TRIVIALITY OF THE HECKE ACTION ON ORDINARY DRINFELD CUSPFORMS OF LEVEL $\Gamma_1(t^n)$

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Abstract. Let $k \geqslant 2$ and $n \geqslant 1$ be any integers. In this paper, we prove that all Hecke operators act trivially on the space of ordinary Drinfeld cuspforms of level $\Gamma_1(t^n)$ and weight k.

1. Introduction

Let p be a rational prime, $q > 1$ a p-power integer, $A = \mathbb{F}_q[t]$, $K = \mathbb{F}_q(t)$ and $K_\infty = \mathbb{F}_q((1/t))$. Let $\mathbb{C}_\infty$ be the $(1/t)$-adic completion of an algebraic closure of K_∞ and put $\Omega = \mathbb{C}_\infty \backslash K_\infty$, which has a natural structure as a rigid analytic variety over K_∞. For any non-zero element $\mathfrak{n} \in A$, we put

$$\Gamma_1(\mathfrak{n}) = \left\{ \gamma \in SL_2(A) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{n} \right\}.$$

For any arithmetic subgroup Γ of $SL_2(A)$ and integer $k \geqslant 2$, a rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ is called a Drinfeld modular form of level Γ and weight k if it satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and a certain regularity condition at cusps. A Drinfeld modular form is called a cuspform if it vanishes at all cusps, and a double cuspform if it vanishes twice at all cusps. They form $\mathbb{C}_\infty$-vector spaces $S_k(\Gamma)$ and $S_k^{(2)}(\Gamma)$, respectively. These spaces admit a natural action of Hecke operators.

Let $\wp \in A$ be an irreducible polynomial, $K_\wp$ the $\wp$-adic completion of K and $\mathbb{C}_\wp$ the $\wp$-adic completion of an algebraic closure of $K_\wp$. For the algebraic closure $\bar{K}$ of K in $\mathbb{C}_\infty$, we fix an embedding $\iota_\wp : \bar{K} \to \mathbb{C}_\wp$.

Suppose that $\wp$ divides $\mathfrak{n}$. The Hecke operator at $\wp$ acting on $S_k(\Gamma_1(\mathfrak{n}))$ is denoted by $U_\wp$. Note that any eigenvalue of $U_\wp$ is an element of $\bar{K}$. We say $f \in S_k(\Gamma_1(\mathfrak{n}))$ is ordinary (with respect to $\iota_\wp$) if f is in the generalized eigenspace belonging to an eigenvalue $\lambda \in \bar{K}$

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satisfying $\iota_\wp(\lambda) \in \mathcal{O}_{\mathbb{C}_\wp}^\times$. We denote the subspace of ordinary Drinfeld cuspforms by $S_k^{\mathrm{ord}}(\Gamma_1(\mathfrak{n}))$. It is an analogue of the notion of ordinariness for elliptic modular forms studied in [Hid].

Let us focus on the case $\mathfrak{n} = t^n$ and $\wp = t$ with some integer $n \geqslant 1$. In this case, the structure of $S_k^{\mathrm{ord}}(\Gamma_1(t^n))$ seems quite simple. For $n = 1$, it is known that all Hecke operators act trivially on the one-dimensional $\mathbb{C}_\infty$-vector space $S_k^{\mathrm{ord}}(\Gamma_1(t))$ [Hat3, Proposition 4.3]. In this paper, we prove that this holds in general, as follows.

Theorem 1.1 (Theorem 4.9)**.** *Let $k \geqslant 2$ and $n \geqslant 1$ be any integers. Then we have*
$$\dim_{\mathbb{C}_\infty} S_k^{\mathrm{ord}}(\Gamma_1(t^n)) = q^{n-1}$$
and all Hecke operators act trivially on $S_k^{\mathrm{ord}}(\Gamma_1(t^n))$.

This suggests that Hida families for Drinfeld cuspforms should be trivial for the level $\Gamma_1(t^n)$.

For Drinfeld modular forms, it is well-known that the weak multiplicity one, which states that any Hecke eigenform is determined up to a scalar multiple by the Hecke eigenvalues, is false. Gekeler [Gek, §7] raised a question if the property holds when we fix the weight. Theorem 1.1 gives a negative answer to it (see also [Böc, Examples 15.4 and 15.7] for a variant ignoring Hecke eigenvalues at places dividing the level).

For the proof of Theorem 1.1, we study a subspace S'_k of $S_k = S_k(\Gamma_1(t^n))$ containing $S_k^{(2)} = S_k^{(2)}(\Gamma_1(t^n))$. It consists of cuspforms which vanish twice at unramified cusps (§3.3). We show that all Hecke operators act trivially on S_k/S'_k and U_t is nilpotent on $S'_k/S_k^{(2)}$ (Lemma 3.9 and Proposition 3.10). Then, using the constancy of the dimension of $S_k^{\mathrm{ord}}(\Gamma_1(t^n))$ with respect to k [Hat3, Proposition 3.4], we reduce Theorem 1.1 to showing that the dimension of $S_2^{\mathrm{ord}}(\Gamma_1(t^n))$ is no more than q^{n-1} (Theorem 3.11).

Consider the multiplicative group $\Theta_n = 1 + tA/t^nA$, which acts on $S_k(\Gamma_1(t^n))$ via the diamond operator. To obtain the upper bound of the dimension, the key point is the freeness of $S_2(\Gamma_1(t^n))$ as a module over the group ring $\mathbb{C}_\infty[\Theta_n]$ (Proposition 4.8): From the fact that $S_2^{\mathrm{ord}}(\Gamma_1(t))$ is one-dimensional [Hat2, Lemma 2.4] and another constancy result of the dimension of the ordinary subspace [Hat3, Proposition 3.5], we see that the Θ_n-fixed part of $S_2^{\mathrm{ord}}(\Gamma_1(t^n))$ is also one-dimensional. Thus the freeness implies that it injects into a single component $\mathbb{C}_\infty[\Theta_n]$ of the free $\mathbb{C}_\infty[\Theta_n]$-module $S_2(\Gamma_1(t^n))$, which gives the desired bound.

The paper is organized as follows. In §2, we will recall the definition of Hecke operators and study their effect on Fourier expansions of

Drinfeld cuspforms at cusps. In §3, we will define the subspace S'_k and study its properties analytically. In §4, using the description of Drinfeld cuspforms via harmonic cocycles on the Bruhat–Tits tree [Tei, Böc], we will give an explicit basis of the $\mathbb{C}_\infty$-vector space $S_2(\Gamma_1(t^n))$ and a description of the diamond operator in terms of the basis. These enable us to show the freeness and Theorem 1.1.

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2. Drinfeld cuspforms of level $\Gamma_1(\mathfrak{n})$

Let $k \geqslant 2$ be any integer and $\mathfrak{n}$ any element of $A\backslash\mathbb{F}_q$. In this section, we study Hecke operators acting on $S_k(\Gamma_1(\mathfrak{n}))$. For any group Γ acting on a set X, we denote the stabilizer of $x \in X$ in Γ by $\mathrm{Stab}(\Gamma, x)$.

2.1. **Cusps and uniformizers.** Consider the action of $SL_2(A)$ on $\mathbb{P}^1(\mathbb{C}_\infty)$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

We refer to any element of $\mathbb{P}^1(K)$ as a cusp. For any arithmetic subgroup Γ of $SL_2(A)$, put

$$\mathrm{Cusps}(\Gamma) = \Gamma\backslash\mathbb{P}^1(K).$$

We abusively identify an element of $\mathrm{Cusps}(\Gamma)$ with a cusp representing it.

Next we recall the definition of the uniformizer at each cusp [GR, (2.7)], following the normalization of [Gek, (4.1)]. Let C be the Carlitz module. It is the Drinfeld module of rank one over A defined by the homomorphism of $\mathbb{F}_q$-algebras

$$A = \mathbb{F}_q[t] \to \mathrm{End}(\mathbb{G}_\mathrm{a}), \quad t \mapsto (Z \mapsto tZ + Z^q),$$

where we put $\mathbb{G}_\mathrm{a} = \mathrm{Spec}(A[Z])$. For any $a \in A$, we denote by $\Phi_a^C(Z)$ the element of $A[Z]$ such that the image of a by the map above is defined by $(Z \mapsto \Phi_a^C(Z))$.

For any subgroup $\mathfrak{b}$ of A containing a non-zero ideal of A, we define

$$e_{\mathfrak{b}}(z) = z \prod_{0 \neq b \in \mathfrak{b}} \left(1 - \frac{z}{b}\right),$$

which is an entire function on Ω. Let $\bar{\pi} \in \mathbb{C}_\infty$ be a Carlitz period, so that

$$\Phi_t^C(\bar{\pi} e_A(z)) = \bar{\pi} e_A(tz). \tag{2.1}$$

For any integer $l \geqslant 0$, we put

$$u_{\mathfrak{b}}(z) = \frac{1}{\bar{\pi} e_{\mathfrak{b}}(z)}, \quad u(z) = u_A(z), \quad u_l(z) = u_{(t^l)}(z) = \frac{1}{t^l} u\left(\frac{z}{t^l}\right).$$

Since $\mathfrak{n} \in A \backslash \mathbb{F}_q$, the group $\Gamma_1(\mathfrak{n})$ is p'-torsion free. For any cusp $s \in \mathbb{P}^1(K)$, choose $\nu_s \in SL_2(A)$ satisfying $\nu_s(\infty) = s$ and put

$$\mathfrak{b}_s = \left\{ b \in A \;\middle|\; \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathrm{Stab}(\nu_s^{-1} \Gamma_1(\mathfrak{n}) \nu_s, \infty) \right\} \supseteq (\mathfrak{n}).$$

Then we refer to the function

$$u_s(z) := u_{\mathfrak{b}_s}(z)$$

as the uniformizer at s for $\Gamma_1(\mathfrak{n})$. Note that $\mathfrak{b}_s$ depends only on s up to a multiple of an element of $\mathbb{F}_q^\times$. Thus $\mathfrak{b}_s$ and $u_s(z)$ are independent of the choice of ν_s if $\mathfrak{b}_s$ is an ideal of A for some choice of ν_s. For example, we have $\mathfrak{b}_\infty = A$ for any choice of ν_∞ and the uniformizer at ∞ is $u(z)$.

For any function f on Ω, integer $k \geqslant 2$ and $\gamma \in GL_2(K)$, we define the slash operator by

$$\left(f|_k \gamma\right)(z) = \det(\gamma)^{k-1} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, for any $f \in S_k(\Gamma_1(\mathfrak{n}))$, we can write

$$(f|_k \nu_s)(z) = \sum_{i \geqslant 1} a_i u_s(z)^i, \quad a_i \in \mathbb{C}_\infty$$

when the $(1/t)$-adic absolute value $|u_s(z)|$ of $u_s(z)$ is sufficiently small. We refer to it as the Fourier expansion of f at the cusp s and put

$$\mathrm{ord}(s, f) = \min\{i \geqslant 1 \mid a_i \neq 0\}.$$

The latter is independent of the choice of ν_s.

Lemma 2.1. *Let $\mathfrak{m} \in A$ be any monic irreducible polynomial and $i \geqslant 1$ any integer.*

(1) $\sum_{\deg(\beta) < \deg(\mathfrak{m})} u\left(\frac{z+\beta}{\mathfrak{m}}\right) = \mathfrak{m} u(z)$.

(2) If $i \geqslant 2$, then $\sum_{\deg(\beta) < \deg(\mathfrak{m})} u\left(\frac{z+\beta}{\mathfrak{m}}\right)^i \in \mathfrak{m} u(z)^2 A[u(z)]$.

(3) $u(\mathfrak{m} z) \in u(z)^2 A[[u(z)]]$.

Here the sum $\sum_{\deg(\beta) < \deg(\mathfrak{m})}$ *runs over the set of* $\beta \in A$ *satisfying* $\deg(\beta) < \deg(\mathfrak{m})$.

Proof. Put $r = \deg(\mathfrak{m})$ and $\Phi^C_{\mathfrak{m}}(Z) = \mathfrak{m}Z + c_1 Z^q + \cdots + c_{r-1}Z^{q^{r-1}} + Z^{q^r}$. We denote by $C[\mathfrak{m}]$ the kernel of the multiplication by $\mathfrak{m}$ on C and by $C[\mathfrak{m}](\mathbb{C}_\infty)$ the A-module of $\mathbb{C}_\infty$-valued points of it. Then we have

$$z \prod_{0 \neq b \in C[\mathfrak{m}](\mathbb{C}_\infty)} \left(1 - \frac{z}{b}\right) = \mathfrak{m}^{-1}\Phi^C_{\mathfrak{m}}(z). \tag{2.2}$$

Let α_i be the coefficient of Z^{q^i} in $\mathfrak{m}^{-1}\Phi^C_{\mathfrak{m}}(Z)$. By [Hat1, Lemma 3.2], we have $\alpha_i \in A$ for any $0 \leqslant i \leqslant r-1$ and $\alpha_r = \mathfrak{m}^{-1}$.

Let $G_{i,\mathfrak{m}}(X)$ be the i-th Goss polynomial with respect to the $\mathbb{F}_q$-vector space $C[\mathfrak{m}](\mathbb{C}_\infty)$. Then [Gek, computation above (7.3)] gives

$$\sum_{\deg(\beta) < \deg(\mathfrak{m})} u\left(\frac{z+\beta}{\mathfrak{m}}\right)^i = G_{i,\mathfrak{m}}(\mathfrak{m}u(z)). \tag{2.3}$$

For $i = 1$, we have $G_{i,\mathfrak{m}}(X) = X$ and 1 follows. For $i \geqslant 2$, [Gek, (3.8)] and (2.2) show that $G_{i,\mathfrak{m}}(X)$ has no linear term and $G_{i,\mathfrak{m}}(\mathfrak{m}X) \in \mathfrak{m}A[X]$, which yields 2. Moreover, we have

$$u(\mathfrak{m}z) = \frac{u(z)^{q^r}}{1 + c_{r-1}u(z)^{q^r - q^{r-1}} + \cdots + \mathfrak{m}u(z)^{q^r - 1}},$$

which implies 3. □

Put $\zeta_{t^l} = \bar{\pi} e_A\left(\frac{1}{t^l}\right) \in \mathbb{C}_\infty$, so that $\Phi^C_{t^l}(\zeta_{t^l}) = 0$ by (2.1).

Lemma 2.2. *Let $l \geqslant 1$ be any integer. For any $\beta \in \mathbb{F}_q$, we have*

$$u_{l-1}\left(\frac{z+\beta}{t}\right) \in t u_l(z) A[\zeta_{t^l}][[u_l(z)]].$$

Here $A[\zeta_{t^l}]$ is the A-subalgebra of $\mathbb{C}_\infty$ generated by ζ_{t^l}.

Proof. This follows from

$$u_{l-1}\left(\frac{z+\beta}{t}\right) = \frac{t}{t^l \bar{\pi} e_A\left(\frac{z+\beta}{t^l}\right)} = \frac{t}{t^l \bar{\pi} e_A\left(\frac{z}{t^l}\right)} \cdot \frac{1}{1 + \frac{t^l \bar{\pi} e_A\left(\frac{\beta}{t^l}\right)}{t^l \bar{\pi} e_A\left(\frac{z}{t^l}\right)}}$$
$$= t u_l(z) \cdot \frac{1}{1 + t^l \beta \zeta_{t^l} u_l(z)}.$$

□

2.2. **Hecke operators.** Now we recall the definition of Hecke operators (for example, see [Hat3, §3.1]). Let $\mathfrak{m} \in A$ be any monic irreducible polynomial. Then the Hecke operator $T_{\mathfrak{m}}$ acting on $S_k(\Gamma_1(\mathfrak{n}))$ is defined as

$$T_{\mathfrak{m}} f = \sum_{\xi} f|_k \xi,$$

where ξ runs over any complete set of representatives of the coset space

$$\Gamma_1(\mathfrak{n})\backslash\Gamma_1(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & \mathfrak{m}\end{pmatrix}\Gamma_1(\mathfrak{n}). \tag{2.4}$$

When $\mathfrak{m}|\mathfrak{n}$, we write $T_\mathfrak{m}$ also as $U_\mathfrak{m}$.

Let $\mathfrak{a}\in A$ be any element which is prime to $\mathfrak{n}$. Take any matrix $\eta_{\mathfrak{a},\diamond}\in SL_2(A)$ satisfying

$$\eta_{\mathfrak{a},\diamond}\equiv\begin{pmatrix}* & *\\ 0 & \mathfrak{a}\end{pmatrix} \bmod \mathfrak{n}$$

and put

$$\xi_{\mathfrak{a},\diamond}=\eta_{\mathfrak{a},\diamond}\begin{pmatrix}\mathfrak{a} & 0\\ 0 & 1\end{pmatrix}.$$

Note that we have

$$\eta_{\mathfrak{a},\diamond}\Gamma_1(\mathfrak{n})\eta_{\mathfrak{a},\diamond}^{-1}=\Gamma_1(\mathfrak{n}),\quad \xi_{\mathfrak{a},\diamond}\Gamma_1(\mathfrak{a}\mathfrak{n})\xi_{\mathfrak{a},\diamond}^{-1}\subseteq\Gamma_1(\mathfrak{n}).$$

Hence we obtain

$$f\in S_k(\Gamma_1(\mathfrak{n}))\Rightarrow f|_k\eta_{\mathfrak{a},\diamond}\in S_k(\Gamma_1(\mathfrak{n})),\ f|_k\xi_{\mathfrak{a},\diamond}\in S_k(\Gamma_1(\mathfrak{a}\mathfrak{n})). \tag{2.5}$$

For any $\alpha\in(A/(\mathfrak{n}))^\times$, we choose a lift $\mathfrak{a}\in A$ of α and put

$$\langle\alpha\rangle_\mathfrak{n}f=f|_k\eta_{\mathfrak{a},\diamond}$$

for any $f\in S_k(\Gamma_1(\mathfrak{n}))$, which is independent of the choices of $\mathfrak{a}$ and $\eta_{\mathfrak{a},\diamond}$. Then $\alpha\mapsto\langle\alpha\rangle_\mathfrak{n}$ defines an action of the group $(A/(\mathfrak{n}))^\times$ on $S_k(\Gamma_1(\mathfrak{n}))$.

Lemma 2.3. *For any $\alpha\in(A/(\mathfrak{n}))^\times$, the diamond operator $\langle\alpha\rangle_\mathfrak{n}$ commutes with all Hecke operators.*

Proof. Let $\mathfrak{m}\in A$ be any monic irreducible polynomial. First suppose $\mathfrak{m}\mid\mathfrak{n}$. Write

$$\eta_{\mathfrak{a},\diamond}=\begin{pmatrix}S & S'\\ T & T'\end{pmatrix}$$

with some $S,S',T,T'\in A$ satisfying $T\equiv 0$, $T'\equiv\mathfrak{a} \bmod \mathfrak{n}$ and $ST'-S'T=1$. Since S is prime to $\mathfrak{n}$, there exists $\beta\in A$ satisfying $\beta S\equiv S' \bmod \mathfrak{n}$. Then we have

$$\eta_{\mathfrak{a},\diamond}^{-1}\begin{pmatrix}1 & 0\\ 0 & \mathfrak{m}\end{pmatrix}\eta_{\mathfrak{a},\diamond}\in\Gamma_1(\mathfrak{n})\begin{pmatrix}1 & \beta\\ 0 & \mathfrak{m}\end{pmatrix}=\Gamma_1(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & \mathfrak{m}\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix},$$

which yields

$$\Gamma_1(\mathfrak{n})\eta_{\mathfrak{a},\diamond}^{-1}\begin{pmatrix}1 & 0\\ 0 & \mathfrak{m}\end{pmatrix}\eta_{\mathfrak{a},\diamond}\Gamma_1(\mathfrak{n})=\Gamma_1(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & \mathfrak{m}\end{pmatrix}\Gamma_1(\mathfrak{n}). \tag{2.6}$$

The lemma in this case follows from this equality.

Next suppose $\mathfrak{m} \nmid \mathfrak{n}$. Note that the natural map

$$SL_2(A) \to SL_2(A/(\mathfrak{n})) \times SL_2(A/(\mathfrak{m}))$$

is surjective. Since $\langle \alpha \rangle_{\mathfrak{n}}$ is independent of the choices of $\mathfrak{a}$ and $\eta_{\mathfrak{a},\diamond}$, we may assume that $\eta_{\mathfrak{a},\diamond}$ satisfies

$$\eta_{\mathfrak{a},\diamond} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{m}.$$

Then we have

$$\eta_{\mathfrak{a},\diamond}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{m} \end{pmatrix} \eta_{\mathfrak{a},\diamond} \in \Gamma_1(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{m} \end{pmatrix}$$

and (2.6) holds also in this case, which yields the lemma. $\square$

Let us give an explicit description of the Hecke operator $T_{\mathfrak{m}}$. For any $\beta \in A$ satisfying $\deg(\beta) < \deg(\mathfrak{m})$, put

$$\xi_{\mathfrak{m},\beta} = \begin{pmatrix} 1 & \beta \\ 0 & \mathfrak{m} \end{pmatrix}.$$

When $\mathfrak{m} = t$, we also write ξ_β for $\xi_{t,\beta}$. Then the operator $U_{\mathfrak{m}}$ for $\mathfrak{m} \mid \mathfrak{n}$ is given by

$$(U_{\mathfrak{m}} f)(z) = \sum_{\deg(\beta)<\deg(\mathfrak{m})} (f|_k \xi_{\mathfrak{m},\beta})(z) = \frac{1}{\mathfrak{m}} \sum_{\deg(\beta)<\deg(\mathfrak{m})} f\left(\frac{z+\beta}{\mathfrak{m}}\right).$$

When $\mathfrak{m} \nmid \mathfrak{n}$, the set

$$\{\xi_{\mathfrak{m},\beta} \mid \deg(\beta) < \deg(\mathfrak{m})\} \cup \{\xi_{\mathfrak{m},\diamond}\}$$

forms a complete set of representatives of the coset space (2.4) and thus

$$T_{\mathfrak{m}} f = \sum_{\deg(\beta)<\deg(\mathfrak{m})} f|_k \xi_{\mathfrak{m},\beta} + f|_k \xi_{\mathfrak{m},\diamond}.$$

3. U_t-operator of level $\Gamma_1(t^n)$

Let $k \geqslant 2$ and $n \geqslant 1$ be any integers. In the rest of the paper, we assume $\mathfrak{n} = t^n$.

In this section, we study the operator U_t acting on $S_k(\Gamma_1(t^n))$, and prove a criterion, in terms of U_t, for all Hecke operators to act trivially on $S_k^{\mathrm{ord}}(\Gamma_1(t^n))$ (Theorem 3.11). We denote $S_k(\Gamma_1(t^n))$ and $S_k^{(2)}(\Gamma_1(t^n))$ also by S_k and $S_k^{(2)}$, respectively.

Put $A_n = A/(t^n)$. Let v_t be the t-adic valuation on K normalized as $v_t(t) = 1$. For any $c \in A_{n-1}$, take any lift $\tilde{c} \in A$ of c and put

$$\bar{v}_t(c) = \min\{v_t(\tilde{c}), n-1\},$$

which is independent of the choice of $\tilde{c}$.

3.1. **Cusps of $\Gamma_1(t^n)$.** For any $c, d \in A_{n-1}$, put

$$\bar{h}_{(c,d)} = \begin{pmatrix} \frac{1}{1+td} & 0 \\ tc & 1+td \end{pmatrix} \in SL_2(A_n).$$

Since the natural map $SL_2(A) \to SL_2(A_n)$ is surjective, we can take a lift $h_{(c,d)} \in \Gamma_1(t)$ of $\bar{h}_{(c,d)}$ by this map.

Lemma 3.1. *Let (c, d) be any element of A_{n-1}^2. Suppose that an element $h \in \Gamma_1(t)$ satisfies*

$$h \equiv \begin{pmatrix} * & * \\ tc & 1+td \end{pmatrix} \bmod t^n.$$

Then $h \in \Gamma_1(t^n) h_{(c,d)}$.

Proof. We have $\det(hh_{(c,d)}^{-1}) = 1$ and

$$hh_{(c,d)}^{-1} \equiv \begin{pmatrix} * & * \\ tc & 1+td \end{pmatrix} \begin{pmatrix} 1+td & 0 \\ -tc & \frac{1}{1+td} \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod t^n.$$

Hence the $(1,1)$-entry of $hh_{(c,d)}^{-1}$ is also congruent to one modulo t^n and thus $hh_{(c,d)}^{-1} \in \Gamma_1(t^n)$. □

From Lemma 3.1, we see that the set

$$\{h_{(c,d)} \mid c, d \in A_{n-1}\}$$

forms a complete set of representatives of $\Gamma_1(t^n) \backslash \Gamma_1(t)$.

Note that for

$$SB(A) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL_2(A) \right\},$$

the map

$$\Gamma_1(t) \backslash SL_2(A)/SB(A) \to \Gamma_1(t) \backslash \mathbb{P}^1(K), \quad \gamma \mapsto \gamma(\infty)$$

is bijective. Hence we obtain

$$\mathrm{Cusps}(\Gamma_1(t)) = \{\infty, 0\}.$$

Consider the natural map

$$\mathrm{Cusps}(\Gamma_1(t^n)) \to \mathrm{Cusps}(\Gamma_1(t)).$$

For $\bullet \in \{\infty, 0\}$, we denote by $\mathrm{Cusps}_\bullet(\Gamma_1(t^n))$ the inverse image of $\bullet$ by this map. Then we have a bijection

$$\Gamma_1(t^n) \backslash \Gamma_1(t)/\mathrm{Stab}(\Gamma_1(t), \bullet) \to \mathrm{Cusps}_\bullet(\Gamma_1(t^n)), \quad \gamma \mapsto \gamma(\bullet).$$

From the equalities

$$\mathrm{Stab}(\Gamma_1(t),\infty) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in A \right\},$$
$$\mathrm{Stab}(\Gamma_1(t),0) = \left\{ \begin{pmatrix} 1 & 0 \\ tc & 1 \end{pmatrix} \;\middle|\; c \in A \right\},$$

we can show the following lemma.

Lemma 3.2. *(1) Let Λ_∞ be a subset of A_{n-1}^2 which forms a complete set of representatives for the equivalence relation*

$$(c,d) \sim (c',d') \Leftrightarrow c = c' \text{ and } d' - d \in cA_{n-1}.$$

Then the set

$$\{h_{(c,d)}(\infty) \mid (c,d) \in \Lambda_\infty\}$$

forms a complete set of representatives of $\mathrm{Cusps}_\infty(\Gamma_1(t^n))$.

(2) The set

$$\{h_{(0,d)}(0) \mid d \in A_{n-1}\}$$

forms a complete set of representatives of $\mathrm{Cusps}_0(\Gamma_1(t^n))$.

Lemma 3.3. *Let (c,d) be any element of A_{n-1}^2. Put $m = \bar{v}_t(c) \in [0, n-1]$.*

(1) For $s = h_{(c,d)}(\infty)$, we have

$$\mathfrak{b}_s = (t^{n-1-m}), \quad u_s(z) = u_{n-1-m}(z) = \frac{1}{t^{n-1-m}} u\left(\frac{z}{t^{n-1-m}}\right).$$

(2) For $s = h_{(0,d)}(0)$, we have

$$\mathfrak{b}_s = (t^n), \quad u_s(z) = u_n(z) = \frac{1}{t^n} u\left(\frac{z}{t^n}\right).$$

Proof. For any $x \in A$, the element

$$h_{(c,d)} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} h_{(c,d)}^{-1} \in SL_2(A) \tag{3.1}$$

is congruent modulo t^n to

$$\begin{pmatrix} 1 - \frac{tcx}{1+td} & \frac{x}{(1+td)^2} \\ -t^2c^2x & 1 + \frac{tcx}{1+td} \end{pmatrix}$$

and thus the element of (3.1) lies in $\Gamma_1(t^n)$ if and only if

$$\bar{v}_t(x) \geqslant \max\{n-1-m, n-2-2m\} = n-1-m,$$

which yields 1.

For 2, observe

$$h_{(0,d)}(0)=h_{(0,d)}J(\infty),\quad J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

Since

$$h_{(0,d)}J\begin{pmatrix}1&x\\0&1\end{pmatrix}J^{-1}h_{(0,d)}^{-1}\equiv\begin{pmatrix}1&0\\-x(1+td)^2&1\end{pmatrix}\bmod t^n,$$

the element of the left-hand side lies in $\Gamma_1(t^n)$ if and only if $x\in(t^n)$. This concludes the proof. □

3.2. Hecke operators of level $\Gamma_1(t^n)$.

Lemma 3.4. *For any $f\in S_k(\Gamma_1(t^n))$, monic irreducible polynomial $\mathfrak{m}\in A$ and $d\in A_{n-1}$, we have*

$$(T_{\mathfrak{m}}f)|_k h_{(0,d)}=\begin{cases}\sum_{\deg(\beta)<\deg(\mathfrak{m})}f|_k h_{(0,d)}\xi_\beta & (\mathfrak{m}=t),\\ \sum_{\deg(\beta)<\deg(\mathfrak{m})}f|_k h_{(0,d)}\xi_{\mathfrak{m},\beta}+f|_k h_{(0,d)}\xi_{\mathfrak{m},\diamond} & (\mathfrak{m}\neq t).\end{cases}$$

Moreover, when $\mathfrak{m}\neq t$, we can write

$$(f|_k h_{(0,d)}\xi_{\mathfrak{m},\diamond})(z)=\sum_{i\geqslant 2}c_i u(z)^i,\quad c_i\in\mathbb{C}_\infty$$

if $|u(z)|$ is sufficiently small.

Proof. Since $f|_k h_{(0,d)}=\langle 1+td\rangle_{t^n}f$, Lemma 2.3 shows the former assertion.

Let us show the latter assertion for $\mathfrak{m}\neq t$. We have

$$(f|_k h_{(0,d)}\xi_{\mathfrak{m},\diamond})(z)=\mathfrak{m}^{k-1}(f|_k h_{(0,d)}\eta_{\mathfrak{m},\diamond})(\mathfrak{m}z).$$

For any $x\in A$, observe

$$h_{(0,d)}\eta_{\mathfrak{m},\diamond}\begin{pmatrix}1&x\\0&1\end{pmatrix}(h_{(0,d)}\eta_{\mathfrak{m},\diamond})^{-1}\in\Gamma_1(t^n),$$

which shows that the uniformizer at the cusp $h_{(0,d)}\eta_{\mathfrak{m},\diamond}(\infty)$ is $u(z)$. Then we can write

$$(f|_k h_{(0,d)}\eta_{\mathfrak{m},\diamond})(z)=\sum_{i\geqslant 1}b_i u(z)^i,\quad b_i\in\mathbb{C}_\infty,$$

and the assertion follows from Lemma 2.1 3. □

Lemma 3.5. *Let $\beta\in\mathbb{F}_q$ and $(c,d)\in A_{n-1}^2$ be any elements.*

(1) $\xi_\beta h_{(c,d)}\in\Gamma_1(t^n)h_{(tc,d-\beta c)}\xi_\beta$.

(2) If $\beta\neq 0$, then

$$\xi_\beta h_{(c,d)}J\in\Gamma_1(t^n)h_{(\beta^{-1}(1+td),d-\beta c)}\begin{pmatrix}1&0\\0&t\end{pmatrix}\begin{pmatrix}\beta&-1\\0&\beta^{-1}\end{pmatrix}.$$

(3) $\xi_0 h_{(c,d)} J \in \Gamma_1(t^n) h_{(tc,d)} J \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. Write

$$h_{(c,d)} = \begin{pmatrix} P & t^n Q \\ tR & S \end{pmatrix}, \quad P, Q, R, S \in A.$$

Since $S \equiv P \bmod t$, we have $t^{-1}(S-P) \in A$ and the element

$$\xi_\beta h_{(c,d)} \xi_\beta^{-1} = \begin{pmatrix} P + t\beta R & \beta\left(\frac{S-P}{t}\right) - \beta^2 R + t^{n-1} Q \\ t^2 R & S - t\beta R \end{pmatrix} \in \Gamma_1(t)$$

satisfies

$$\xi_\beta h_{(c,d)} \xi_\beta^{-1} \equiv \begin{pmatrix} * & * \\ t^2 c & 1 + t(d - \beta c) \end{pmatrix} \bmod t^n.$$

Thus Lemma 3.1 shows 1.

For 2, the matrix $\xi_\beta h_{(c,d)} J$ equals

$$\begin{pmatrix} t^n \beta^{-1} Q + S & \beta\left(\frac{S-P}{t}\right) + t^{n-1} Q - \beta^2 R \\ t\beta^{-1} S & S - t\beta R \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} \beta & -1 \\ 0 & \beta^{-1} \end{pmatrix}.$$

The first matrix lies in $\Gamma_1(t)$, and it is congruent modulo t^n to

$$\begin{pmatrix} * & * \\ t\beta^{-1}(1 + td) & 1 + t(d - \beta c) \end{pmatrix}.$$

By Lemma 3.1, this matrix is contained in $\Gamma_1(t^n) h_{(\beta^{-1}(1+td), d - \beta c)}$ and 2 follows.

For 3, the matrix $\xi_0 h_{(c,d)} J$ equals

$$\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} P & t^n Q \\ tR & S \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} P & t^{n-1} Q \\ t^2 R & S \end{pmatrix} J \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.$$

The first matrix of the right-hand side lies in $\Gamma_1(t)$, and it is congruent modulo t^n to

$$\begin{pmatrix} * & * \\ t^2 c & 1 + td \end{pmatrix},$$

from which 3 follows by Lemma 3.1. □

Lemma 3.6. *Let $a, c, d \in A_{n-1}$ be any elements. Take any lift $\mathfrak{a} \in A$ of $1 + ta \in A_n$. Then we have*

$$\eta_{\mathfrak{a},\diamond} h_{(c,d)} \in \Gamma_1(t^n) h_{((1+ta)c, a+d+tad)}.$$

Proof. Since $\mathfrak{a} \equiv 1 \bmod t$, the matrix $\eta_{\mathfrak{a},\diamond}$ lies in $\Gamma_1(t)$. Thus the lemma follows from

$$\eta_{\mathfrak{a},\diamond} h_{(c,d)} \equiv \begin{pmatrix} * & * \\ t(1 + ta)c & (1 + ta)(1 + td) \end{pmatrix} \bmod t^n.$$

□

3.3. **Unramified double cuspforms.** Put

$$S'_k = \{f \in S_k \mid \mathrm{ord}\big(h_{(0,d)}(\infty), f\big) \geqslant 2 \text{ for any } d \in A_{n-1}\}.$$

Lemma 3.7. *S'_k is stable under all Hecke operators.*

Proof. Let f be any element of S'_k and $\mathfrak{m} \in A$ any monic irreducible polynomial. By Lemma 3.3 1 the uniformizer at the cusp $h_{(0,d)}(\infty)$ is $u(z)$ and we can write

$$(f|_k h_{(0,d)})(z) = \sum_{i \geqslant 2} a_i u(z)^i, \quad a_i \in \mathbb{C}_\infty.$$

Then Lemma 2.1 2 shows that the term

$$\sum_{\deg(\beta)<\deg(\mathfrak{m})} f|_k h_{(0,d)} \xi_{\mathfrak{m},\beta}$$

in the equality of Lemma 3.4 has no linear term of $u(z)$. Thus the lemma follows from the latter assertion of Lemma 3.4. □

For any $f \in S_k$ and $d \in A_{n-1}$, we write

$$(f|_k h_{(0,d)})(z) = \sum_{i \geqslant 1} a_i u(z)^i, \quad a_i \in \mathbb{C}_\infty$$

and put $L_d(f) = a_1$. Then the $\mathbb{C}_\infty$-linear map

$$L : S_k/S'_k \to \bigoplus_{d \in A_{n-1}} \mathbb{C}_\infty, \quad f \mapsto (L_d(f))_d$$

is injective.

Lemma 3.8.

$$\dim_{\mathbb{C}_\infty} S_k/S'_k = q^{n-1}.$$

In particular, the map L is bijective.

Proof. We denote $\mathrm{Cusps}(\Gamma_1(t^n))$ also by Cusps. By Lemma 3.2 1, the points

$$h_{(0,d)}(\infty), \quad d \in A_{n-1}$$

form a subset Cusps$'$ of cardinality q^{n-1} of Cusps. We abusively identify Cusps and Cusps$'$ with the reduced divisors they define on the Drinfeld modular curve $X_1(t^n)_{\mathbb{C}_\infty}$ over $\mathbb{C}_\infty$, and put $D = \mathrm{Cusps} + \mathrm{Cusps}'$. Let g be the genus of $X_1(t^n)_{\mathbb{C}_\infty}$ and h the number of cusps. Since $0 \in \mathrm{Cusps}\backslash\mathrm{Cusps}'$, we have $h > q^{n-1}$.

Let $\bar{\omega}$ be the Hodge bundle on $X_1(t^n)_{\mathbb{C}_\infty}$, so that $\deg(\bar{\omega}^{\otimes 2}) = 2g - 2 + 2h$ and $\deg(\bar{\omega}) \geqslant 0$ (see for example [Hat1, Corollary 4.2] with $\Delta = \{1\}$). For $k \geqslant 2$, we have

$$\begin{aligned}\deg(\bar{\omega}^{\otimes k}(-D)) &= k\deg(\bar{\omega}) - \deg(D)\\ &= (k-2)\deg(\bar{\omega}) + 2g - 2 + h - q^{n-1} \geqslant 2g - 1.\end{aligned}$$

Since S'_k can be identified with $H^0(X_1(t^n)_{\mathbb{C}_\infty}, \bar{\omega}^{\otimes k}(-D))$, the Riemann–Roch theorem implies

$$\dim_{\mathbb{C}_\infty} S'_k = \deg(\bar{\omega}^{\otimes k}(-D)) + 1 - g = (k-1)(g-1+h) - q^{n-1}.$$

From $\dim_{\mathbb{C}_\infty} S_k = (k-1)(g-1+h)$ [Böc, Proposition 5.4], we obtain $\dim_{\mathbb{C}_\infty} S_k/S'_k = q^{n-1}$. Since the both sides of the injection L have the same dimension, it is a bijection. □

Lemma 3.9. *All Hecke operators act trivially on S_k/S'_k.*

Proof. Let $\mathfrak{m} \in A$ be any monic irreducible polynomial. Take any $f \in S_k$. By Lemma 2.1 and Lemma 3.4, we obtain $L_d(T_\mathfrak{m} f) = L_d(f)$ for any $d \in A_{n-1}$ and the injectivity of the map L shows $T_\mathfrak{m} f \equiv f \bmod S'_k$. This concludes the proof. □

3.4. Nilpotency of U_t on $S'_k/S_k^{(2)}$.

For any integer i, put

$$C_i = \{(c,d) \in A_{n-1}^2 \mid \bar{v}_t(c) \geqslant i\}.$$

To study the U_t-action on S'_k, we define

$$S'_{k,i} = \{f \in S_k \mid \mathrm{ord}(h_{(c,d)}(\infty), f) \geqslant 2 \text{ for any } (c,d) \in C_i\}$$

so that

$$S'_k = S'_{k,n-1} \supseteq S'_{k,n-2} \supseteq \cdots \supseteq S'_{k,0} = S'_{k,-1} \supseteq S_k^{(2)}.$$

Proposition 3.10. *Let $i \in [0, n-1]$ be any integer.*

(1) $U_t(S'_{k,i}) \subseteq S'_{k,i-1}$.
(2) $U_t(S'_{k,0}) \subseteq S_k^{(2)}$.

In particular, the operator U_t acting on $S'_k/S_k^{(2)}$ is nilpotent.

Proof. For the assertion 1, take any $f \in S'_{k,i}$ and $(c,d) \in C_{i-1}$. We need to show

$$\mathrm{ord}(h_{(c,d)}(\infty), U_t f) \geqslant 2. \tag{3.2}$$

Since the case of $c = 0$ follows from Lemma 3.7, we may assume $c \neq 0$. Put $m = \bar{v}_t(c)$. For any $\beta \in \mathbb{F}_q$, we have $(tc, d - \beta c) \in C_i$ and the assumption yields $\bar{v}_t(tc) = m + 1$. By Lemma 3.3 1, we can write

$$(f|_k h_{(tc,d-\beta c)})(z) = \sum_{j \geqslant 2} a_j^{(\beta)} u_{n-2-m}(z)^j, \quad a_j^{(\beta)} \in \mathbb{C}_\infty$$

and Lemma 3.5 1 yields

$$((U_t f)|_k h_{(c,d)})(z) = \sum_{\beta \in \mathbb{F}_q} (f|_k \xi_\beta h_{(c,d)})(z) = \sum_{\beta \in \mathbb{F}_q} (f|_k h_{(tc, d-\beta c)} \xi_\beta)(z)$$
$$= \frac{1}{t} \sum_{\beta \in \mathbb{F}_q} \sum_{j \geqslant 2} a_j^{(\beta)} u_{n-2-m} \left(\frac{z+\beta}{t} \right)^j.$$

Since the uniformizer at $h_{(c,d)}(\infty)$ is $u_{n-1-m}(z)$, Lemma 2.2 gives the inequality (3.2).

Let us show the assertion 2. Take any $f \in S'_{k,0}$ and $d \in A_{n-1}$. Since we already know $U_t f \in S'_{k,0}$ by 1, it is enough to show

$$\operatorname{ord}(h_{(0,d)}(0), U_t f) \geqslant 2. \tag{3.3}$$

By Lemma 3.3 2, the uniformizer at $h_{(0,d)}(0) = h_{(0,d)} J(\infty)$ is $u_n(z)$.

Consider the equality

$$(U_t f)|_k h_{(0,d)} J = \sum_{\beta \in \mathbb{F}_q^\times} f|_k \xi_\beta h_{(0,d)} J + f|_k \xi_0 h_{(0,d)} J. \tag{3.4}$$

For the first term in the right-hand side of (3.4), we have

$$\bar{v}_t(\beta^{-1}(1+td)) = 0$$

and by Lemma 3.3 1 we can write

$$(f|_k h_{(\beta^{-1}(1+td),d)})(z) = \sum_{j \geqslant 2} a_j u_{n-1}(z)^j, \quad a_j \in \mathbb{C}_\infty.$$

Then Lemma 3.5 2 gives

$$(f|_k \xi_\beta h_{(0,d)} J)(z) = t^{k-1} (\beta^{-1} t)^{-k} \sum_{j \geqslant 2} a_j u_{n-1} \left(\frac{\beta z - 1}{\beta^{-1} t} \right)^j$$
$$= \frac{\beta^k}{t} \sum_{j \geqslant 2} a_j \beta^{-2j} u_{n-1} \left(\frac{z - \beta^{-1}}{t} \right)^j$$

and by Lemma 2.2 this term lies in $u_n(z)^2 \mathbb{C}_\infty[[u_n(z)]]$.

For the second term in the right-hand side of (3.4), write

$$(f|_k h_{(0,d)} J)(z) = \sum_{j \geqslant 1} a_j u_n(z)^j, \quad a_j \in \mathbb{C}_\infty.$$

By Lemma 3.5 3, we have

$$(f|_k \xi_0 h_{(0,d)} J)(z) = t^{k-1} (f|_k h_{(0,d)} J)(tz) = t^{k-1} \sum_{j \geqslant 1} a_j u_n(tz)^j.$$

Since Lemma 2.1 3 shows

$$u_n(tz) \in u_n(z)^2 \mathbb{C}_\infty[[u_n(z)]],$$

we obtain the inequality (3.3). This concludes the proof of the proposition. $\square$

Recall that we fixed an embedding $\iota_t : \bar{K} \to \mathbb{C}_t$. We say $\lambda \in \bar{K}$ is a t-adic unit if $\iota_t(\lambda) \in \mathcal{O}_{\mathbb{C}_t}^\times$.

Theorem 3.11. *For any integer $k \geqslant 2$, the following are equivalent.*

(1) U_t acting on $S_k^{(2)}(\Gamma_1(t^n))$ has no t-adic unit eigenvalue.
(2) U_t acting on S'_k has no t-adic unit eigenvalue.
(3) $\dim_{\mathbb{C}_\infty} S_k^{\mathrm{ord}}(\Gamma_1(t^n)) \leqslant q^{n-1}$.
(4) U_t acting on $S_2^{(2)}(\Gamma_1(t^n))$ is nilpotent.
(5) U_t acting on S'_2 is nilpotent.
(6) $\dim_{\mathbb{C}_\infty} S_2^{\mathrm{ord}}(\Gamma_1(t^n)) \leqslant q^{n-1}$.

If these equivalent conditions hold, then for any $k \geqslant 2$ we have

$$\dim_{\mathbb{C}_\infty} S_k^{\mathrm{ord}}(\Gamma_1(t^n)) = q^{n-1}$$

and all Hecke operators act trivially on $S_k^{\mathrm{ord}}(\Gamma_1(t^n))$.

Proof. The equivalence of 1 and 2 follows from Proposition 3.10. Note that the multiplicity μ of t-adic unit eigenvalues of U_t acting on S_k is equal to $\dim_{\mathbb{C}_\infty} S_k^{\mathrm{ord}}(\Gamma_1(t^n))$. By Lemma 3.8 and Lemma 3.9, the only t-adic unit eigenvalue of U_t acting on S_k/S'_k is one, with multiplicity q^{n-1}. Hence $\mu \geqslant q^{n-1}$, and the equality holds if and only if there is no other t-adic unit eigenvalue on S_k. The latter condition means that 2 holds. This implies that 2 and 3 are equivalent, and that 3 is equivalent to $\dim_{\mathbb{C}_\infty} S_k^{\mathrm{ord}}(\Gamma_1(t^n)) = q^{n-1}$.

By [Hat3, (2.6) and Proposition 2.2], any eigenvalue of U_t acting on $S_2(\Gamma_1(t^n))$ is algebraic over $\mathbb{F}_q$. Thus U_t acts on a subspace of $S_2(\Gamma_1(t^n))$ without t-adic unit eigenvalue if and only if the action is nilpotent. This shows the equivalence of 4–6. The equivalence of 3 and 6 follows from [Hat3, Proposition 3.4 (1)].

If these conditions hold, then we have $\dim_{\mathbb{C}_\infty} S_k^{\mathrm{ord}}(\Gamma_1(t^n)) = q^{n-1}$ and the natural map

$$S_k^{\mathrm{ord}}(\Gamma_1(t^n)) \to S_k/S'_k$$

is an isomorphism compatible with all Hecke operators. Now the last assertion follows from Lemma 3.9. $\square$

Since $X_1(t)_{\mathbb{C}_\infty}$ is of genus zero, we have $S_2^{(2)}(\Gamma_1(t)) = 0$ and the nilpotency of U_t acting on it holds trivially. Thus Theorem 3.11 yields the following corollary, which reproves [Hat2, Lemma 2.4] and [Hat3, Proposition 4.3] without using the theory of A-expansions [Pet] or Bandini–Valentino's formula [BV, (4.2)].

Corollary 3.12. *For any $k \geqslant 2$, we have*

$$\dim_{\mathbb{C}_\infty} S_k^{\mathrm{ord}}(\Gamma_1(t)) = 1$$

and all Hecke operators act trivially on $S_k^{\mathrm{ord}}(\Gamma_1(t))$.

Note that by [GN, Corollary 5.7] the genus of $X_1(t^n)_{\mathbb{C}_\infty}$ is

$$1 + q^{2n-2} - (n+1)q^{n-1} + (n-1)q^{n-2} \tag{3.5}$$

and for $n \geqslant 2$ it is zero only if $n = q = 2$. Thus the nilpotency of U_t acting on $S_2^{(2)}(\Gamma_1(t^n))$ seems non-trivial in general. We will prove it in a rather indirect way (Corollary 4.10).

4. Freeness and triviality

In this section, we prove the triviality of the Hecke action on $S_k(\Gamma_1(t^n))$ for any $k \geqslant 2$ and $n \geqslant 1$ (Theorem 4.9). Put $\Theta_n = 1 + tA_n \subseteq A_n^\times$. The key point of the proof is to show that $S_2(\Gamma_1(t^n))$, which we consider as a $\mathbb{C}_\infty[\Theta_n]$-module via the diamond operator, is the direct sum of copies of $\mathbb{C}_\infty[\Theta_n]$ (Proposition 4.8). For this, we need a description of $S_2(\Gamma_1(t^n))$ using harmonic cocycles on the Bruhat–Tits tree.

4.1. **Bruhat–Tits tree and $\Gamma_1(t^n)$.** We consider K_∞^2 as the set of row vectors, and define an action $\circ$ of $GL_2(K_\infty)$ on K_∞^2 by

$$\gamma \circ (x_1, x_2) = (x_1, x_2)\gamma^{-1}.$$

Let $\mathcal{T}$ be the Bruhat–Tits tree for $SL_2(K_\infty)$ (see for example [Ser, Ch. II, §1], [GN, §1] and [Böc, §3.1]). Recall that the set $\mathcal{T}_0$ of vertices of $\mathcal{T}$ is by definition the set of $K_\infty^\times$-equivalence classes of $\mathcal{O}_{K_\infty}$-lattices in K_∞^2, where $\mathcal{O}_{K_\infty}$ is the ring of integers of K_∞. The action $\circ$ induces an action of $GL_2(K_\infty)$ on the tree $\mathcal{T}$, and also on the oriented tree $\mathcal{T}^o$ associated to $\mathcal{T}$. We denote by $\mathcal{T}_1^o$ the set of oriented edges. For any $e \in \mathcal{T}_1^o$, the origin, the terminus and the opposite edge of e are denoted by $o(e)$, $t(e)$ and $-e$, respectively. Then the group $\{\pm 1\}$ acts on $\mathcal{T}_1^o$ by $(-1)e = -e$, which commutes with the action of $GL_2(K_\infty)$.

Put $\pi = 1/t$, which is a uniformizer of K_∞. For any integer i, let v_i be the class of the lattice $\mathcal{O}_{K_\infty}(\pi^i, 0) \oplus \mathcal{O}_{K_\infty}(0,1)$. Then we have $\begin{pmatrix} \pi^{-i} & 0 \\ 0 & 1 \end{pmatrix} v_0 = v_i$. We denote by e_i the oriented edge with origin v_i and terminus v_{i+1}.

For any subgroup Γ of $SL_2(A)$, we say $e \in \mathcal{T}_1^o$ is Γ-stable if $\mathrm{Stab}(\Gamma, e) = \{1\}$, and Γ-unstable otherwise. We define Γ-stability of a vertex similarly. The set of Γ-stable edges is denoted by $\mathcal{T}_1^{o,\Gamma\text{-st}}$. For $\Gamma = \Gamma_1(t)$, we know [LM, §7] that the set of $\Gamma_1(t)$-stable edges is equal to $\Gamma_1(t)J(\pm e_0)$ with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 4.1. *A complete set of representatives of* $\Gamma_1(t^n)\backslash\mathcal{T}_1^{o,\Gamma_1(t)\text{-st}}/\{\pm 1\}$ *is given by*

$$\Lambda_{1,n} = \{h_{(c,d)}Je_0 \mid c,d \in A_{n-1}\}.$$

Proof. Consider the natural map

$$\Lambda_{1,n} \to \Gamma_1(t^n)\backslash\mathcal{T}_1^{o,\Gamma_1(t)\text{-st}}/\{\pm 1\}, \tag{4.1}$$

which is surjective since $\mathcal{T}_1^{o,\Gamma_1(t)\text{-st}} = \Gamma_1(t)J(\pm e_0)$. Suppose that $(c,d),(c',d') \in A_{n-1}^2$ satisfy

$$\gamma h_{(c,d)}Je_0 = h_{(c',d')}Je_0 \quad \text{or} \quad \gamma h_{(c,d)}J(-e_0) = h_{(c',d')}Je_0$$

with some $\gamma \in \Gamma_1(t^n)$. For the former case, since Je_0 is $\Gamma_1(t)$-stable we have $\gamma h_{(c,d)} = h_{(c',d')}$ and thus $(c,d) = (c',d')$. For the latter case, we have

$$J^{-1}h_{(c',d')}^{-1}\gamma h_{(c,d)}Jv_0 = v_1.$$

Since the distance of v_0 and v_1 is one, it contradicts [Ser, Ch. II, §1.2, Corollary]. Hence the map (4.1) is also injective. □

4.2. **Harmonic cocycles.** In this subsection, we recall a description of Drinfeld cuspforms using harmonic cocycles due to Teitelbaum [Tei], following [Böc] and [Hat3].

Let $k \geqslant 2$ be any integer. We denote by $H_{k-2}(\mathbb{C}_\infty)$ the $\mathbb{C}_\infty$-subspace of homogeneous polynomials of degree $k-2$ in the polynomial ring $\mathbb{C}_\infty[X,Y]$. We consider the left action $\circ$ of $GL_2(K)$ on it defined by

$$\gamma \circ (X,Y) = (X,Y)\gamma.$$

We put $V_k(\mathbb{C}_\infty) = \mathrm{Hom}_{\mathbb{C}_\infty}(H_{k-2}(\mathbb{C}_\infty),\mathbb{C}_\infty)$, on which $GL_2(K)$ acts naturally. For $\xi \in GL_2(K)$, $\omega \in V_k(\mathbb{C}_\infty)$ and $P(X,Y) \in H_{k-2}(\mathbb{C}_\infty)$, the action is given by

$$(\xi \circ \omega)(P(X,Y)) = \omega(\xi^{-1} \circ P(X,Y)) = \omega(P((X,Y)\xi^{-1})).$$

Definition 4.2. A map $c : \mathcal{T}_1^o \to V_k(\mathbb{C}_\infty)$ is called a harmonic cocycle of weight k over $\mathbb{C}_\infty$ if the following two conditions hold:

(1) For any $v \in \mathcal{T}_0$, we have

$$\sum_{e \in \mathcal{T}_1^o,\ t(e)=v} c(e) = 0.$$

(2) For any $e \in \mathcal{T}_1^o$, we have $c(-e) = -c(e)$.

For any arithmetic subgroup Γ of $SL_2(A)$, we say c is Γ-equivariant if $c(\gamma e) = \gamma \circ c(e)$ for any $\gamma \in \Gamma$ and $e \in \mathcal{T}_1^o$. We denote the $\mathbb{C}_\infty$-vector space of Γ-equivariant harmonic cocycles of weight k over $\mathbb{C}_\infty$ by $C_k^{\mathrm{har}}(\Gamma)$.

Let Γ be an arithmetic subgroup of $SL_2(A)$ which is p'-torsion free. In this case, for any Γ-unstable vertex v, the group $\mathrm{Stab}(\Gamma, v)$ fixes a unique rational end which we denote by $b(v)$.

Definition 4.3. A Γ-stable edge $e' \in \mathcal{T}_1^o$ is called a Γ-source of an edge $e \in \mathcal{T}_1^o$ if the following conditions hold.

(1) If e is Γ-stable, then $e' = e$.
(2) If e is Γ-unstable, then a vertex v of e' is Γ-unstable, e lies on the unique half line from v to $b(v)$ and e has the same orientation as e' with respect to this half line.

The set of Γ-sources of e is denoted by $\mathrm{src}_\Gamma(e)$.

For any harmonic cocycle $c : \mathcal{T}_1^o \to V_k(\mathbb{C}_\infty)$ of weight k over $\mathbb{C}_\infty$, we have

$$c(e) = \sum_{e' \in \mathrm{src}_\Gamma(e)} c(e'). \tag{4.2}$$

We denote by $S_k(\Gamma)$ the $\mathbb{C}_\infty$-vector space of Drinfeld cuspforms of level Γ and weight k. Then, for any rigid analytic function f on Ω and $e \in \mathcal{T}_1^o$, Teitelbaum defined an element $\mathrm{Res}(f)(e) \in V_k(\mathbb{C}_\infty)$, which gives a natural isomorphism of $\mathbb{C}_\infty$-vector spaces

$$\mathrm{Res}_\Gamma : S_k(\Gamma) \to C_k^{\mathrm{har}}(\Gamma), \quad f \mapsto (e \mapsto \mathrm{Res}(f)(e)) \tag{4.3}$$

[Tei, Theorem 16]. Note that we are following the normalization in [Böc, Theorem 5.10]. Moreover, by [Böc, (17)], the slash operator can be read off via the corresponding harmonic cocycle by

$$\mathrm{Res}(f|_k \gamma)(e) = \gamma^{-1} \circ \mathrm{Res}(f)(\gamma e). \tag{4.4}$$

Lemma 4.4.

$$\dim_{\mathbb{C}_\infty} C_2^{\mathrm{har}}(\Gamma_1(t^n)) = q^{2(n-1)}.$$

Proof. By [GN, Proposition 5.6] (or Lemma 3.2), the number h of cusps of $X_1(t^n)_{\mathbb{C}_\infty}$ equals

$$h = (n+1)q^{n-1} - (n-1)q^{n-2}.$$

Thus the lemma follows from [Böc, Proposition 5.4], (3.5) and (4.3). $\square$

Lemma 4.5. *Let c be any element of $C_2^{\mathrm{har}}(\Gamma_1(t^n))$.*

(1) For any $\gamma \in \Gamma_1(t^n)$ and $e \in \mathcal{T}_1^o$, we have $c(\gamma e) = c(e)$.

(2) c is determined by its restriction to the subset $\Lambda_{1,n}$ of Lemma 4.1.

Proof. Since the group $GL_2(K)$ acts trivially on $V_2(\mathbb{C}_\infty) = \mathbb{C}_\infty$, we have $c(\gamma e) = \gamma \circ c(e) = c(e)$ and the assertion 1 follows.

For the assertion 2, it suffices to show that if the restriction of c to $\Lambda_{1,n}$ is zero, then $c(e) = 0$ for any $e \in \mathcal{T}_1^o$. By (4.2), we may assume that e is $\Gamma_1(t)$-stable. Then it is written as $e = \pm\gamma e'$ with some $e' \in \Lambda_{1,n}$ and $\gamma \in \Gamma_1(t^n)$, which yields $c(e) = \pm\gamma \circ c(e') = 0$. This concludes the proof. $\square$

Corollary 4.6. *The $\mathbb{C}_\infty$-linear map*

$$C_2^{\mathrm{har}}(\Gamma_1(t^n)) \to \bigoplus_{e \in \Lambda_{1,n}} \mathbb{C}_\infty, \quad c \mapsto (c(e))_{e \in \Lambda_{1,n}}$$

is an isomorphism.

Proof. By Lemma 4.5 2, the map is injective. Since $\sharp\Lambda_{1,n} = q^{2(n-1)}$, Lemma 4.4 implies that it is an isomorphism. $\square$

By Corollary 4.6, there exists a unique element $[c,d] \in C_2^{\mathrm{har}}(\Gamma_1(t^n))$ satisfying

$$[c,d](h_{(c',d')}Je_0) = \begin{cases} 1 & \text{if } (c',d') = (c,d), \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{[c,d] \mid c,d \in A_{n-1}\}$ forms a basis of the $\mathbb{C}_\infty$-vector space $C_2^{\mathrm{har}}(\Gamma_1(t^n))$.

4.3. **Proof of the main theorem.** Consider the subgroup $\Theta_n = 1 + tA_n$ of $A_n^\times$. Via the isomorphism $\mathrm{Res}_{\Gamma_1(t^n)}$ of (4.3), the diamond operator $\langle\alpha\rangle_{t^n}$ acting on $S_2(\Gamma_1(t^n))$ induces an operator on $C_2^{\mathrm{har}}(\Gamma_1(t^n))$, which we also denote by $\langle\alpha\rangle_{t^n}$. In particular, the group Θ_n acts on $C_2^{\mathrm{har}}(\Gamma_1(t^n))$ via $\alpha \mapsto \langle\alpha\rangle_{t^n}$.

Lemma 4.7. *For any $a,c,d \in A_{n-1}$, the action of $1+ta \in \Theta_n$ on $[c,d]$ is given by*

$$\langle 1+ta\rangle_{t^n}[c,d] = [(1+ta)^{-1}c, (1+ta)^{-1}(d-a)].$$

Proof. By (4.4) and Lemma 3.6, for any $c',d' \in A_{n-1}$ we have

$$(\langle 1+ta\rangle_{t^n}[c,d])\,(h_{(c',d')}Je_0) = [c,d](h_{((1+ta)c', a+d'+tad')}Je_0),$$

which is equal to one if $(c',d') = ((1+ta)^{-1}c, (1+ta)^{-1}(d-a))$ and zero otherwise. This concludes the proof. $\square$

Proposition 4.8. *The $\mathbb{C}_\infty[\Theta_n]$-module $S_2(\Gamma_1(t^n))$ is isomorphic to the direct sum of q^{n-1} copies of $\mathbb{C}_\infty[\Theta_n]$.*

Proof. It suffices to show the assertion for $C_2^{\mathrm{har}}(\Gamma_1(t^n))$. Take any $(c,d) \in A_{n-1}^2$. We claim that the Θ_n-orbit

$$\{\langle 1+ta\rangle_{t^n}[c,d] \mid a \in A_{n-1}\}$$

of $[c,d]$ is of cardinality q^{n-1}. Indeed, if $\langle 1+ta\rangle_{t^n}[c,d] = \langle 1+ta'\rangle_{t^n}[c,d]$ for some $a, a' \in A_{n-1}$, then Lemma 4.7 yields

$$(1+ta)^{-1}(d-a) = (1+ta')^{-1}(d-a'),$$

which is equivalent to $(1+td)(a'-a) = 0$ and we obtain $a' = a$.

We denote by $V(c,d)$ the $\mathbb{C}_\infty$-subspace of $C_2^{\mathrm{har}}(\Gamma_1(t^n))$ spanned by the Θ_n-orbit of $[c,d]$. Then $V(c,d)$ is stable under the Θ_n-action and $\dim_{\mathbb{C}_\infty} V(c,d) = q^{n-1}$. Consider the map

$$\mathbb{C}_\infty[\Theta_n] \to V(c,d), \quad \alpha \mapsto \langle \alpha\rangle_{t^n}[c,d].$$

It is a homomorphism of $\mathbb{C}_\infty[\Theta_n]$-modules which is surjective. Since the both sides have the same dimension, it is an isomorphism. Since the $\mathbb{C}_\infty$-vector space $C_2^{\mathrm{har}}(\Gamma_1(t^n))$ is the direct sum of $V(c,d)$'s, the proposition follows from Lemma 4.4. □

Theorem 4.9. *We have*

$$\dim_{\mathbb{C}_\infty} S_2^{\mathrm{ord}}(\Gamma_1(t^n)) = q^{n-1}$$

and all Hecke operators act trivially on $S_k^{\mathrm{ord}}(\Gamma_1(t^n))$ for any $k \geqslant 2$.

Proof. By Theorem 3.11, it is enough to show $\dim_{\mathbb{C}_\infty} S_2^{\mathrm{ord}}(\Gamma_1(t^n)) \leqslant q^{n-1}$. Put

$$\Gamma_0^p(t^n) = \left\{ \gamma \in SL_2(A) \;\middle|\; \gamma \bmod t^n \in \begin{pmatrix} 1+tA_n & A_n \\ 0 & 1+tA_n \end{pmatrix} \right\},$$

as in [Hat3, §3]. Then the Θ_n-fixed part of $S_2(\Gamma_1(t^n))$ is $S_2(\Gamma_0^p(t^n))$. Since the Hecke operator U_t commutes with the action of Θ_n and it is defined by the same formula for the levels $\Gamma_1(t^n)$ and $\Gamma_0^p(t^n)$ [Hat3, §3.1], we see that $S_2^{\mathrm{ord}}(\Gamma_1(t^n))$ is stable under the Θ_n-action and

$$S_2^{\mathrm{ord}}(\Gamma_1(t^n))^{\Theta_n} = S_2^{\mathrm{ord}}(\Gamma_0^p(t^n)),$$

where the right-hand side is the ordinary subspace of $S_2(\Gamma_0^p(t^n))$ defined similarly to the case of $S_2(\Gamma_1(t^n))$. Then [Hat3, Proposition 3.5] and Corollary 3.12 yield

$$\dim_{\mathbb{C}_\infty} S_2^{\mathrm{ord}}(\Gamma_1(t^n))^{\Theta_n} = \dim_{\mathbb{C}_\infty} S_2^{\mathrm{ord}}(\Gamma_0^p(t^n)) = \dim_{\mathbb{C}_\infty} S_2^{\mathrm{ord}}(\Gamma_1(t)) = 1.$$

On the other hand, Proposition 4.8 gives an injection of $\mathbb{C}_\infty[\Theta_n]$-modules

$$S_2^{\mathrm{ord}} := S_2^{\mathrm{ord}}(\Gamma_1(t^n)) \to \bigoplus_{i=1}^{q^{n-1}} V_i, \quad V_i = \mathbb{C}_\infty[\Theta_n].$$

Let I be the set of integers $M \in [1, q^{n-1}]$ such that there exists an injection of $\mathbb{C}_\infty[\Theta_n]$-modules $S_2^{\mathrm{ord}} \to \bigoplus_{i=1}^M V_i$. Then I is nonempty and let m be its minimal element.

Now we reduce ourselves to showing $m = 1$. Suppose $m > 1$ and consider an injection $S_2^{\mathrm{ord}} \to \bigoplus_{i=1}^m V_i$. Since Θ_n is an abelian p-group and $\mathbb{C}_\infty$ contains no non-trivial p-power root of unity, Schur's lemma implies that the only irreducible representation of Θ_n over $\mathbb{C}_\infty$ is the trivial representation. Since both of

$$S_2^{\mathrm{ord}} \cap V_1, \quad S_2^{\mathrm{ord}} \cap \bigoplus_{i=2}^m V_i$$

are $\mathbb{C}_\infty[\Theta_n]$-submodules of S_2^{ord}, if one of them is non-zero then it contains the trivial representation. Since the $\mathbb{C}_\infty$-vector space $(S_2^{\mathrm{ord}})^{\Theta_n}$ is one-dimensional, we see that either of them is zero. Thus either of the induced maps

$$S_2^{\mathrm{ord}} \to \left(\bigoplus_{i=1}^m V_i\right)/V_1 \simeq \bigoplus_{i=1}^{m-1} V_i, \quad S_2^{\mathrm{ord}} \to \left(\bigoplus_{i=1}^m V_i\right)/\left(\bigoplus_{i=2}^m V_i\right) \simeq V_1$$

is injective, which contradicts the minimality of m. This concludes the proof of the theorem. □

Theorem 3.11 and Theorem 4.9 yield the following corollary.

Corollary 4.10. *The operator U_t acting on $S_2^{(2)}(\Gamma_1(t^n))$ is nilpotent.*

Remark 4.11. By Theorem 3.11, if we could prove the nilpotency of U_t acting on $S_2^{(2)}(\Gamma_1(t^n))$ directly, then Theorem 4.9 would follow. As the proof of Theorem 4.9 indicates, the reason we can bypass it is that we know the dimension of $S_2^{\mathrm{ord}}(\Gamma_1(t))$ because $X_1(t)_{\mathbb{C}_\infty}$ is of genus zero. The author has no idea of how to show the nilpotency directly.

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