DIMENSION VARIATION OF GOUVÊA-MAZUR TYPE FOR DRINFELD CUSPFORMS OF LEVEL $\Gamma_1(t)$

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Abstract. Let p be a rational prime and $q > 1$ a p-power. Let $S_k(\Gamma_1(t))$ be the space of Drinfeld cuspforms of level $\Gamma_1(t)$ and weight k for $\mathbb{F}_q[t]$. For any non-negative rational number α, we denote by $d(k, \alpha)$ the dimension of the slope α generalized eigenspace for the U-operator acting on $S_k(\Gamma_1(t))$. In this paper, we prove a function field analogue of the Gouvêa-Mazur conjecture for this setting. Namely, we show that for any $\alpha \leqslant m$ and $k_1, k_2 > \alpha + 1$, if $k_1 \equiv k_2 \bmod p^m$, then $d(k_1, \alpha) = d(k_2, \alpha)$.

1. Introduction

Let p be a rational prime, $q > 1$ a p-power, $A = \mathbb{F}_q[t]$ and $\wp \in A$ a monic irreducible polynomial. For $K_\infty = \mathbb{F}_q((1/t))$, we denote by $\mathbb{C}_\infty$ the $(1/t)$-adic completion of an algebraic closure of K_∞. Then the Drinfeld upper half plane $\Omega = \mathbb{C}_\infty \backslash K_\infty$ has a natural structure of a rigid analytic variety over K_∞.

Let k be an integer and Γ a subgroup of $SL_2(A)$. Then a Drinfeld modular form of level Γ and weight k is a rigid analytic function $f : \Omega \to \mathbb{C}_\infty$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \text{ for any } z \in \Omega,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and a holomorphy condition at cusps. The notion of Drinfeld modular form can be considered as a function field analogue of that of elliptic modular form and the former often has properties which are parallel to the latter. However, despite that the theory of p-adic families of elliptic modular forms is highly developed and has been yielding many applications, $\wp$-adic properties of Drinfeld modular forms are not well-understood yet. A typical difficulty in the Drinfeld case seems that a naïve analogue of the universal character $\mathbb{Z}_p^\times \to \mathbb{Z}_p[[\mathbb{Z}_p^\times]]^\times$ is not locally analytic by [Jeo, Lemma 2.5] and thus similar constructions to those in the classical case including [AIP] will not immediately produce

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an analytic family of invertible sheaves interpolating automorphic line bundles.

Still, there seem to exist interesting structures in $\wp$-adic properties of Drinfeld modular forms. In [BV1, BV2], Bandini-Valentino studied an analogue of the classical Atkin U-operator, which we also denote by U, acting on the space $S_k(\Gamma_1(t))$ of Drinfeld cuspforms of level $\Gamma_1(t)$ and weight k. The operator U is defined by

$$(Uf)(z) = \frac{1}{t} \sum_{\beta \in \mathbb{F}_q} f\left(\frac{z+\beta}{t}\right). \tag{1.1}$$

The normalized t-adic valuation of an eigenvalue of U is called slope. Note that here we adopt the different normalization from that of Bandini-Valentino, and as a result our notion of slope is smaller than theirs by one. For a non-negative rational number α, we denote by $d(k,\alpha)$ the dimension of the generalized eigenspace of U acting on $S_k(\Gamma_1(t))$ for the eigenvalues of slope α. Then they proposed a conjecture on a p-adic variation of $d(k,\alpha)$ with respect to k [BV2, Conjecture 6.1] which can be regarded as a function field analogue of the Gouvêa-Mazur conjecture [GM1, Conjecture 1]. In this paper, we will prove it.

Theorem 1.1. *(Theorem 2.10) Let $m \geqslant 0$ be an integer and α a non-negative rational number. Suppose $\alpha \leqslant m$. Then the dimension $d(k,\alpha)$ of the slope α generalized eigenspace in $S_k(\Gamma_1(t))$ satisfies*

$$k_1, k_2 > \alpha + 1,\ k_1 \equiv k_2 \bmod p^m \Rightarrow d(k_1,\alpha) = d(k_2,\alpha).$$

For the proof, put

$$P^{(k)}(X) = \det(I - XU \mid S_k(\Gamma_1(t))).$$

First note that, as is mentioned in [Wan, §4, Remarks], the arguments of [GM2] and [Wan] can be generalized over suitable Drinfeld modular curves (including $X_1^{\Delta}(\mathfrak{n})$ of [Hat]). In particular, the characteristic power series of U acting on the spaces of $\wp$-adic overconvergent Drinfeld modular forms of weight k_1 and k_2 are congruent modulo $\wp^{p^m}$. Also in our setting, we can show the congruence $P^{(k_1)}(X) \equiv P^{(k_2)}(X) \bmod t^{p^m}$ up to some factor. However, though with this we can prove Theorem 1.1 for $p \geqslant 3$, it is not enough to settle the case of $p = 2$ on which Bandini-Valentino stated their conjecture.

Instead, we investigate the formula of the representing matrix of U given by Bandini-Valentino [BV1, (3.1)] more closely. Luckily, the representing matrix is of very special form: each entry on the j-th column (with the normalization that the leftmost column is the zeroth) is an element of $\mathbb{F}_q t^j$. Thanks to this fact, we can give a lower bound of elementary divisors of the representing matrix (Lemma 2.2). Then a perturbation argument shows that the n-th coefficients of $P^{(k)}(X)$ and $P^{(k+p^m)}(X)$ are much more congruent than modulo t^{p^m} up to some factor of slope $\geqslant k-1$ (Corollary 2.7), which is enough to yield Theorem 1.1 for any p.

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2. Dimension variation

Let $k \geqslant 2$ be an integer. Put

$$\Gamma_1(t) = \left\{ \gamma \in GL_2(A) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod t \right\} \subseteq SL_2(A).$$

On the space $S_k(\Gamma_1(t))$ of Drinfeld cuspforms of level $\Gamma_1(t)$ and weight k, we consider the U-operator for t defined by (1.1). Note that we follow the usual normalization of the U-operator which differs from that of [BV1, §2.4] by $1/t$. Then Bandini-Valentino [BV1, (3.1)] explicitly describe the action of U with respect to some basis $\mathbf{c}_0^{(k)}, \ldots, \mathbf{c}_{k-2}^{(k)}$, which reads as follows with our normalization:

$$\begin{aligned} U(\mathbf{c}_j^{(k)}) =& (-t)^j \binom{k-2-j}{j} \mathbf{c}_j^{(k)} - t^j \sum_{h \in \mathbb{Z}, h \neq 0} \left\{ \binom{k-2-j-h(q-1)}{-h(q-1)} \right. \\ & \left. + (-1)^{j+1} \binom{k-2-j-h(q-1)}{j} \right\} \mathbf{c}_{j+h(q-1)}^{(k)}. \end{aligned} \tag{2.1}$$

Here it is understood that the binomial coefficient $\binom{c}{d}$ is zero if any of $c, d, c-d$ is negative and the terms involving $\mathbf{c}_{j+h(q-1)}^{(k)}$ are zero if $j+h(q-1) \notin [0, k-2]$. We denote by $U^{(k)} = (U_{i,j}^{(k)})_{0 \leqslant i,j \leqslant k-2}$ the representing matrix of U for this basis. Then we have $U^{(k)} \in M_{k-1}(A)$. We identify the t-adic completion of A with $\mathbb{F}_q[[t]]$ naturally and consider $U^{(k)}$ as an element of $M_{k-1}(\mathbb{F}_q[[t]])$.

Definition 2.1. Let $B = (B_{i,j})_{0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant n-1}$ be an element of $M_{m,n}(\mathbb{F}_q[[t]])$. We say B is glissando if $B_{i,j} \in \mathbb{F}_q t^j$ for any i, j.

By (2.1), the matrix $U^{(k)}$ is glissando.

Lemma 2.2. *Let $B = (B_{i,j})_{0 \leqslant i \leqslant m-1, 0 \leqslant j \leqslant n-1}$ be a glissando matrix in $M_{m,n}(\mathbb{F}_q[[t]])$. Let $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_r$ be the elementary divisors of B*

(namely, they are integers or $+\infty$ such that t^{s_i} is the $(i-1,i-1)$-entry of the Smith normal form of B). Then we have $s_l \geqslant l-1$ for any l.

Proof. We prove the lemma by induction on n. For $n=1$, we have $s_1=0$ if $B\neq O$ and $s_1=+\infty$ otherwise. For $n>1$, we may assume $B\neq O$ and let c be the integer with $0\leqslant c\leqslant n-1$ such that the leftmost non-zero column of B is the c-th one. Since B is glissando, the first elementary divisor of B is $c\geqslant 0$ and the rest are equal to the elementary divisors of a matrix $t^{c+1}B'$, where B' is also glissando with $n-1$ columns. Let $s'_1\leqslant\cdots\leqslant s'_{r'}$ be the elementary divisors of B'. By the induction hypothesis, we have $s'_l\geqslant l-1$ and thus $s_l=c+1+s'_{l-1}\geqslant l-1$ for $l\geqslant 2$. This concludes the proof. □

Let v_t be the t-adic additive valuation normalized as $v_t(t)=1$. For any element $P(X)=\sum_{n=0}^{\infty}p_nX^n\in\mathbb{F}_q[[t]][[X]]$, the Newton polygon of $P(X)$ is by definition the lower convex hull of the set

$$\{(n,v_t(p_n))\mid n\geqslant 0\}.$$

Lemma 2.3. *Let $B\in M_m(\mathbb{F}_q[[t]])$ be a glissando matrix. For any non-negative integer l, put*

$$P(X)=\det(I-t^lXB)=\sum_{n=0}^{m}p_nX^n\in\mathbb{F}_q[[t]][X].$$

(1) $v_t(p_n)\geqslant ln+\frac{1}{2}n(n-1)$.
(2) Any slope of the Newton polygon of $P(X)$ is no less than l.

Proof. First note that, for the characteristic polynomial $Q(X)=\det(XI-t^lB)$, we have $P(X)=X^mQ(X^{-1})$ and thus p_n is, up to a sign, equal to the sum of the principal $n\times n$ minors of t^lB. Since B is glissando, this shows (1). Since $p_0=1$, the resulting inequality $v_t(p_n)\geqslant ln$ implies (2). □

Now we put

$$P^{(k)}(X)=\det(I-XU^{(k)})=\sum_{n=0}^{k-1}a_n^{(k)}X^n$$

and $a_n^{(k)}=0$ for any $n\geqslant k$. Let $y=N^{(k)}(x)$ be the Newton polygon of $P^{(k)}(X)$. For any non-negative rational number α, we denote by $d(k,\alpha)$ the dimension of the generalized eigenspace for the eigenvalues of normalized t-adic valuation α. Then $d(k,\alpha)$ is equal to the width of the segment of slope α in the Newton polygon $N^{(k)}$.

Lemma 2.4. $d(k,0)=1$.

Proof. By (2.1), we have $U^{(k)}_{0,0} = \binom{k-2}{0} = 1$. On the other hand, since $U^{(k)}$ is glissando, we have $v_t(U^{(k)}_{i,j}) \geqslant j$ and

$$a^{(k)}_1 = -\sum_{j=0}^{k-2} U^{(k)}_{j,j} \equiv -1 \bmod t.$$

Moreover, from Lemma 2.3 (1) we obtain $v_t(a^{(k)}_n) > 0$ for any $n \geqslant 2$. This yields the lemma. □

Lemma 2.5. *Let a and b be non-negative integers. Let $m \geqslant 1$ be an integer. Then we have*

$$\tbinom{a+p^m}{b} \equiv \tbinom{a}{b} + \tbinom{a}{b-p^m} \bmod p.$$

Here it is understood that $\binom{c}{d} = 0$ if any of $c, d, c-d$ is negative.

Proof. This follows from

$$(X+1)^{a+p^m} \equiv (X+1)^a (X^{p^m} + 1) \bmod p.$$

□

Proposition 2.6. *Let $m \geqslant 1$ be an integer. Then there exist glissando matrices $C \in M_{p^m,k-1}(A)$ and $D \in M_{p^m,p^m-k+1}(A)$ satisfying*

$$U^{(k+p^m)} \equiv \left(\begin{array}{c|c|c} U^{(k)} & O & O \\ C & t^{k-1}D & O \end{array} \right) \bmod t^{p^m}.$$

Here it is understood that the middle blocks are empty if $p^m \leqslant k-1$.

Proof. Let j be an integer satisfying $0 \leqslant j \leqslant k + p^m - 2$. By (2.1), the element $U(\mathbf{c}^{(k+p^m)}_j)$ is equal to

$$\begin{aligned} &(-t)^j \tbinom{k+p^m-2-j}{j} \mathbf{c}^{(k+p^m)}_j \\ &- t^j \sum_{h\in\mathbb{Z}, h\neq 0} \left\{ \tbinom{k+p^m-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1} \tbinom{k+p^m-2-j-h(q-1)}{j} \right\} \mathbf{c}^{(k+p^m)}_{j+h(q-1)}. \end{aligned}$$

Note that both of $U^{(k+p^m)}_{i,j}$ and $U^{(k)}_{i,j}$ are divisible by t^{p^m} for $j \geqslant p^m$. Since $U^{(k+p^m)}$ is glissando, what we need to show is

(1) For any $j \leqslant \min\{k-2, p^m - 1\}$ and $i \in [0, k-2]$, we have $U^{(k+p^m)}_{i,j} = U^{(k)}_{i,j}$, and
(2) If $k \leqslant p^m$, then for any $j \in [k-1, p^m - 1]$ and $i \in [0, k-2]$, we have $U^{(k+p^m)}_{i,j} = 0$.

First we suppose $j \leqslant \min\{k-2, p^m-1\}$. By Lemma 2.5, the element $U(\mathbf{c}_j^{(k+p^m)})$ equals

$$
\begin{aligned}
&(-t)^j \left(\binom{k-2-j}{j} + \binom{k-2-j}{j-p^m}\right) \mathbf{c}_j^{(k+p^m)} \\
&- t^j \sum_{\substack{h\in\mathbb{Z}, h\neq 0 \\ j+h(q-1)\in[0,k-2]}} \left\{ \binom{k-2-j-h(q-1)}{-h(q-1)} + \binom{k-2-j-h(q-1)}{-h(q-1)-p^m} \right. \\
&\left. +(-1)^{j+1} \left(\binom{k-2-j-h(q-1)}{j} + \binom{k-2-j-h(q-1)}{j-p^m} \right) \right\} \mathbf{c}_{j+h(q-1)}^{(k+p^m)} \\
&- t^j \sum_{\substack{h\in\mathbb{Z}, h\neq 0 \\ j+h(q-1)\geqslant k-1}} \left\{ \binom{k+p^m-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1} \binom{k+p^m-2-j-h(q-1)}{j} \right\} \mathbf{c}_{j+h(q-1)}^{(k+p^m)}.
\end{aligned}
$$

Hence $U(\mathbf{c}_j^{(k+p^m)})$ agrees with

$$
\begin{aligned}
&\sum_{i=0}^{k-2} U_{i,j}^{(k)} \mathbf{c}_i^{(k+p^m)} + (-t)^j \binom{k-2-j}{j-p^m} \mathbf{c}_j^{(k+p^m)} \\
&- t^j \sum_{\substack{h\in\mathbb{Z}, h\neq 0 \\ j+h(q-1)\in[0,k-2]}} \left\{ \binom{k-2-j-h(q-1)}{-h(q-1)-p^m} + (-1)^{j+1} \binom{k-2-j-h(q-1)}{j-p^m} \right\} \mathbf{c}_{j+h(q-1)}^{(k+p^m)} \\
&- t^j \sum_{\substack{h\in\mathbb{Z}, h\neq 0 \\ j+h(q-1)\geqslant k-1}} \left\{ \binom{k+p^m-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1} \binom{k+p^m-2-j-h(q-1)}{j} \right\} \mathbf{c}_{j+h(q-1)}^{(k+p^m)}.
\end{aligned}
$$

Since $j < p^m$, we have $\binom{k-2-j}{j-p^m} = 0$. For the case of $j + h(q-1) \in [0, k-2]$, we also have $-h(q-1)-p^m \leqslant j-p^m < 0$ and $\binom{k-2-j-h(q-1)}{-h(q-1)-p^m} = \binom{k-2-j-h(q-1)}{j-p^m} = 0$. This proves (1).

Next we suppose $k \leqslant p^m$ and $j \in [k-1, p^m-1]$. For any $i \in [0, k-2]$, the element $U_{i,j}^{(k+p^m)}$ is equal to

$$
-t^j \left\{ \binom{k+p^m-2-j-h(q-1)}{-h(q-1)} + (-1)^{j+1} \binom{k+p^m-2-j-h(q-1)}{j} \right\}
$$

if we can write $i = j + h(q-1)$ with some $h \neq 0$, and zero otherwise. Since $i \leqslant k-2$, we have $k-2-j-h(q-1) \geqslant 0$ and Lemma 2.5 implies

$$
\begin{aligned}
\binom{k-2-j-h(q-1)+p^m}{-h(q-1)} &= \binom{k-2-j-h(q-1)}{-h(q-1)} + \binom{k-2-j-h(q-1)}{-h(q-1)-p^m}, \\
\binom{k-2-j-h(q-1)+p^m}{j} &= \binom{k-2-j-h(q-1)}{j} + \binom{k-2-j-h(q-1)}{j-p^m}.
\end{aligned}
$$

Since $i = j + h(q-1) \in [0, k-2]$ and $j < p^m$, we have $\binom{k-2-j-h(q-1)}{-h(q-1)-p^m} = \binom{k-2-j-h(q-1)}{j-p^m} = 0$ as is seen above. Since $j \geqslant k-1$, we also have $\binom{k-2-j-h(q-1)}{-h(q-1)} = \binom{k-2-j-h(q-1)}{j} = 0$. This proves (2) and the proposition follows. $\square$

Let $V \in M_{k+p^m-1}(A)$ be the matrix of the right-hand side of Proposition 2.6. Let D' be the upper $(p^m - k + 1) \times (p^m - k + 1)$ block of D if $k \leqslant p^m$ and $D' = O$ otherwise. Then D' is also glissando. Put

$$\tilde{P}(X) = \det(I - XV) = P^{(k)}(X) \det(I - t^{k-1} X D')$$

and write $\tilde{P}(X) = \sum_{n=0}^{k+p^m-1} \tilde{a}_n X^n$. We denote by $\tilde{N}$ the Newton polygon of $\tilde{P}(X)$.

Corollary 2.7. *Let m and n be integers satisfying $m \geqslant 1$ and $0 \leqslant n \leqslant k + p^m - 1$. Then we have*

$$v_t(a_n^{(k+p^m)} - \tilde{a}_n) \geqslant p^m + \sum_{l=1}^{n-1} \min\{l-1, p^m\}.$$

Proof. Write

$$U^{(k+p^m)} = V + t^{p^m} W$$

with some $W \in M_{k+p^m-1}(A)$. Let $s_1 \leqslant \cdots \leqslant s_{k+p^m-1}$ be the elementary divisors of V. Since V is glissando, by Lemma 2.2 we obtain $s_l \geqslant l-1$ for any l. Then [Ked, Theorem 4.4.2] shows

$$v_t(a_n^{(k+p^m)} - \tilde{a}_n) \geqslant p^m + \sum_{l=1}^{n-1} \min\{s_l, p^m\} \geqslant p^m + \sum_{l=1}^{n-1} \min\{l-1, p^m\}.$$

$\square$

Lemma 2.8. *Let m and n be integers satisfying $m \geqslant 1$ and $n \geqslant 2$. Then we have*

$$p^m + \sum_{l=1}^{n-1} \min\{l-1, p^m\} > m(n-1).$$

Proof. First we assume $n - 2 \geqslant p^m$. Then the left-hand side of the lemma is equal to

$$(2.2)\quad p^m + \sum_{l=1}^{p^m+1} (l-1) + \sum_{l=p^m+2}^{n-1} p^m = \frac{1}{2} p^m (p^m + 3) + p^m (n - 2 - p^m).$$

For $m \geqslant 1$, we have $\frac{1}{2} p^m \geqslant m$ and thus $\frac{1}{2} p^m (p^m + 3) \geqslant m(p^m + 2)$. Hence the right-hand side of (2.2) is greater than $m(n-1)$.

Next we assume $n-2<p^m$. In this case, the left-hand side of the lemma equals $p^m+\frac{1}{2}(n-1)(n-2)$. It is greater than $m(n-1)$ if and only if

$$\left(n-\left(m+\frac{3}{2}\right)\right)^2+2p^m-m(m+1)-\frac{1}{4}>0.$$

Since m and n are integers, the first term is no less than $\frac{1}{4}$. Since we can show $2p^m>m(m+1)$ for any p and $m\geqslant 1$, the lemma follows also for this case. $\square$

Lemma 2.9. *The part of the Newton polygon $\tilde{N}$ of $\tilde{P}(X)$ of slope less than $k-1$ agrees with that of $N^{(k)}$.*

Proof. For any $Q(X)\in\mathbb{F}_q[[t]][X]$ and any non-negative rational number α, the Newton polygon of $Q(X)$ has a segment of slope α and width l if and only if it has exactly l roots of normalized t-adic valuation $-\alpha$. By Lemma 2.3 (2), every root of the polynomial $\det(I-t^{k-1}XD')$ has normalized t-adic valuation no more than $-(k-1)$. Thus, for $\tilde{P}(X)$ and $P^{(k)}(X)$, the sets of roots of normalized t-adic valuation more than $-(k-1)$ agree including multiplicities. This shows the lemma. $\square$

Theorem 2.10. *Let k and m be integers satisfying $k\geqslant 2$ and $m\geqslant 0$. Let α be a non-negative rational number satisfying $\alpha\leqslant m$ and $\alpha<k-1$. Then we have $d(k+p^m,\alpha)=d(k,\alpha)$.*

Proof. Let $\{\alpha_1,\ldots,\alpha_N\}$ be the set of slopes of the Newton polygons $N^{(k+p^m)}$ and $N^{(k)}$ which is no more than m and less than $k-1$, and renumber them so that $\alpha_i<\alpha_{i+1}$ for any i. We proceed by induction, following the proof of [Wan, Lemma 4.1]. By Lemma 2.4, we have $\alpha_1=0$ and $d(k+p^m,0)=d(k,0)=1$. Thus we may assume $m\geqslant 1$ and $N\geqslant 2$.

Suppose that for some $r\leqslant N-1$, the equality $d(k+p^m,\alpha_i)=d(k,\alpha_i)$ holds for any i satisfying $1\leqslant i\leqslant r$. By Lemma 2.9, this means that the Newton polygons $N^{(k)}$, $N^{(k+p^m)}$ and $\tilde{N}$ agree with each other on the part of slope no more than α_r. Put $\alpha=\alpha_{r+1}>\alpha_1=0$ and let us prove $d(k+p^m,\alpha)=d(k,\alpha)$. We choose $k'\in\{k,k+p^m\}$ such that the slope α occurs in $N^{(k')}$ and let k'' be the other. Let β be the slope of $N^{(k'')}$ on the right of α_r. Then $\beta\geqslant\alpha$.

Let $(n,v_t(a_n^{(k')}))$ be the endpoint of the segment of $N^{(k')}$ of slope α. Since the Newton polygon $N^{(k')}$ has a segment of slope zero, we have $n\geqslant 2$ and

$$N^{(k')}(n)=v_t(a_n^{(k')})\leqslant\alpha(n-1)\leqslant m(n-1).$$

Then Corollary 2.7 and Lemma 2.8 imply

$$v_t(a_n^{(k')}) < v_t(a_n^{(k+p^m)} - \tilde{a}_n). \tag{2.3}$$

If $k' = k$, then Lemma 2.9 shows $v_t(a_n^{(k')}) = v_t(a_n^{(k)}) = v_t(\tilde{a}_n)$ and from (2.3) we obtain $v_t(a_n^{(k+p^m)}) = v_t(\tilde{a}_n) = v_t(a_n^{(k)})$. This equality implies $\alpha = \beta$ and $d(k, \alpha) \leqslant d(k + p^m, \alpha)$. In particular, the slope α also occurs in $N^{(k+p^m)}$.

If $k' = k+p^m$, then (2.3) gives $v_t(\tilde{a}_n) = v_t(a_n^{(k+p^m)})$. Let γ be the slope of the Newton polygon $\tilde{N}$ on the right of α_r. Then this equality implies $\gamma \leqslant \alpha < k-1$. By Lemma 2.9, we have $\beta = \gamma \leqslant \alpha$. Therefore, we have $\alpha = \beta = \gamma$ and the width of the segment of slope α in $\tilde{N}$ is no less than that in $N^{(k+p^m)}$. Thus Lemma 2.9 again shows $d(k, \alpha) \geqslant d(k + p^m, \alpha)$. In particular, the slope α also occurs in $N^{(k)}$. Combining these two cases, we obtain $d(k, \alpha) = d(k + p^m, \alpha)$. This concludes the proof of Theorem 2.10. □

3. Remarks

Computations using (2.1) with Pari/GP indicate that the slopes appearing in $S_k(\Gamma_1(t))$ have some patterns (see also [BV2, §6]). The below is a table of the case $p = q = 2$, where the bold numbers denote multiplicities.

k	slopes	k	slopes
2	$0^{\mathbf{1}}$	13	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, 4^{\mathbf{1}}, \frac{11}{2}^{\mathbf{4}}, +\infty^{\mathbf{4}}$
3	$0^{\mathbf{1}}, +\infty^{\mathbf{1}}$	14	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, 5^{\mathbf{1}}, 6^{\mathbf{5}}, +\infty^{\mathbf{4}}$
4	$0^{\mathbf{1}}, 1^{\mathbf{1}}, +\infty^{\mathbf{1}}$	15	$0^{\mathbf{1}}, 2^{\mathbf{1}}, \frac{5}{2}^{\mathbf{2}}, 6^{\mathbf{1}}, \frac{13}{2}^{\mathbf{4}}, +\infty^{\mathbf{5}}$
5	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, +\infty^{\mathbf{1}}$	16	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{3}}, 7^{\mathbf{5}}, +\infty^{\mathbf{5}}$
6	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, +\infty^{\mathbf{2}}$	17	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, \frac{7}{2}^{\mathbf{2}}, \frac{15}{2}^{\mathbf{6}}, +\infty^{\mathbf{5}}$
7	$0^{\mathbf{1}}, 2^{\mathbf{1}}, \frac{5}{2}^{\mathbf{2}}, +\infty^{\mathbf{2}}$	18	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{3}}, 8^{\mathbf{5}}, +\infty^{\mathbf{6}}$
8	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{3}}, +\infty^{\mathbf{2}}$	19	$0^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{1}}, \frac{9}{2}^{\mathbf{2}}, 8^{\mathbf{1}}, \frac{17}{2}^{\mathbf{6}}, +\infty^{\mathbf{6}}$
9	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, \frac{7}{2}^{\mathbf{2}}, +\infty^{\mathbf{3}}$	20	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{1}}, 4^{\mathbf{1}}, 5^{\mathbf{1}}, 8^{\mathbf{1}}, 9^{\mathbf{7}}, +\infty^{\mathbf{6}}$
10	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{3}}, +\infty^{\mathbf{3}}$	21	$0^{\mathbf{1}}, \frac{3}{2}^{\mathbf{2}}, 4^{\mathbf{1}}, \frac{11}{2}^{\mathbf{2}}, 8^{\mathbf{1}}, \frac{19}{2}^{\mathbf{6}}, +\infty^{\mathbf{7}}$
11	$0^{\mathbf{1}}, 2^{\mathbf{1}}, 4^{\mathbf{1}}, \frac{9}{2}^{\mathbf{4}}, +\infty^{\mathbf{3}}$	22	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 2^{\mathbf{1}}, 5^{\mathbf{1}}, 6^{\mathbf{1}}, 8^{\mathbf{1}}, 9^{\mathbf{1}}, 10^{\mathbf{7}}, +\infty^{\mathbf{7}}$
12	$0^{\mathbf{1}}, 1^{\mathbf{1}}, 3^{\mathbf{1}}, 4^{\mathbf{1}}, 5^{\mathbf{3}}, +\infty^{\mathbf{4}}$	23	$0^{\mathbf{1}}, 2^{\mathbf{1}}, \frac{5}{2}^{\mathbf{2}}, 6^{\mathbf{1}}, 8^{\mathbf{1}}, 10^{\mathbf{1}}, \frac{21}{2}^{\mathbf{8}}, +\infty^{\mathbf{7}}$

From the table, it seems that only small denominators are allowed for slopes: In the author's computation, as is already mentioned in [BV2, §1], the only case a non-trivial denominator appears is the case of $p = 2$ and the denominator is at most 2. Moreover, it seems likely that the finite slopes of $S_k(\Gamma_1(t))$ are less than $k - 1$, and that for any n, the

n-th smallest finite slope of $S_k(\Gamma_1(t))$ is bounded independently of k (say, by q^{n-1}). If the latter observations hold in general, then combined with Theorem 2.10 it follows that for any n, the n-th smallest finite slopes of $S_k(\Gamma_1(t))$ are periodic of p-power period with respect to k including multiplicities. For example, it seems from the table that the third smallest finite slopes of $S_k(\Gamma_1(t))$ in the case of $p = q = 2$ are the repetition of

$$2^{\mathbf{1}}, \tfrac{5}{2}^{\mathbf{2}}, 3^{\mathbf{3}}, \tfrac{7}{2}^{\mathbf{2}}, 2^{\mathbf{1}}, 4^{\mathbf{1}}, 3^{\mathbf{1}}, 4^{\mathbf{1}}.$$

This could be thought of as a function field analogue of Emerton's theorem [Eme] which asserts that the minimal slopes of $S_k(\Gamma_0(2))$ are periodic of period 8.

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