0)$ is compatible with g_c and $\tilde{g}_c$. Consider the isomorphism μ_∞ of Lemma 4.4. We have $\tilde{g}_c(e_\Lambda(\eta)) = e_\Lambda(c\eta)$ in B_0 and this yields $c\mu_\infty = g_c^*(\mu_\infty)$. □

5. Structure around cusps I

Suppose moreover that R_0 is regular. Note that Lemma 4.1 implies

$$j_t(\mathrm{TD}^\triangledown(\Lambda)) \in y^{-1}R_0[[y]]^\times. \tag{5.1}$$

We define a scheme $\widehat{\mathrm{Cusps}}_{R_0}^\Delta$ by the cartesian diagram

$$\begin{array}{ccc} \widehat{\mathrm{Cusps}}_{R_0}^\Delta & \longrightarrow & X_1^\Delta(\mathfrak{n})_{R_0} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R_0[[\frac{1}{j}]]) & \longrightarrow & \mathbb{P}^1_{R_0} \end{array}$$

and put $\mathrm{Cusps}_{R_0}^\Delta = (\widehat{\mathrm{Cusps}}_{R_0}^\Delta|_{V(1/j)})_{\mathrm{red}}$. Since $Y_1^\Delta(\mathfrak{n})_{R_0}$ is regular and (5.1) implies that the map j_t induces an isomorphism

$$y^\triangledown : S_0 = \mathrm{Spec}(R_0((y))) \to \mathrm{Spec}(R_0((1/j))),$$

we see as in the proof of [KM, Lemma 8.11.4] that $\widehat{\mathrm{Cusps}}^{\Delta}_{R_0}$ is isomorphic to the normalization of $\mathcal{S}_0 = \mathrm{Spec}(R_0[[y]])$ in the scheme $Y_1^{\Delta}(\mathfrak{n})_{\mathcal{S}_0}$ defined by the cartesian diagram

$$\begin{array}{ccc} Y_1^{\Delta}(\mathfrak{n})_{\mathcal{S}_0} & \longrightarrow & Y_1^{\Delta}(\mathfrak{n})_{R_0} \\ \downarrow & & \downarrow \\ \mathcal{S}_0 \xrightarrow[y^{\triangledown}]{} \mathrm{Spec}(R_0((1/j))) & \longrightarrow & \mathbb{A}^1_{R_0}. \end{array}$$

For $\bullet \in \{\emptyset, \Delta\}$, let us consider the functor sending a scheme S over S_0 to the set of $\Gamma_1^{\bullet}(\mathfrak{n})$-structures on $\mathrm{TD}^{\triangledown}(\Lambda)|_S$, which is representable by a finite etale scheme $[\Gamma_1^{\bullet}(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}}$ over S_0. By Lemma 3.2 and Lemma 3.3, as in the proof of [KM, Corollary 8.4.4] we obtain a natural isomorphism

$$[\Gamma_1^{\Delta}(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}}/\mathbb{F}_q^{\times} \to Y_1^{\Delta}(\mathfrak{n})_{\mathcal{S}_0},$$

where $\mathbb{F}_q^{\times}$ acts as the automorphism group of $\mathrm{TD}^{\triangledown}(\Lambda)$. Thus $\widehat{\mathrm{Cusps}}^{\Delta}_{R_0}$ is isomorphic to the quotient $\mathcal{Z}^{\Delta}_{R_0}/\mathbb{F}_q^{\times}$ of the normalization $\mathcal{Z}^{\Delta}_{R_0}$ of $\mathcal{S}_0$ in $[\Gamma_1^{\Delta}(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}}$ by the induced action of $\mathbb{F}_q^{\times}$. Note that we have a natural identification

$$[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}} \times_{S_0} T_0 = [\Gamma_1(\mathfrak{n})]_{\mathrm{TD}},$$

where the right-hand side is a similar finite etale scheme over T_0 for $\mathrm{TD}(\Lambda)$. We also put $\mathcal{T}_0 = \mathrm{Spec}(R_0[[x]])$. It is normal since R_0 is regular.

Lemma 5.1. *There exists a natural isomorphism over S_0*

$$[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}} = [\Gamma_1(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}} \times_{S_0} T_0 \to [\Gamma_1^{\Delta}(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}}$$

which is compatible with actions of $\mathbb{F}_q^{\times} = \mathrm{Aut}_{A,S_0}(\mathrm{TD}^{\triangledown}(\Lambda))$. Here this group acts on the left-hand side diagonally.

Proof. Let λ be the universal $\Gamma_1(\mathfrak{n})$-structure on $\mathrm{TD}^{\triangledown}(\Lambda)$ over $[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}}$. Taking the determinant of locally constant etale sheaves of locally free $A/(\mathfrak{n})$-modules, we obtain a natural isomorphism of A-module schemes $\iota : \mathcal{H}^{\triangledown}_{\infty}|_{[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}}} \to \mathrm{TD}^{\triangledown}(\Lambda)[\mathfrak{n}]/\mathrm{Im}(\lambda)$. Then, by Lemma 4.5, the map

$$(I^{\triangledown}_{\infty}/\Delta)|_{[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}}} \to [\Gamma_1^{\Delta}(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}}, \quad [(j : \underline{A/(\mathfrak{n})} \to \mathcal{H}^{\triangledown}_{\infty})] \mapsto [\iota \circ j]$$

gives the desired isomorphism. □

Lemma 5.2. *The scheme $\mathcal{Z}^{\Delta}_{R_0}$ over $\mathcal{S}_0$ is decomposed as*

$$\mathcal{Z}^{\Delta}_{R_0} = \mathcal{Z}^{\Delta,0}_{R_0} \sqcup \mathcal{Z}^{\Delta,\neq 0}_{R_0}, \quad \mathcal{Z}^{\Delta,0}_{R_0} = \coprod_{(A/(\mathfrak{n}))^{\times}} \mathcal{T}_0.$$

Moreover, the group $\mathbb{F}_q^\times = \mathrm{Aut}_{A,S_0}(\mathrm{TD}^\nabla(\Lambda))$ *induces free actions on the two components of the former decomposition.*

Proof. First note that, for any scheme S over $A_\mathfrak{n}$ and any finite etale A-module scheme $\mathcal{G}$ over S, the big fppf sheaf $\mathscr{H}om_{A,S}(C[\mathfrak{n}],\mathcal{G})$ is representable by a finite etale A-module scheme over S and thus its zero section is a closed and open immersion.

Since $\mathcal{T}_0$ is normal, Lemma 5.1 implies that $\mathcal{Z}^\Delta_{R_0}$ is identified with the normalization of $\mathcal{T}_0$ in the finite etale scheme $[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}}$ over T_0. For any scheme T over T_0, we have an exact sequence of finite etale A-module schemes over T

$$0 \longrightarrow C[\mathfrak{n}]|_T \xrightarrow{\lambda_\infty} \mathrm{TD}(\Lambda)[\mathfrak{n}]|_T \xrightarrow{\pi_\infty} \underline{A/(\mathfrak{n})}|_T \longrightarrow 0.$$

Any $\Gamma_1(\mathfrak{n})$-structure $\lambda : C[\mathfrak{n}]|_T \to \mathrm{TD}(\Lambda)[\mathfrak{n}]|_T$ over T induces an A-linear homomorphism $\pi_\infty \circ \lambda : C[\mathfrak{n}]|_T \to \underline{A/(\mathfrak{n})}|_T$. This gives a morphism over T_0

$$[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}} \to \mathscr{H}om_{A,T_0}(C[\mathfrak{n}],\underline{A/(\mathfrak{n})}) = T_0 \sqcup U,$$

where U is the complement of the zero section. Let $[\Gamma_1(\mathfrak{n})]^0_{\mathrm{TD}}$ be the inverse image of T_0. It is isomorphic to $\mathscr{A}ut_{A,T_0}(C[\mathfrak{n}]) = \underline{(A/(\mathfrak{n}))^\times}$.

Since $\mathscr{H}om_{A,\mathcal{T}_0}(C[\mathfrak{n}],\underline{A/(\mathfrak{n})})$ is also a finite etale A-module scheme over $\mathcal{T}_0$, it agrees with the normalization of $\mathcal{T}_0$ in $\mathscr{H}om_{A,T_0}(C[\mathfrak{n}],\underline{A/(\mathfrak{n})})$. Moreover, it is etale locally isomorphic to $\underline{A/(\mathfrak{n})}$. Thus we obtain a map

$$\mathcal{Z}^\Delta_{R_0} \to \mathscr{H}om_{A,\mathcal{T}_0}(C[\mathfrak{n}],\underline{A/(\mathfrak{n})}) = \mathcal{T}_0 \sqcup \mathcal{U},$$

where $\mathcal{U}$ is the complement of the zero section. Since $\mathcal{U}$ is etale locally isomorphic to $\underline{A/(\mathfrak{n})\backslash\{0\}}$, the group $\mathbb{F}_q^\times$ acts freely on $\mathcal{U}$.

Let $\mathcal{Z}^{\Delta,0}_{R_0}$ and $\mathcal{Z}^{\Delta,\neq 0}_{R_0}$ be the inverse images of $\mathcal{T}_0$ and $\mathcal{U}$, respectively. Since the component $\mathcal{Z}^{\Delta,0}_{R_0}$ is the normalization of $\mathcal{T}_0$ in $[\Gamma_1(\mathfrak{n})]^0_{\mathrm{TD}}$, the latter decomposition of the lemma follows. Hence we also obtain the freeness of the $\mathbb{F}_q^\times$-actions as in the lemma. □

The tuple $(\mathrm{TD}(\Lambda),\lambda_\infty,[\mu_\infty])$ over T_0 gives a map $T_0 \to Y_1^\Delta(\mathfrak{n})_{R_0}$. Since the ring $R_0[[x]]$ is normal, this extends to a map

$$x^\Delta_\infty : \mathcal{T}_0 \to X_1^\Delta(\mathfrak{n})_{R_0}.$$

The R_0-algebra homomorphism defined by $x \mapsto 0$ gives a point $P^\Delta_\infty \in X_1^\Delta(\mathfrak{n})_{R_0}$, which we refer to as the ∞-cusp. We write the complete local ring at this point as $\hat{\mathcal{O}}_{X_1^\Delta(\mathfrak{n})_{R_0},P^\Delta_\infty}$.

Theorem 5.3. *Suppose that* R_0 *is a flat* $A_\mathfrak{n}$*-algebra which is an excellent regular domain.*

(1) *The map* x_∞^Δ *induces an isomorphism of complete local rings*
$$(x_\infty^\Delta)^* : \hat{\mathcal{O}}_{X_1^\Delta(\mathfrak{n})_{R_0}, P_\infty^\Delta} \to R_0[[x]].$$
(2) *The invertible sheaf* $\omega_{\mathrm{un}}^\Delta$ *on* $Y_1^\Delta(\mathfrak{n})_{R_0}$ *extends to an invertible sheaf* $\bar{\omega}_{\mathrm{un}}^\Delta$ *on* $X_1^\Delta(\mathfrak{n})_{R_0}$ *satisfying*
$$(x_\infty^\Delta)^*(\bar{\omega}_{\mathrm{un}}^\Delta) = R_0[[x]]dX,$$
where dX *denotes the invariant differential form of* $\mathrm{TD}^\triangledown(\Lambda)$ *associated to its parameter* X.
(3) *The formation of* $\bar{\omega}_{\mathrm{un}}^\Delta$ *is compatible with any base change* $R_0 \to R_0'$ *of flat* $A_\mathfrak{n}$*-algebras which are excellent regular domains.*
(4) *The natural action of* $\mathbb{F}_q^\times$ *on* $\omega_{\mathrm{un}}^\Delta$ *via* $c \mapsto [c]_\Delta$ *extends to an action on* $\bar{\omega}_{\mathrm{un}}^\Delta$ *covering its action on* $X_1^\Delta(\mathfrak{n})_{R_0}$.

Proof. The assertion (1) follows from Lemma 5.2. Moreover, Lemma 5.2 also implies that the trivial invertible sheaf $\mathcal{O}_{\mathcal{Z}_{R_0}^\Delta} dX$, with the natural $\mathbb{F}_q^\times$-action via $X \mapsto cX$ which covers the action on $\mathcal{Z}_{R_0}^\Delta$, descends to the quotient $\mathcal{Z}_{R_0}^\Delta/\mathbb{F}_q^\times \simeq \widehat{\mathrm{Cusps}}_{R_0}^\Delta$ and we obtain $\bar{\omega}_{\mathrm{un}}^\Delta$ by gluing. (3) follows from the uniqueness of the descended sheaf.

For (4), Lemma 5.1 implies that $[c]_\Delta$ acts on
$$\mathcal{P} := [\Gamma_1^\Delta(\mathfrak{n})]_{\mathrm{TD}^\triangledown} \simeq [\Gamma_1(\mathfrak{n})]_{\mathrm{TD}^\triangledown} \times_{S_0} T_0$$
via $1 \times g_c^*$. Thus, for the universal $\Gamma_1(\mathfrak{n})$-structure $\lambda_{\mathrm{un}}^\triangledown$ on $\mathrm{TD}^\triangledown(\Lambda)$ over $[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}^\triangledown}$, we have
$$[c]_\Delta^*(\mathrm{TD}^\triangledown(\Lambda)|_\mathcal{P}, \lambda_{\mathrm{un}}^\triangledown|_\mathcal{P}) = (\mathrm{TD}^\triangledown(\Lambda)|_\mathcal{P}, \lambda_{\mathrm{un}}^\triangledown|_\mathcal{P}).$$
Since any $\Gamma_1(\mathfrak{n})$-structure has no non-trivial automorphism, the natural action of $[c]_\Delta$ on $\omega_{\mathrm{un}}^\Delta|_{\mathcal{P}/\mathbb{F}_q^\times}$ is the descent of the map given by
$$[c]_\Delta^*(\mathcal{O}_\mathcal{P} dX) \to \mathcal{O}_\mathcal{P} dX, \quad dX \otimes 1 \mapsto dX.$$
Hence it extends to the sheaf $\mathcal{O}_{\mathcal{Z}_{R_0}^\Delta} dX$, and thus to $\bar{\omega}_{\mathrm{un}}^\Delta$. □

6. Structure around cusps II

Let $W_\mathfrak{n}(X)$ be the $\mathfrak{n}$-th Carlitz cyclotomic polynomial, namely the unique monic prime factor of $\Phi_\mathfrak{n}^C(X)$ in $A[X]$ which does not divide $\Phi_\mathfrak{m}^C(X)$ for any non-trivial divisor $\mathfrak{m}$ of $\mathfrak{n}$ [Car, §3]. Then
$$I = \mathscr{I}\!som_{A,R_0}(\underline{A/(\mathfrak{n})}, C[\mathfrak{n}])$$
is represented by $\mathrm{Spec}(R_0[X]/(W_\mathfrak{n}(X)))$, which is finite etale over R_0. For any scheme S over R_0, we put $I_S = I \times_{R_0} S$. Let $R_\mathfrak{n}$ be the affine ring of a connected component of I, which is a finite etale domain over R_0. We denote by ζ the image of X in $R_\mathfrak{n}$. In this section, we give an explicit

description of the scheme $[\Gamma_1^{\Delta}(\mathfrak{n})]_{\mathrm{TD}^{\triangledown}}$ over $S_{\mathfrak{n}} = \mathrm{Spec}(R_{\mathfrak{n}}((y)))$, from which we obtain more precise information on the formal completion along cusps.

Put $T_{\mathfrak{n}} = \mathrm{Spec}(R_{\mathfrak{n}}((x)))$. By Lemma 5.1, it is enough to describe the restriction

$$[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}|_{T_{\mathfrak{n}}}} = [\Gamma_1(\mathfrak{n})]_{\mathrm{TD}} \times_{T_0} T_{\mathfrak{n}}.$$

For this, we denote by $\mathscr{H}$ the set of A-linear surjections $(A/(\mathfrak{n}))^2 \to A/(\mathfrak{n})$. By the map $(a,b) \mapsto ({}^t(u,v) \mapsto (a,b)^t(u,v))$, we identify the set $\mathscr{H}$ with $\{(a,b) \in (A/(\mathfrak{n}))^2 \mid (a,b) = (1)\}$. As in [KM, Proposition 10.2.4], for any $\Xi \in \mathscr{H}$ we denote by k_Ξ the unique generator of $\mathrm{Ker}(\Xi)$ satisfying $\Xi(l) = \det(k_\Xi, l)$ for any $l \in (A/(\mathfrak{n}))^2$. We also choose $l_\Xi \in (A/(\mathfrak{n}))^2$ satisfying $\Xi(l_\Xi) = 1$. Then, for any $g \in GL_2(A/(\mathfrak{n}))$ there exists a unique $n(g,\Xi) \in A/(\mathfrak{n})$ satisfying

$$l_{\Xi\circ g} = g^{-1}(l_\Xi) + n(g,\Xi)g^{-1}(k_\Xi). \tag{6.1}$$

Put $\mathrm{Fix}(\Xi) = \{g \in GL_2(A/(\mathfrak{n})) \mid \Xi \circ g = \Xi\}$. Considering the representing matrix for g with respect to the ordered basis (k_Ξ, l_Ξ), we have an isomorphism

$$\mathrm{Fix}(\Xi) \to \left\{ \begin{pmatrix} \det(g) & n(g,\Xi) \\ 0 & 1 \end{pmatrix} \;\middle|\; g \in \mathrm{Fix}(\Xi) \right\}. \tag{6.2}$$

We denote by $[\Gamma(\mathfrak{n})]_{\mathrm{TD}|_{T_{\mathfrak{n}}}}$ the scheme representing the functor over $T_{\mathfrak{n}}$ sending a $T_{\mathfrak{n}}$-scheme T to the set of $\Gamma(\mathfrak{n})$-structures on $\mathrm{TD}(\Lambda)|_T$. It is finite etale over $T_{\mathfrak{n}}$. By (4.7), to give $\alpha \in [\Gamma(\mathfrak{n})]_{\mathrm{TD}|_{T_{\mathfrak{n}}}}(T)$ satisfying $\pi_\infty \circ \alpha = \Xi$ is the same as to give $\alpha(k_\Xi) \in C[\mathfrak{n}](T)$ inducing an A-linear isomorphism $\underline{A/(\mathfrak{n})} \to C[\mathfrak{n}]$ and $\alpha(l_\Xi) \in \pi_\infty^{-1}([1])(T)$, where $[1]$ is the section $T_{\mathfrak{n}} \to \underline{A/(\mathfrak{n})}$ corresponding to $1 \in A/(\mathfrak{n})$.

By taking the determinant, we have an A-linear isomorphism of etale sheaves of locally free $A/(\mathfrak{n})$-modules

$$\omega : \bigwedge^2 \mathrm{TD}(\Lambda)[\mathfrak{n}] \to C[\mathfrak{n}],$$

which defines a map $[\Gamma(\mathfrak{n})]_{\mathrm{TD}|_{T_{\mathfrak{n}}}} \to I$ by $(\alpha \mapsto \omega \circ \wedge^2\alpha)$. For any scheme T over $T_{\mathfrak{n}}$, we say an element $\alpha \in [\Gamma(\mathfrak{n})]_{\mathrm{TD}|_{T_{\mathfrak{n}}}}(T)$ is canonical if the map $\omega \circ \wedge^2\alpha : T \to I$ is equal to the structure map $T \to T_{\mathfrak{n}} \to I$. The subfunctor of canonical elements is represented by a finite etale scheme $[\Gamma(\mathfrak{n})]^{\mathrm{can}}_{\mathrm{TD}|_{T_{\mathfrak{n}}}}$ over $T_{\mathfrak{n}}$.

Lemma 6.1. *Put $B_{\mathfrak{n}} = R_{\mathfrak{n}}((x))[\eta]/(\Phi^C_{\mathfrak{n}}(\eta) - 1/x)$. Then the map over $T_{\mathfrak{n}}$*

$$\coprod_{\Xi\in\mathscr{H}} \mathrm{Spec}(B_{\mathfrak{n}}) \to [\Gamma(\mathfrak{n})]^{\mathrm{can}}_{\mathrm{TD}|_{T_{\mathfrak{n}}}},$$

which is defined on the Ξ*-component by the canonical* $\Gamma(\mathfrak{n})$*-structure* $(k_\Xi, l_\Xi) \mapsto (e_\Lambda(\zeta), e_\Lambda(\eta))$ *over* $B_\mathfrak{n}$*, is an isomorphism.*

Proof. The element $e_\Lambda(\eta) \in B_\mathfrak{n}$ defines a map $\mathrm{Spec}(B_\mathfrak{n}) \to \pi_\infty^{-1}([1])$. Since it is $C[\mathfrak{n}]$-equivariant, it is an isomorphism of $C[\mathfrak{n}]$-torsors over $T_\mathfrak{n}$ and the lemma follows. □

Put $\bar{\Gamma}_1 = \left\{\begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix} \in GL_2(A/(\mathfrak{n}))\right\}$ and $\bar{\Gamma}_1^1 = \bar{\Gamma}_1 \cap SL_2(A/(\mathfrak{n}))$. For any element $f \neq 0 \in A$, we define

$$G_f(w) = w^{q^{\deg(f)}} - xw^{q^{\deg(f)}}\Phi_f^C\left(\frac{1}{w}\right) \in R_0[[x]][w].$$

Then we have natural maps

$$R_0[[w]] \longrightarrow R_0[[x]][w]/(G_f(w)) \longrightarrow R_0[[w]]$$

which are isomorphisms. Moreover, for any $b \in A/(\mathfrak{n})$, let f_b be the monic generator of the ideal $\mathrm{Ann}_A(b(A/(\mathfrak{n})))$. Then f_b divides $\mathfrak{n}$ and $f_b \in A_\mathfrak{n}^\times$.

Lemma 6.2. *The scheme* $[\Gamma_1^\Delta(\mathfrak{n})]_{\mathrm{TD}^\triangledown}$ *over* $S_\mathfrak{n}$ *is decomposed as*

$$[\Gamma_1^\Delta(\mathfrak{n})]_{\mathrm{TD}^\triangledown} = \coprod_{(a,b)} \mathrm{Spec}(R_\mathfrak{n}((x))[w]/(G_{f_b}(w))) \simeq \coprod_{(a,b)} \mathrm{Spec}(R_\mathfrak{n}((w))).$$

Here the direct sum is taken over a complete representative of the set

$$\{(a,b) \in (A/(\mathfrak{n}))^2 \mid (a,b) = (1)\}/\bar{\Gamma}_1^1.$$

Proof. For any scheme T over $T_\mathfrak{n}$, any $\Gamma(\mathfrak{n})$-structure α on $\mathrm{TD}(\Lambda)|_T$ defines a $\Gamma_1(\mathfrak{n})$-structure $\zeta \mapsto \alpha({}^t(0,1))$. Since we have $SL_2(A/(\mathfrak{n}))/\bar{\Gamma}_1^1 = GL_2(A/(\mathfrak{n}))/\bar{\Gamma}_1$, Lemma 6.1 yields

$$[\Gamma_1(\mathfrak{n})]_{\mathrm{TD}|_{T_\mathfrak{n}}} = [\Gamma(\mathfrak{n})]^{\mathrm{can}}_{\mathrm{TD}|_{T_\mathfrak{n}}}/\bar{\Gamma}_1^1 = \coprod_{\Xi \in \mathscr{H}/\bar{\Gamma}_1^1} \mathrm{Spec}(B_\mathfrak{n}^{\bar{\Gamma}_1^1 \cap \mathrm{Fix}(\Xi)}).$$

Note that, via the isomorphism of Lemma 6.1, any element $g \in \bar{\Gamma}_1^1 \cap \mathrm{Fix}(\Xi)$ acts on $B_\mathfrak{n}$ of the Ξ-component by

$$\eta \mapsto \eta + \Phi^C_{n(g,\Xi)}(\zeta).$$

For $\Xi = (a,b)$, we have $k_\Xi = {}^t(b,-a)$ and

$$\bar{\Gamma}_1^1 \cap \mathrm{Fix}(\Xi) = \left\{\begin{pmatrix} 1 & 0 \\ (f_b)/(\mathfrak{n}) & 1 \end{pmatrix}\right\}.$$

By the isomorphism (6.2), the additive subgroup

$$n(\Xi) = \{n(g,\Xi) \in A/(\mathfrak{n}) \mid g \in \bar{\Gamma}_1^1 \cap \mathrm{Fix}(\Xi)\}$$

is isomorphic to $(f_b)/(\mathfrak{n})$. In particular, they have the same cardinality. On the other hand, for any $g \in \bar{\Gamma}_1^1 \cap \mathrm{Fix}(\Xi)$, (6.1) yields $bn(g,\Xi) = 0$ and thus $n(\Xi) \subseteq (f_b)/(\mathfrak{n})$. Hence they are equal.

Put $h_b = \mathfrak{n}/f_b$. Consider the map

$$R_{\mathfrak{n}}((x))[\eta']/(\Phi^C_{f_b}(\eta') - 1/x) \to B_{\mathfrak{n}}, \quad \eta' \mapsto \Phi^C_{h_b}(\eta).$$

Note that the left-hand side is isomorphic to $R_{\mathfrak{n}}((1/\eta'))$ and thus normal. Hence this map identifies the left-hand side with $B_{\mathfrak{n}}^{\bar{\Gamma}_1^1 \cap \mathrm{Fix}(\Xi)}$. By changing the variable as $w = 1/\eta'$, Lemma 5.1 yields the decomposition as in the lemma. □

Theorem 6.3. *Suppose that R_0 is a flat $A_{\mathfrak{n}}$-algebra which is an excellent regular domain.*

(1) We have a natural isomorphism over $R_{\mathfrak{n}}[[y]]$

$$\widehat{\mathrm{Cusps}}^{\Delta}_{R_{\mathfrak{n}}} = \coprod_{(a,b)} \mathrm{Spec}(R_{\mathfrak{n}}[[x]][w]/(G_{f_b}(w))) \simeq \coprod_{(a,b)} \mathrm{Spec}(R_{\mathfrak{n}}[[w]]).$$

Here the direct sum is taken over a complete representative of the set

$$\mathbb{F}_q^{\times} \backslash \{(a,b) \in (A/(\mathfrak{n}))^2 \mid (a,b) = (1)\}/\bar{\Gamma}_1^1.$$

(2) $\mathrm{Cusps}^{\Delta}_{R_0}$ is finite etale over R_0. In particular, it defines an effective Cartier divisor of $X_1^{\Delta}(\mathfrak{n})_{R_0}$ over R_0.

(3) At each point of $\mathrm{Cusps}^{\Delta}_{R_0}$, the invertible sheaf

$$\Omega^1_{X_1^{\Delta}(\mathfrak{n})_{R_0}/R_0}(2\mathrm{Cusps}^{\Delta}_{R_0})$$

is locally generated by the section dx/x^2.

Proof. Note that the ring $R_{\mathfrak{n}}[[w]]$ is normal. Since the group $\mathbb{F}_q^{\times}$ acts freely on the index set of the decomposition of Lemma 6.2, we obtain the assertion (1), which implies the assertion (2) since we have $\mathrm{Cusps}^{\Delta}_{R_{\mathfrak{n}}} = \mathrm{Cusps}^{\Delta}_{R_0} \times_{R_0} \mathrm{Spec}(R_{\mathfrak{n}})$.

For the assertion (3), by a base change it is enough to show it over $R_{\mathfrak{n}}$. Put $e = \deg(f_b)$ and $G_{f_b}(w) = w^{q^e} - xH(w)$. Then we have $H(w)dx = xf_b w^{q^e-2}dw$ in $\Omega^1_{R_{\mathfrak{n}}[[w]]/R_{\mathfrak{n}}}$ and

$$\frac{dw}{w^2} = \frac{H(w)}{f_b}\frac{dx}{xw^{q^e}} = \frac{1}{f_b}\frac{dx}{x^2},$$

which concludes the proof. □

On the component of $\widehat{\mathrm{Cusps}}^{\Delta}_{R_{\mathfrak{n}}}$ corresponding to $\Xi = (a,b)$, the pullback of $\mathrm{TD}^{\nabla}(\Lambda)$ agrees with $\mathrm{TD}(f_b\Lambda)$ over $R_{\mathfrak{n}}((w))$ with a universal $\Gamma_1^{\Delta}(\mathfrak{n})$-structure $(\lambda, [\mu])$. Let us describe them explicitly. We set $T'_{\mathfrak{n}} =$

$\mathrm{Spec}(R_{\mathfrak{n}}((w)))$, and consider the ring $R_{\mathfrak{n}}((w))$ as a subring of $B_{\mathfrak{n}}$ as in the proof of Lemma 6.2. Put

$$(P_\Xi, Q_\Xi) = (e_{f_b\Lambda}(\zeta), e_{f_b\Lambda}(\eta))(k_\Xi, l_\Xi)^{-1}.$$

Then we have $Q_\Xi \in \mathrm{TD}(f_b\Lambda)[\mathfrak{n}](T'_{\mathfrak{n}})$ and

$$\lambda : C[\mathfrak{n}](T'_{\mathfrak{n}}) \to \mathrm{TD}(f_b\Lambda)[\mathfrak{n}](T'_{\mathfrak{n}}), \quad \zeta \mapsto Q_\Xi. \tag{6.3}$$

On the other hand, taking the determinant as in the proof of Lemma 5.1 yields

$$C[\mathfrak{n}] \otimes (\mathrm{TD}(f_b\Lambda)[\mathfrak{n}]/\mathrm{Im}(\lambda)) \to \bigwedge^2 \mathrm{TD}(f_b\Lambda)[\mathfrak{n}]$$
$$\zeta \otimes (P_\Xi \bmod \mathrm{Im}(\lambda)) \mapsto Q_\Xi \wedge P_\Xi$$

and similarly for $\lambda^{f_b\Lambda}_{\infty,\mathfrak{n}}$. Since $\det(k_\Xi, l_\Xi) = 1$, we obtain an isomorphism

$$\iota : \mathcal{H}_\infty|_{T'_{\mathfrak{n}}} \to \mathrm{TD}(f_b\Lambda)[\mathfrak{n}]/\mathrm{Im}(\lambda)$$

defined by $e_{f_b\Lambda}(\eta) \bmod \mathrm{Im}(\lambda^{f_b\Lambda}_{\infty,\mathfrak{n}}) \mapsto -P_\Xi \bmod \mathrm{Im}(\lambda)$. Then we have $\mu = \iota \circ \mu^{f_b\Lambda}_{\infty,\mathfrak{n}}$, which is given by

$$\mu : \underline{A/(\mathfrak{n})} \to \mathrm{TD}(f_b\Lambda)[\mathfrak{n}]/\mathrm{Im}(\lambda), \quad 1 \mapsto -P_\Xi \bmod \mathrm{Im}(\lambda). \tag{6.4}$$

7. Case of level $\Gamma_1^\Delta(\mathfrak{n}, \wp)$

For the structure around cusps of $X_1^\Delta(\mathfrak{n}, \wp)$, we first note that $Y_1^\Delta(\mathfrak{n}, \wp)_{R_0}$ is normal near infinity in the sense of [KM, (8.6.2)] by Lemma 3.1. Thus the description around cusps using Tate-Drinfeld modules and normalization as in the beginning of §5 is also valid in this case.

The closed immersion $\lambda^\Lambda_{\infty,\wp} : C[\wp] \to \mathrm{TD}(\Lambda)$ defines a $\Gamma_0(\wp)$-structure on $\mathrm{TD}(\Lambda)$ over T_0. Hence we have a map

$$x^{\Delta,\wp}_\infty : \mathcal{T}_0 \to X_1^\Delta(\mathfrak{n}, \wp)_{R_0}$$

and a point $P^{\Delta,\wp}_\infty \in X_1^\Delta(\mathfrak{n}, \wp)_{R_0}$.

More generally, for any $\Xi = (a, b) \in \mathscr{H}$, consider the map $R_0((x)) \to R_0((w)) = R_0((x))[w]/(G_{f_b}(w))$ and the Tate-Drinfeld module $\mathrm{TD}(f_b\Lambda)$ over $R_0((w))$. The latter has a canonical $\Gamma_0(\wp)$-structure $\mathcal{C}$ given by the closed immersion $\lambda^{f_b\Lambda}_{\infty,\wp}$ of Lemma 4.2. We denote by $Z = [\Gamma_0(\wp)]_{\mathrm{TD}(f_b\Lambda)}$ the scheme representing the functor sending each scheme T over $R_0((w))$ to the set of $\Gamma_0(\wp)$-structures on $\mathrm{TD}(f_b\Lambda)|_T$. It is finite over $R_0((w))$ and thus Noetherian. We denote by $\mathcal{G}_{\mathrm{un}}$ the universal $\Gamma_0(\wp)$-structure on Z.

For any Noetherian scheme T over $R_0((w))$ and any $\Gamma_0(\wp)$-structure $\mathcal{G}$ on $\mathrm{TD}(f_b\Lambda)|_T$, the theory of Hilbert schemes shows that the functor $\mathscr{H}om_{T,A}(\mathcal{G}, \underline{A/(\wp)})$ is representable, locally of finite presentation and separated over T. From the etaleness of $\underline{A/(\wp)}$, we see that the group scheme $\mathscr{H}om_{T,A}(\mathcal{G}, \underline{A/(\wp)})$ is also formally etale over T. Hence it is etale over T and thus its zero section is a closed and open immersion. We write its complement as U_T.

By composing with $\pi_{\infty,\wp}^{f_b\Lambda} : \mathrm{TD}(f_b\Lambda)[\wp] \to \underline{A/(\wp)}$, the universal $\Gamma_0(\wp)$-structure $\mathcal{G}_{\mathrm{un}}$ gives a map

$$Z = [\Gamma_0(\wp)]_{\mathrm{TD}(f_b\Lambda)} \to \mathscr{H}om_{Z,A}(\mathcal{G}_{\mathrm{un}}, \underline{A/(\wp)}) = Z \sqcup U_Z.$$

Hence the left-hand side is decomposed accordingly, and the component over Z agrees with the section $\mathrm{Spec}(R_0((w))) \to Z$ given by $\mathcal{C}$. From this, we can show that we have the same description of the complete local ring at $P_\infty^{\Delta,\wp} \in X_1^\Delta(\mathfrak{n}, \wp)_{R_0}$ and a similar extended invertible sheaf $\bar{\omega}_{\mathrm{un}}^{\Delta,\wp}$ which is compatible with $\bar{\omega}_{\mathrm{un}}^{\Delta}$, as in Theorem 5.3. Furthermore, after passing to $R_{\mathfrak{n}}((w))$, we can also show that the formal completion of $X_1^\Delta(\mathfrak{n}, \wp)_{R_{\mathfrak{n}}}$ along the cusp corresponding to $\mathcal{C}$ over the component of Ξ is isomorphic to $R_{\mathfrak{n}}[[w]]$ via the projection to $X_1^\Delta(\mathfrak{n})_{R_{\mathfrak{n}}}$. It can be considered as a Drinfeld analogue of the unramified cusp of the modular curve $X_0(p)$.

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