Tame characters and ramification of finite flat group schemes

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Abstract.

In this paper, for a complete discrete valuation field K of mixed characteristic $(0,p)$ and a finite flat group scheme $\mathcal{G}$ of p-power order over $\mathcal{O}_K$, we determine the tame characters appearing in the Galois representation $\mathcal{G}(\bar{K})$ in terms of the ramification theory of Abbes and Saito, without any restriction on the absolute ramification index of K or the embedding dimension of $\mathcal{G}$.

Key words. Galois representation; Group scheme; Ramification

1 Introduction

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with residue field F which may be imperfect, G_K be its absolute Galois group, I_K be its inertia subgroup and P_K be its wild inertia subgroup. Consider a finite G_K-module M of p-power order. Since P_K acts unipotently on M, there is a filtration $M = M_0 \supsetneq M_1 \supsetneq \dots M_m \supsetneq 0$ of M by G_K-submodules M_i such that every graded piece M_i/M_{i+1} is tame and killed by p. Then the I_K-module $M_i/M_{i+1} \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ is the direct sum of some powers of fundamental characters ([15]). When V comes from a geometrical object, such as a group scheme or the etale cohomology of a proper smooth variety over $\mathcal{O}_K$, then these exponents of tame characters often can be described more precisely.

For example, let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ of p-power order. When $\mathcal{G}$ is monogenic, that is to say, when the affine algebra of $\mathcal{G}$ is generated over $\mathcal{O}_K$ by one element, it is well-known that the tame characters appearing in the I_K-module $\mathcal{G}(\bar{K})$ are determined by the slopes of the Newton polygon of a defining equation of $\mathcal{G}$ ([15, Proposition 10]).

On the other hand, for an elliptic modular form f of level N prime to p, we also have a description of the tame characters of the associated mod p Galois representation $\bar{\rho}_f$ ([9, Theorem 2.5, Theorem 2.6], [8, Section 4.3]). This is based on Raynaud's theory of prolongations of finite flat group schemes or the integral p-adic Hodge theory. However, for an analogous study of the associated mod p Galois representation of a Hilbert modular form over a totally real number field, we encounter a local field of higher absolute ramification index. In this case, these two theories no longer work well and we need some other techniques.

In this paper, to propose a new approach toward this problem, we generalize the aforementioned result of [15] to the higher dimensional case using the ramification theory of Abbes and Saito ([2,3]) and determine the tame characters appearing in $\mathcal{G}(\bar{K})$ in terms of the ramification of $\mathcal{G}$ without any restriction on the absolute ramification index of K. Namely, we show the following theorem.

Theorem 1.1 *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. Write $\{\mathcal{G}^j\}_{j\in\mathbb{Q}_{>0}}$ for the ramification filtration of $\mathcal{G}$ in the sense of* [2, 3]. *Then the graded piece $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ is killed by p and the I_K-module $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ is the direct sum of fundamental characters of level j.*

In view of the following corollary, we can regard this theorem as a counterpart for finite flat group schemes of the Hasse-Arf theorem in the classical ramification theory.

Corollary 1.2 *Let L be an abelian extension of K. Suppose that there exists* *a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ of p-power order such that $\mathcal{O}_L$ is a $\mathcal{G}$-torsor over $\mathcal{O}_K$. Then the denominator of every jump of the upper numbering ramification filtration* $\{\mathrm{Gal}(L/K)^j\}_{j\in\mathbb{Q}_{>0}}$ ([2]) *is a power of p.*

These results are proved in Section 4 (Theorem 4.7, Theorem 4.8, Corollary 4.10). To prove the main theorem, we firstly show that the tubular neighborhood of $\mathcal{G}$ can be chosen to have a rigid analytic group structure. Passing to the closed fiber, we realize the graded piece of the ramification filtration of $\mathcal{G}$ as the kernel of an etale isogeny of the additive groups $\bar{\mathbb{G}}_{\mathrm{a}}^r$ over $\bar{F}$. Then we determine the tame characters by comparing the I_K-action on the graded piece with the $\bar{\mathbb{G}}_{\mathrm{m}}$-action on $\bar{\mathbb{G}}_{\mathrm{a}}^r$ given by the multiplication. As an appendix, we also give an explicit formula for the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$ ([14]).

2 Review of the ramification theory of Abbes and Saito

Let K be a complete discrete valuation field with residue field F which may be imperfect. Set $\pi = \pi_K$ to be an uniformizer of K. The separable closure of K is denoted by $\bar{K}$ and the absolute Galois group of K by G_K. Let $\mathfrak{m}_{\bar{K}}$ and $\bar{F}$ be the maximal ideal and the residue field of $\mathcal{O}_{\bar{K}}$ respectively. We extend the valuation v_K of K to $\bar{K}$ and normalize this as $v_K(\pi) = 1$. In [2,3], Abbes and Saito defined the ramification theory of a finite flat $\mathcal{O}_K$-algebra of relative complete intersection. Using this, Abbes and Mokrane studied the ramification of finite flat group schemes over $\mathcal{O}_K$ ([1]). In this section, we gather the necessary definitions and briefly recall their theory.

Let A be a finite flat $\mathcal{O}_K$-algebra and $\mathbb{A}$ be a complete Noetherian semi-local ring (with its topology defined by $\mathrm{rad}(\mathbb{A})$) which is of formally smooth over $\mathcal{O}_K$ and whose quotient ring $\mathbb{A}/\mathrm{rad}(\mathbb{A})$ is of finite type over F. A surjection of $\mathcal{O}_K$-algebras $\mathbb{A} \to A$ is called an embedding if $\mathbb{A}/\mathrm{rad}(\mathbb{A}) \to A/\mathrm{rad}(A)$ is an isomorphism. A morphism of embeddings $(f, \mathbf{f}) : (\mathbb{A} \to A) \to (\mathbb{A}' \to A')$ is defined to be a pair of $\mathcal{O}_K$-homomorphisms $f : A \to A'$ and $\mathbf{f} : \mathbb{A} \to \mathbb{A}'$ which commutes the following diagram.

$$\begin{array}{ccc} \mathbb{A} & \longrightarrow & A \\ {\scriptstyle \mathbf{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbb{A}' & \longrightarrow & A' \end{array}$$

A morphism $(f, \mathbf{f})$ is said to be finite flat if $\mathbf{f}$ is finite flat and the map $\mathbb{A}' \otimes_{\mathbb{A}} A \to A'$ is an isomorphism.

For an embedding $(\mathbb{A} \to A)$ and $j \in \mathbb{Q}_{>0}$, the j-th tubular neighborhood of $(\mathbb{A} \to A)$ is the smooth K-affinoid variety $X^j(\mathbb{A} \to A)$ constructed as follows. Write $j = k/l$ with k, l non-negative integers. Put $I = \mathrm{Ker}(\mathbb{A} \to A)$ and $\mathcal{A}_0^{k,l} = \mathbb{A}[I^l/\pi^k]^\wedge$, where $\wedge$ means the π-adic completion. Then $\mathcal{A}_0^{k,l}$ is a quotient ring of the Tate algebra $\mathcal{O}_K\langle T_1, \dots, T_r\rangle$ for some r. Its generic fiber $\mathcal{A}_K^j = \mathcal{A}_0^{k,l} \otimes_{\mathcal{O}_K} K$ is independent of the choice of a representation $j = k/l$ ([3, Lemma 1.4]) and set $X^j(\mathbb{A} \to A) = \mathrm{Sp}(\mathcal{A}_K^j)$. This defines the functor X^j from the category of embeddings to the category of smooth K-affinoid varieties, which sends a finite flat map of embeddings to a finite flat map of K-affinoid varieties ([3, Lemma 1.6]).

Note that the functor X^j is compatible with the tensor product in the following sense. For embeddings $(\mathbb{A} \to A)$ and $(\mathbb{B} \to B)$, put $\mathbb{C} = \mathbb{A}\hat{\otimes}_{\mathcal{O}_K}\mathbb{B}$ and $C = A \otimes_{\mathcal{O}_K} B$. Then $(\mathbb{C} \to C)$ is also an embedding and we have the canonical isomorphism of K-affinoid varieties

$$X^j(\mathbb{C} \to C) \to X^j(\mathbb{A} \to A) \times_K X^j(\mathbb{B} \to B).$$

We put $F(A) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg.}}(A, \mathcal{O}_{\bar{K}})$ and $F^j(A) = \varprojlim \pi_0(X^j(\mathbb{A} \to A)_{\bar{K}})$. Here $\pi_0(X_{\bar{K}})$ denotes the set of geometric connected components of a K-affinoid variety X and the projective limit is taken in the category of embeddings of A. Note that the projective family $\pi_0(X^j(\mathbb{A} \to A)_{\bar{K}})$ is constant ([3, Section 1.2]). These define contravariant functors F and F^j from the category of finite flat $\mathcal{O}_K$-algebras to the category of finite G_K-sets. Moreover, there are morphisms of functors $F \to F^j$ and $F^{j'} \to F^j$ for $j' \geq j > 0$.

By the finiteness theorem of Grauert and Remmert ([6, Theorem 1.3]), there exists a finite separable extension L of K such that the geometric closed fiber of the unit disc $\mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L}$ for the supremum norm in $X^j(\mathbb{A} \to A)_L = X^j(\mathbb{A} \to A) \times_K L$ is reduced. Then for any finite separable extension L' of L, the π_L-adic formal scheme $\mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L} \times_{\mathcal{O}_L} \mathcal{O}_{L'}$ coincides with the unit disc for the supremum norm in $X^j(\mathbb{A} \to A)_{L'}$ and thus is normal. The π_L-adic formal scheme $\mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L}$ is referred as the stable normalized integral model of $X^j(\mathbb{A} \to A)$ over L and its geometric closed fiber is denoted by $\bar{X}^j(\mathbb{A} \to A)$, which is independent of the choice of L. If L/K is Galois, the right action of the Galois group $\mathrm{Gal}(L/K)$ on L defined by $\sigma.z = \sigma^{-1}(z)$ induces the L-semilinear left action on $X^j(\mathbb{A} \to A)_L$, which also extends by the functoriality of the unit disc for the supremum norm to the $\mathcal{O}_L$-semilinear left action on $\mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L}$. This is compatible with a base extension of such L and thus defines the canonical $\bar{F}$-semilinear G_K-action on $\bar{X}^j(\mathbb{A} \to A)$. Then we have the G_K-equivariant isomorphism

$$\pi_0(\bar{X}^j(\mathbb{A} \to A)) \to \pi_0(X^j(\mathbb{A} \to A)_{\bar{K}}),$$

where the former is the set of connected components of $\bar{X}^j(\mathbb{A} \to A)$ ([3, Corollary 1.11]).

Suppose that A is of relative complete intersection over $\mathcal{O}_K$ and $A \otimes_{\mathcal{O}_K} K$ is etale over K. Then the natural map $F(A) \to F^j(A)$ is surjective. The family $\{F(A) \to F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ is separated, exhaustive and its jumps are rational ([2, Proposition 6.4]). The conductor of A is defined to be

$$c(A) = \inf\{j \in \mathbb{Q}_{>0} \mid F(A) \to F^j(A) \text{ is an isomorphism}\}.$$

If B is the affine algebra of a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ which is generically etale, then B is of relative complete intersection (for example, [7, Proposition 2.2.2]) and the theory above can all be applied to B. By the functoriality, $F^j(B)$ is endowed with a G_K-module structure ([1, Lemme 2.1.1]) and the natural map $\mathcal{G}(\bar{K}) = F(B) \to F^j(B)$ is a G_K-homomorphism. Let $\mathcal{G}^j$ denote the schematic closure ([14]) in $\mathcal{G}$ of the kernel of this homomorphism. It is called the j-th ramification filtration of $\mathcal{G}$. We refer $c(B)$ as the conductor of $\mathcal{G}$, which is denoted also by $c(\mathcal{G})$. We put $\mathcal{G}^{j+}(\bar{K}) = \cup_{j'>j} \mathcal{G}^{j'}(\bar{K})$ and define $\mathcal{G}^{j+}$ to be the schematic closure of $\mathcal{G}^{j+}(\bar{K})$ in $\mathcal{G}$.

As in the classical ramification theory, the ramification filtration $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$ of $\mathcal{G}$ is not compatible with closed subgroups but with quotients. Namely, we have the following variant of the classical Herbrand theorem.

Proposition 2.1 ([1, *Lemme* 2.3.2]) *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ and $\mathcal{H}$ be its closed finite flat subgroup scheme over $\mathcal{O}_K$. Then the natural map $\mathcal{G}^j(\bar{K}) \to (\mathcal{G}/\mathcal{H})^j(\bar{K})$ is surjective for $j \in \mathbb{Q}_{>0}$.*

In closing this section, we state the following corollary of Proposition 2.1 which is used in Section 4.

Lemma 2.2 *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ which is generically etale and $j > 0$ be a jump of the ramification filtration $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$ of $\mathcal{G}$. Then we have $c(\mathcal{G}/\mathcal{G}^{j+}) = j$ and $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) = (\mathcal{G}/\mathcal{G}^{j+})^j(\bar{K})$.*

PROOF. From the definition of j, we see that the subgroup $\mathcal{G}^j(\bar{K}) \subseteq \mathcal{G}(\bar{K})$ has a non-trivial image in $(\mathcal{G}/\mathcal{G}^{j+})(\bar{K})$. By Proposition 2.1, the natural map $\mathcal{G}^t(\bar{K}) \to (\mathcal{G}/\mathcal{G}^{j+})^t(\bar{K})$ is surjective for any $t > 0$. We have $(\mathcal{G}/\mathcal{G}^{j+})^t = 0$ for $t > j$ and $(\mathcal{G}/\mathcal{G}^{j+})^j \neq 0$. This concludes the lemma. □

3 Group structure on the tubular neighborhood of a finite flat group scheme

For the rest of the paper, we assume that K is of characteristic 0. Let $\mathcal{G} = \mathrm{Spf}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$. We define a formal resolution of $\mathcal{G}$ to be a closed immersion $\mathcal{G} \to \Gamma$ of (profinite) formal group schemes over $\mathcal{O}_K$, where $\Gamma = \mathrm{Spf}(\mathbb{B})$ is connected and smooth. Such an immersion can be constructed as follows. By a theorem of Raynaud ([4, Théorème 3.1.1]), we can find an abelian scheme $\mathcal{V}$ over $\mathcal{O}_K$ and a closed immersion of group schemes $\mathcal{G} \to \mathcal{V}$. Taking the formal completion of $\mathcal{V}$ along the zero section, we get a formal resolution of $\mathcal{G}$. We refer the relative dimension of Γ over $\mathcal{O}_K$ as the dimension of a formal resolution $(\mathcal{G} \to \Gamma)$. We define a morphism of formal resolutions to be a pair of group homomorphisms $(f, \mathbf{f})$ which makes the following diagram commutative.

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d, "f"'] & \Gamma \arrow[d, "\mathbf{f}"] \\ \mathcal{G}' \arrow[r] & \Gamma' \end{tikzcd}$$

Note that a formal resolution of $\mathcal{G}$ is also an embedding of B. We say $(f, \mathbf{f})$ is finite flat if this is finite flat in the sense of Section 2. For $j \in \mathbb{Q}_{>0}$, let us consider the j-th tubular neighborhood $X^j(\mathbb{B} \to B)$ of the embedding $(\mathbb{B} \to B)$, which we also write as $X^j(\mathcal{G} \to \Gamma)$, or $X^j_{\mathcal{G}}$ by abuse of notation. Its stable normalized integral model over L and geometric closed fiber are also denoted abusively by $\mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L}$ and $\bar{X}^j_{\mathcal{G}}$. The following lemma enables us to introduce group structures on these varieties.

Lemma 3.1 *Let $(\mathbb{A} \to A)$ and $(\mathbb{B} \to B)$ be embeddings of finite flat $\mathcal{O}_K$-algebras. Put $\mathbb{C} = \mathbb{A}\hat{\otimes}_{\mathcal{O}_K}\mathbb{B}$ and $C = A \otimes_{\mathcal{O}_K} B$. Then the canonical isomorphism*

$$X^j(\mathbb{C} \to C) \to X^j(\mathbb{A} \to A) \times_K X^j(\mathbb{B} \to B)$$

extends to a canonical isomorphism between their stable normalized integral models

$$\mathfrak{X}^j(\mathbb{C} \to C)_{\mathcal{O}_L} \to \mathfrak{X}^j(\mathbb{A} \to A)_{\mathcal{O}_L} \times_{\mathcal{O}_L} \mathfrak{X}^j(\mathbb{B} \to B)_{\mathcal{O}_L}$$

for any finite extension L over K where these three models are defined.

PROOF. We write the affinoid algebras of $X^j(\mathbb{A} \to A)$, $X^j(\mathbb{B} \to B)$ and $X^j(\mathbb{C} \to C)$ as $\mathcal{A}^j_K$, $\mathcal{B}^j_K$ and $\mathcal{C}^j_K$ respectively. Let $\mathring{\mathcal{A}}^j_{\mathcal{O}_L}$ denote the unit disc in $\mathcal{A}^j_L = \mathcal{A}^j_K \hat{\otimes}_K L$ for the supremum norm. Define $\mathring{\mathcal{B}}^j_{\mathcal{O}_L}$ and $\mathring{\mathcal{C}}^j_{\mathcal{O}_L}$ similarly for B and C. From the proof of [6, Theorem 1.3], there exists a continuous surjection $\alpha : L\langle T_1, \ldots, T_{r'}\rangle \to \mathcal{A}^j_L$ such that $||_{\mathrm{sup}} = ||_\alpha$, where $||_\alpha$ is the residue norm induced by α. We also have a surjection $\beta : L\langle U_1, \ldots, U_{s'}\rangle \to \mathcal{B}^j_L$ with the same property for B. Consider the surjection

$$\alpha\hat{\otimes}\beta : L\langle T_1, \ldots, T_{r'}\rangle \hat{\otimes}_L L\langle U_1, \ldots, U_{s'}\rangle \to \mathcal{A}^j_L \hat{\otimes}_L \mathcal{B}^j_L = \mathcal{C}^j_L.$$

The unit disc in $\mathcal{A}^j_L \hat{\otimes}_L \mathcal{B}^j_L$ for the residue norm induced by $\alpha\hat{\otimes}\beta$ is $\mathring{\mathcal{A}}^j_{\mathcal{O}_L} \hat{\otimes}_{\pi_L} \mathring{\mathcal{B}}^j_{\mathcal{O}_L}$, where $\hat{\otimes}_{\pi_L}$ denotes the π_L-adic complete tensor product over $\mathcal{O}_L$. Its geometric closed fiber $(\mathring{\mathcal{A}}^j_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \bar{F}) \otimes_{\bar{F}} (\mathring{\mathcal{B}}^j_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \bar{F})$ is reduced. By [6, Proposition 1.1], we see that the stable normalized integral model $\mathring{\mathcal{C}}^j_{\mathcal{O}_L}$ is equal to $\mathring{\mathcal{A}}^j_{\mathcal{O}_L} \hat{\otimes}_{\pi_L} \mathring{\mathcal{B}}^j_{\mathcal{O}_L}$. $\square$

Corollary 3.2 *The functor of the j-th tubular neighborhood X^j defines a functor from the category of formal resolutions to the category of smooth rigid K-analytic groups. This also induces the two functors below.*

(i) The functor defined by $(\mathcal{G} \to \Gamma) \mapsto \mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L}$, from the full subcategory of the category of formal resolutions which consists of $(\mathcal{G} \to \Gamma)$ such that the stable normalized integral model of $X^j_{\mathcal{G}}$ is defined over L, to the category of smooth π_L-adic formal group schemes over $\mathcal{O}_L$,

(ii) the functor defined by $(\mathcal{G} \to \Gamma) \mapsto \bar{X}^j_{\mathcal{G}}$, from the category of formal resolutions to the category of smooth algebraic groups over $\bar{F}$.

PROOF. By definition, the canonical isomorphisms in the previous lemma are associative. Hence $X^j_{\mathcal{G}}$, $\mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L}$ and $\bar{X}^j_{\mathcal{G}}$ have the group structures induced by that of Γ. The algebraic group $\bar{X}^j_{\mathcal{G}}$ is reduced, hence smooth by [16, Theorem 11.6]. $\square$

Let $\mathcal{G} = \mathrm{Spf}(B)$ be a connected finite flat group scheme over $\mathcal{O}_K$ and $(\mathcal{G} \to \Gamma = \mathrm{Spf}(\mathbb{B}))$ be a formal resolution of dimension r. Consider the quotient $\Gamma/\mathcal{G}$ as a (profinite) formal group scheme ([10]) and set $\Gamma/\mathcal{G} = \mathrm{Spf}(\mathbb{A}')$. We insert here the proof of the following lemma due to the lack of references.

Lemma 3.3 *The ring homomorphism of $\mathcal{O}_K$-algebras $\mathbb{A}' \to \mathbb{B}$ is finite flat. Moreover, there is an isomorphism of $\mathcal{O}_K$-algebras $\mathbb{A} = \mathcal{O}_K[[T_1, \ldots, T_r]] \to \mathbb{A}'$ which maps the ideal $(T_1, \ldots, T_r)$ isomorphically to $J_{\mathbb{A}'} = \mathrm{Ker}(\mathbb{A}' \to \mathcal{O}_K)$.*

PROOF. We know from [10, Théorème 1.4] that $\mathbb{A}'$ is a complete local ring which is a subring of $\mathbb{B}$, that $\mathbb{B}$ is isomorphic to $\prod_{\lambda \in \Lambda} \mathbb{A}'$ for some index set Λ and that we have a natural isomorphism $\mathbb{B} \hat{\otimes}_{\mathbb{A}'} \mathbb{B} \to B \otimes_{\mathcal{O}_K} \mathbb{B}$. Hence Λ is a finite set and $\mathbb{B}$ is finite faithfully flat over $\mathbb{A}'$. By a theorem of Eakin-Nagata ([13, Theorem 3.7]), we see that $\mathbb{A}'$ is Noetherian. From [13, Theorem 23.7], we see that $\mathbb{A}' \otimes_{\mathcal{O}_K} F$ is a regular local ring of dimension r. Since the residue field of $\mathbb{A}'$ is F, we have an isomorphism $F[[T_1, \ldots, T_r]] \to \mathbb{A}' \otimes_{\mathcal{O}_K} F$ which maps the ideal $(T_1, \ldots, T_r)$ onto the image of $J_{\mathbb{A}'}$. Lifting this map to $\mathcal{O}_K[[T_1, \ldots, T_r]] \to \mathbb{A}'$ and using Nakayama's lemma conclude the proof. $\square$

From this lemma, we can regard the zero section $\mathrm{Spf}(\mathcal{O}_K) \to \mathrm{Spf}(\mathbb{A}')$ as a formal resolution of the trivial group. We choose once and for all an isomorphism $\mathbb{A} = \mathcal{O}_K[[T_1, \ldots, T_r]] \to \mathbb{A}'$ as in the lemma. The $\mathcal{O}_K$-algebra $\mathbb{A}$ has the formal group law induced by this isomorphism. By definition of $\mathbb{A}$ and $\mathbb{A}'$, we have a finite flat map of formal resolutions

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathrm{Spf}(\mathbb{B}) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{Spf}(\mathcal{O}_K) & \longrightarrow & \mathrm{Spf}(\mathbb{A}). \end{array}$$

Thus we have a finite flat map of rigid K-analytic groups

$$f^j : X^j_{\mathcal{G}} = X^j(\mathcal{G} \to \Gamma) \to X^j(\mathrm{Spf}(\mathcal{O}_K) \to \mathrm{Spf}(\mathbb{A})) = D^{r,j},$$

where $D^{r,j} = \{(z_1, \ldots, z_r) \in \mathcal{O}^r_{\bar{K}} \mid v_K(z_i) \geq j \text{ for any } i\}$. We call this the affinoid homomorphism associated to a formal resolution $(\mathcal{G} \to \Gamma)$. Write $\mathcal{B}^j_K$ and $\mathcal{A}^j_K$ for the K-affinoid algebras of $X^j_{\mathcal{G}}$ and $D^{r,j}$ respectively. The stable normalized integral model over L of $D^{r,j}$ is denoted by $\mathfrak{D}^{r,j}_{\mathcal{O}_L}$ and its geometric closed fiber by $\bar{D}^{r,j}$. From Corollary 3.2, we also have a homomorphism

$$\mathring{f}^j : \mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathfrak{D}^{r,j}_{\mathcal{O}_L}$$

of smooth π_L-adic formal group schemes over $\mathcal{O}_L$ and a homomorphism

$$\bar{f}^j : \bar{X}^j_{\mathcal{G}} \to \bar{D}^{r,j}$$

of smooth algebraic groups over $\bar{F}$. We also refer them as the associated homomorphisms to $(\mathcal{G} \to \Gamma)$.

Lemma 3.4 *The affinoid homomorphism $f^j : X^j_{\mathcal{G}} \to D^{r,j}$ is etale for any $j > 0$. Moreover, for $j > c(\mathcal{G})$, there exists a finite extension K'/K such that $X^j_{\mathcal{G},K'}$ is isomorphic to the disjoint sum of finitely many copies of $D^{r,j}_{K'}$.*

PROOF. We have $\Omega^1_{\mathcal{B}^j_K/\mathcal{A}^j_K} = \mathcal{B}^j_K \hat{\otimes}_{\mathbb{B}} \hat{\Omega}_{\mathbb{B}/\mathbb{A}}$. It is enough to show that $\hat{\Omega}_{\mathbb{B}/\mathbb{A}}$ is a torsion $\mathcal{O}_K$-module. Let $J_{\mathbb{A}}$ and $J_{\mathbb{B}}$ be the augmentation ideals of $\mathbb{A}$ and $\mathbb{B}$ respectively. Set $I = \mathrm{Ker}(\mathbb{B} \to B)$. Then

$$\hat{\Omega}_{\mathbb{B}/\mathbb{A}} = \mathrm{Coker}(\mathbb{B} \otimes_{\mathbb{A}} \hat{\Omega}_{\mathbb{A}/\mathcal{O}_K} \to \hat{\Omega}_{\mathbb{B}/\mathcal{O}_K})$$

is equal to

$$\mathbb{B} \otimes_{\mathcal{O}_K} \mathrm{Coker}(\mathrm{Cot}(\mathbb{A}) \to \mathrm{Cot}(\mathbb{B})) = \mathbb{B} \otimes_{\mathcal{O}_K} J_{\mathbb{B}}/(I + J^2_{\mathbb{B}}) = \mathbb{B} \otimes_{\mathcal{O}_K} \mathrm{Cot}(B),$$

where Cot means the cotangent space. This shows the first assertion. For the second assertion, take a finite extension K' of K where the geometric connected components of $X^j_{\mathcal{G}}$ are defined. For $j > c(\mathcal{G})$, we have $\deg(f^j) = \sharp\mathcal{G}(\bar{K}) = \sharp\pi_0(X^j_{\mathcal{G},K'})$ and each of the connected components of $X^j_{\mathcal{G},K'}$ is a finite etale cover of $D^{r,j}_{K'}$ whose degree is one. Thus this is isomorphic to $D^{r,j}_{K'}$. $\square$

Take a finite extension L of K where the stable normalized integral models of $X^j_{\mathcal{G}}$ and $D^{r,j}$ are defined. The generic fiber $\mathcal{G}_L = \mathcal{G} \times_{\mathcal{O}_K} L$ can be regarded as a rigid L-analytic subgroup of $X^j_{\mathcal{G},L}$ and we have an exact sequence of rigid K-analytic groups

$$0 \to \mathcal{G}_L \to X^j_{\mathcal{G},L} \to D^{r,j}_L \to 0.$$

Let $\mathcal{H}^j_{\mathcal{O}_L}$ be the kernel of the associated homomorphism $\mathring{f}^j : \mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathfrak{D}^{r,j}_{\mathcal{O}_L}$ as a π_L-adic formal group scheme over $\mathcal{O}_L$.

Lemma 3.5 *The associated homomorphism $\mathring{f}^j : \mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathfrak{D}^{r,j}_{\mathcal{O}_L}$ is finite flat. Thus, the π_L-adic formal group scheme $\mathcal{H}^j_{\mathcal{O}_L}$ can be regarded as a finite flat group scheme over $\mathcal{O}_L$ and there exists an exact sequence of π_L-adic formal group schemes*

$$0 \to \mathcal{H}^j_{\mathcal{O}_L} \to \mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathfrak{D}^{r,j}_{\mathcal{O}_L} \to 0. \tag{1}$$

PROOF. From Lemma 3.4, the associated affinoid map $f^j : X^j_{\mathcal{G}} \to D^{r,j}$ is finite etale. Let $\mathcal{B}^j_K$ and $\mathcal{A}^j_K$ be their affinoid algebras as above. Let $\mathring{\mathcal{B}}^j_{\mathcal{O}_L}$ and $\mathring{\mathcal{A}}^j_{\mathcal{O}_L}$ denote their unit disc for the supremum norms. Since $D^{r,j}$ is integral, we see that f^j is surjective and the ring homomorphism $\mathcal{A}^j_K \to \mathcal{B}^j_K$ is injective. Thus we have an injection $\mathring{\mathcal{A}}^j_{\mathcal{O}_L} \to \mathring{\mathcal{B}}^j_{\mathcal{O}_L}$, which is finite by [5, Corollary 6.4.1/6]. Hence $\bar{f}^j : \bar{X}^j_{\mathcal{G}} \to \bar{D}^{r,j}$ is a surjective homomorphism of algebraic groups over $\bar{F}$. Since $\bar{X}^j_{\mathcal{G}}$ and $\bar{D}^{r,j}$ are regular, we see that $\bar{f}^j$ is faithfully flat by [13, Theorem 23.1]. Since $\mathring{\mathcal{A}}^j_{\mathcal{O}_L}$ and $\mathring{\mathcal{B}}^j_{\mathcal{O}_L}$ is π_L-torsion free, the map $\mathring{f}^j$ is flat by the local criterion of flatness. This concludes the lemma. $\square$

From Lemma 3.4, we see that for $j > c = c(\mathcal{G})$, the map $\mathring{f}^j$ identifies $\mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L}$ with the direct sum of finitely many copies of $\mathfrak{D}^{r,j}_{\mathcal{O}_L}$. More precisely, we have the following.

Lemma 3.6 *Let $c = c(\mathcal{G})$ be the conductor of $\mathcal{G}$. Then the associated homomorphism $\mathring{f}^j : \mathfrak{X}^j_{\mathcal{G},\mathcal{O}_L} \to \mathfrak{D}^{r,j}_{\mathcal{O}_L}$ is finite etale if and only if $j \geq c$.*

PROOF. Let $\bar{0}$ be the zero section of $\bar{D}^{r,j}$. Set $X^{j+}_{\mathcal{G}} = \cup_{j'>j} X^{j'}_{\mathcal{G}}$. From [3, Lemma 1.12], we have

$$\sharp\mathcal{G}(\bar{K}) = \deg(\bar{f}^j) \geq \sharp(\bar{f}^j)^{-1}(\bar{0}) = \sharp\pi_0(X^{j+}_{\mathcal{G},\bar{K}}).$$

This shows that $j \geq c$ if and only if $\bar{f}^j$ is etale. This is also equivalent to the etaleness of $\mathring{f}^j$. $\square$

4 Ramification and the I_K-module structure of a finite flat group scheme

Consider the canonical $\bar{K}$-semilinear left action of G_K on $\bar{X}^j_{\mathcal{G}}$. When we restrict this to I_K, the action is $\bar{F}$-linear. We call this the geometric monodromy action of I_K and write the action of $\sigma \in I_K$ as σ_{geom} (note that here we follow the terminology in [2], and not in [3] where the monodromy action is called arithmetic, since in our case there is no "geometric" action other than the monodromy action). Similarly, we have the geometric monodromy action of I_K on $\bar{D}^{r,j}$.

The latter action is described as follows. Let the additive group (*resp.* multiplicative group) over $\bar{F}$ be denoted by $\bar{\mathbb{G}}_{\mathrm{a}}$ (*resp.* $\bar{\mathbb{G}}_{\mathrm{m}}$). Consider the left action $\bar{\mathbb{G}}_{\mathrm{m}} \times \bar{\mathbb{G}}_{\mathrm{a}}^r \to \bar{\mathbb{G}}_{\mathrm{a}}^r$ given by the multiplication. Write this action of $\lambda \in \bar{F}^\times$ as $[\lambda]$. This action is defined by $T_i \mapsto \lambda T_i$, where $\bar{\mathbb{G}}_{\mathrm{a}}^r = \mathrm{Spec}(\bar{F}[T_1, \dots, T_r])$. For $j \in \mathbb{Q}_{>0}$, we define a tame character θ_j to be $\theta_{l'}^{k'}$, where k'/l' is the prime-to-p-denominator part of j mod $\mathbb{Z}$ ([15]). In other words, we set

$$\theta_j(\sigma) = (\sigma(\pi^{1/l'})/\pi^{1/l'})^{k'} \bmod \mathfrak{m}_{\bar{K}} \in \bar{F}.$$

Note that, for $j = k/l$ and $l = p^m l_0$ with $(k, l) = 1$ and $p \nmid l_0$, we have $\theta_j = \theta_{l_0}^{kp^{-m}}$. We call any of $\mathbb{F}_p$-conjugates of θ_j the fundamental character of level j.

Lemma 4.1 *There is an isomorphism $\bar{D}^{r,j} \to \bar{\mathbb{G}}_{\mathrm{a}}^r$ of algebraic groups over $\bar{F}$ such that the geometric monodromy action σ_{geom} on $\bar{D}^{r,j}$ for any $\sigma \in I_K$ corresponds by this isomorphism to the multiplication $[\theta_j(\sigma)]$ on $\bar{\mathbb{G}}_{\mathrm{a}}^r$.*

PROOF. Recall that $D^{r,j} = X^j(\mathbb{A} \to \mathcal{O}_K)$, where $\mathbb{A} = \mathcal{O}_K[[T_1, \dots, T_r]]$. Put $j = k/l$. Let L be a finite Galois extension of K containing $\pi^{1/l}$ and $e' = e(L/K)$ be its ramification index over K. Then $e'k/l \in \mathbb{Z}$ and the affine ring of the stable normalized integral model of $D^{r,j}$ over L is

$$\mathring{\mathcal{A}}^j_{\mathcal{O}_L} = \mathcal{O}_L\langle T_1/(\pi_L)^{e'k/l}, \dots, T_r/(\pi_L)^{e'k/l}\rangle.$$

Define an $\mathcal{O}_L$-algebra isomorphism $\mathcal{O}_L\langle W_1, \dots, W_r\rangle \to \mathring{\mathcal{A}}^j_{\mathcal{O}_L}$ by $W_i \mapsto T_i/(\pi^{1/l})^k$. Set $\mu_{\mathbb{A}}$ to be the coproduct of $\mathbb{A}$ and μ to be the coproduct of the algebra $\mathcal{O}_L\langle W_1, \dots, W_r\rangle$ induced by $\mu_{\mathbb{A}}$. We have

$$\mu_{\mathbb{A}}(T_i) = T_i \hat{\otimes} 1 + 1 \hat{\otimes} T_i + \text{(higher degree)}$$

and then $\mu(W_i)$ is equal to

$$W_i \hat{\otimes}_\pi 1 + 1 \hat{\otimes}_\pi W_i + (\pi^{1/l})^k \text{(higher degree)}.$$

This shows the first assertion. In the affine algebra of $\bar{D}^{r,j}$, we have

$$\sigma^*_{\mathrm{geom}}(T_i/(\pi^{1/l})^k) = T_i/\sigma^{-1}((\pi^{1/l})^k) = \theta_j(\sigma)T_i/(\pi^{1/l})^k.$$

This corresponds to the action $[\theta_j(\sigma)]$ on $\bar{\mathbb{G}}^r_{\mathrm{a}} = \mathrm{Spec}(\bar{F}[W_1, \dots, W_r])$. $\square$

Next we consider the geometric monodromy action on $\bar{X}^j_{\mathcal{G}}$ for $j \in \mathbb{Q}_{>0}$. Let $\bar{X}^{j,0}_{\mathcal{G}}$ denote the unit component of the algebraic group $\bar{X}^j_{\mathcal{G}}$ and $\bar{\mathcal{H}}^j$ be the geometric closed fiber of $\mathcal{H}^j_{\mathcal{O}_L}$. We begin with the following lemma.

Lemma 4.2 *If $\psi \in \mathrm{End}(\bar{X}^{j,0}_{\mathcal{G}})$ induces the zero map on $\bar{D}^{r,j}$, then $\psi = 0$.*

PROOF. Put $\bar{\mathcal{H}}^j_0 = \bar{\mathcal{H}}^j \cap \bar{X}^{j,0}_{\mathcal{G}}$. This is the kernel of the faithfully flat map $\bar{X}^{j,0}_{\mathcal{G}} \to \bar{D}^{r,j}$ and by assumption we have the following commutative diagram whose rows are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bar{\mathcal{H}}^j_0 & \longrightarrow & \bar{X}^{j,0}_{\mathcal{G}} & \longrightarrow & \bar{D}^{r,j} & \longrightarrow & 0 \\
 & & \big\downarrow & & \psi\big\downarrow & & 0\big\downarrow & & \\
0 & \longrightarrow & \bar{\mathcal{H}}^j_0 & \longrightarrow & \bar{X}^{j,0}_{\mathcal{G}} & \longrightarrow & \bar{D}^{r,j} & \longrightarrow & 0
\end{array}$$

Thus ψ factors through $\bar{\mathcal{H}}^j_0$. Put $\bar{C} = \mathrm{Im}(\psi)$. Then this is a closed subgroup scheme of $\bar{\mathcal{H}}^j_0$ and the map $\bar{X}^{j,0}_{\mathcal{G}} \to \bar{C}$ is faithfully flat. Since $\bar{X}^{j,0}_{\mathcal{G}}$ is regular and connected, we see that $\bar{C}$ is also regular and connected by [13, Theorem 23.7]. Hence $\bar{C} = 0$ and we have $\psi = 0$. $\square$

Corollary 4.3 *Let $\mathcal{G}$ be a connected finite flat group scheme over $\mathcal{O}_K$. Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension r. Then the algebraic group $\bar{X}^{j,0}_{\mathcal{G}}$ is isomorphic to $\bar{\mathbb{G}}^r_{\mathrm{a}}$.*

PROOF. By the previous lemma and Lemma 4.1, we see that $\bar{X}^{j,0}_{\mathcal{G}}$ is killed by p. Hence the assertion follows from [12, Lemma 1.7.1]. $\square$

Corollary 4.4 *The geometric monodromy action of I_K on $\bar{X}^{j,0}_{\mathcal{G}}$ is tame.*

PROOF. For an element σ of the wild inertia subgroup P_K, the geometric monodromy action σ_{geom} on $\bar{D}^{r,j}$ is trivial. Applying the lemma to $\sigma_{\mathrm{geom}} - \mathrm{id} \in \mathrm{End}(\bar{X}^{j,0}_{\mathcal{G}})$ shows the assertion. $\square$

Corollary 4.5 *Let J be a finite cyclic quotient of I_K through which the tame character θ_j factors and $\tau \in J$. Let $F(t)$ denote the minimal polynomial of*

$\theta_j(\tau) \in \bar{\mathbb{F}}_p$ *over* $\mathbb{F}_p$. *Then the geometric monodromy action of* I_K *on* $\bar{X}_{\mathcal{G}}^{j,0}$ *also factors through* J *and the equation* $F(\tau_{\mathrm{geom}}) = 0$ *holds in* $\mathrm{End}(\bar{X}_{\mathcal{G}}^{j,0})$.

PROOF. The first assertion follows from Lemma 4.2 as the previous corollary. The second assertion also follows from this lemma using Lemma 4.1. □

Let $c = c(\mathcal{G})$ be the conductor of $\mathcal{G}$. The lemma below enables us to realize $\mathcal{G}^c(\bar{K})$ as a subgroup of $\bar{X}_{\mathcal{G}}^{c,0}$.

Lemma 4.6 *The specialization map* $\mathrm{sp}_c : X_{\mathcal{G}}^c \to \bar{X}_{\mathcal{G}}^c$ *induces an* I_K*-equivariant isomorphism* $\mathcal{G}(\bar{K}) \to \bar{\mathcal{H}}^c(\bar{F})$ *and* $\mathcal{G}^c(\bar{K}) \to \bar{\mathcal{H}}_0^c(\bar{F})$. *Here we consider on the left-hand side the natural action as the* $\bar{K}$*-valued points of* $\mathcal{G}$ *(resp.* $\mathcal{G}^c$*) and on the right-hand side the restriction of the geometric monodromy action on* $\bar{X}_{\mathcal{G}}^c$.

PROOF. By definition, the generic fiber of $\mathcal{H}_{\mathcal{O}_L}^c$ is equal to $\mathcal{G}_L$. From the exact sequence (1) and Lemma 3.6, we know that $\mathcal{H}_{\mathcal{O}_L}^c$ is etale over $\mathcal{O}_L$ and there is the following exact sequence of algebraic groups over $\bar{F}$.

$$0 \to \bar{\mathcal{H}}^c \to \bar{X}_{\mathcal{G}}^c \to \bar{D}^{r,j} \to 0 \tag{2}$$

Thus we have a natural isomorphism $\mathcal{H}_{\mathcal{O}_L}^c(\bar{K}) \to \bar{\mathcal{H}}^c(\bar{F})$ and the composite

$$\mathcal{G}(\bar{K}) = \mathcal{H}_{\mathcal{O}_L}^c(\bar{K}) \to \bar{\mathcal{H}}^c(\bar{F}) \to \bar{X}_{\mathcal{G}}^c(\bar{F})$$

coincides with the map sp_c. From the isomorphism $\pi_0(\bar{X}_{\mathcal{G}}^c) \to \pi_0(X_{\mathcal{G},\bar{K}}^c)$, we see that this map sends $\mathcal{G}^c(\bar{K})$ isomorphically onto $\bar{\mathcal{H}}_0^c(\bar{F})$. For $x \in X_{\mathcal{G}}^c(\bar{K})$ and $\sigma \in I_K$, let $\sigma(x)$ denote the natural action of σ on $\bar{K}$-valued points. Then we have $\sigma_{\mathrm{geom}}(x) \circ \sigma = \sigma(x)$. Taking its specialization shows the I_K-equivariance. □

The following theorem can be regarded as a generalization for a finite flat group scheme over $\mathcal{O}_K$ of the structure theorem of the graded pieces of the classical upper numbering ramification filtration.

Theorem 4.7 *Let* $\mathcal{G}$ *be a finite flat group scheme over* $\mathcal{O}_K$ *and* $j \in \mathbb{Q}_{>0}$. *Then the* G_K*-module* $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ *is tame and killed by* p.

PROOF. Since $\mathcal{G}^j = (\mathcal{G}^0)^j$, where $\mathcal{G}^0$ denotes the unit component of $\mathcal{G}$, we may assume $\mathcal{G}$ is connected. From Lemma 2.2, we may also assume $j = c = c(\mathcal{G})$. Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension r and consider its associated affinoid homomorphism $X_{\mathcal{G}}^c \to D^{r,c}$. Then the theorem follows from Corollary 4.3, Corollary 4.4, and Lemma 4.6. □

From this theorem, we see that the inertia subgroup I_K acts on $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})\otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ by the direct sum of tame characters. The theorem below determines these characters up to p-power exponent.

Theorem 4.8 *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ and $j \in \mathbb{Q}_{>0}$. Then I_K acts on $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ by the direct sum of fundamental characters of level j.*

PROOF. The same argument as in the proof of Theorem 4.7 reduces the claim to the case where $\mathcal{G}$ is connected and $j = c = c(\mathcal{G})$. Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension r. Let J be as in Corollary 4.5 and τ be a generator of J. Then Corollary 4.5 and Lemma 4.6 show that every eigenvalue of the action of τ_{geom} on the finite dimensional $\mathbb{F}_p$-vector space $\mathcal{G}^c(\bar{K})$ is a conjugate of $\theta_j(\tau)$ over $\mathbb{F}_p$. Since the order of J is prime to p, we conclude that I_K acts on $\mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ by the direct sum of $\mathbb{F}_p$-conjugates of θ_c. $\square$

Corollary 4.9 *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$. Then the order of the image of the homomorphism $I_K \to \mathrm{Aut}(\mathcal{G}(\bar{K}))$ is a power of p if and only if every jump j of the ramification filtration $\{\mathcal{G}^j\}_{j\in\mathbb{Q}_{>0}}$ is an element of $\mathbb{Z}[1/p]$.*

PROOF. From Theorem 4.8, we see that the jumps are in $\mathbb{Z}[1/p]$ if and only if the graded pieces $\mathcal{G}^j(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ are unramified. By Theorem 4.7, this is equivalent to the condition that $\sharp\mathrm{Im}(I_K \to \mathrm{Aut}(\mathcal{G}(\bar{K})))$ is a p-power. $\square$

Corollary 1.2 is the special case of the following result.

Corollary 4.10 *Let M be an extension of a complete discrete valuation field over K with p-power relative ramification index $e(M/K)$ and L be an abelian extension of M. Suppose that there exists a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ of p-power order such that $\mathcal{O}_L$ is a $\mathcal{G}$-torsor over $\mathcal{O}_M$. Then the denominator of every jump of the upper numbering ramification filtration $\{\mathrm{Gal}(L/M)^j\}_{j\in\mathbb{Q}_{>0}}$ is a power of p.*

PROOF. We have a natural isomorphism

$$\mathcal{G} \times_{\mathcal{O}_K} \mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_L) \times_{\mathcal{O}_M} \mathrm{Spec}(\mathcal{O}_L).$$

From [1, Lemme 2.1.5] and the assumption on $e(M/K)$, it is sufficient to prove that the jumps of $\{\mathcal{G}^j\}_{j\in\mathbb{Q}_{>0}}$ are contained in $\mathbb{Z}[1/p]$. On the other hand, we also have a G_L-equivariant isomorphism

$$\mathrm{Hom}_{M\text{-alg.}}(L, \bar{L}) = \mathrm{Hom}_{L\text{-alg.}}(L \otimes_M L, \bar{L}) \to \mathcal{G}(\bar{L}) = \mathcal{G}(\bar{K}).$$

From this, we see that $\mathrm{Gal}(\bar{K}/L \cap \bar{K})$ acts trivially on $\mathcal{G}(\bar{K})$. Let N be the finite Galois extension of K which corresponds to the kernel of the map $G_K \to \mathrm{Aut}(\mathcal{G}(\bar{K}))$. Since $[L : M]$ is a p-power, $[N : M \cap N]$ is a p-power. From the assumption on $e(M/K)$, we see that $[(M \cap N)^{\mathrm{nr}} : K^{\mathrm{nr}}]$ is also a p-power. Hence the image of $I_K \to \mathrm{Aut}(\mathcal{G}(\bar{K}))$ is a p-group and Corollary 4.9 concludes the proof. $\square$

Example 4.11 *Let X and Y be proper smooth schemes over $\mathcal{O}_K$ with geometrically connected fibers. Suppose that X is a $\mathcal{G}$-torsor over Y with some $\mathcal{G}$ as in Corollary 4.10. Set $L = \mathrm{Frac}(\hat{\mathcal{O}}_{X,s})$ and $M = \mathrm{Frac}(\hat{\mathcal{O}}_{Y,s'})$, where s and s' denote the generic points of the closed fibers of X and Y. Then L and M satisfy the assumption of the corollary.*

When $\mathcal{G}(\bar{K})$ is unramified and killed by p, we have the following reinforcement of Corollary 4.9, which is an easy corollary of Proposition 2.1. The author does not know if every jump is an integer whenever $\mathcal{G}(\bar{K})$ is unramified. If $\mathcal{G}$ is monogenic, then we see that this holds true from [11, Theorem 4].

Proposition 4.12 *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ which is killed by p. Suppose that the G_K-module $\mathcal{G}(\bar{K})$ is unramified. Then every jump j of the ramification filtration $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$ is an element of $p\mathbb{Z}$.*

PROOF. We may assume $K = K^{\mathrm{nr}}$ and G_K acts trivially on $\mathcal{G}(\bar{K})$. There is a quotient W of $\mathcal{G}(\bar{K})/\mathcal{G}^{j+}(\bar{K})$ where $\mathcal{G}^j(\bar{K})$ has a non-trivial image and of rank one over $\mathbb{F}_p$. Taking the schematic closure of $\mathrm{Ker}(\mathcal{G}(\bar{K})/\mathcal{G}^{j+}(\bar{K}) \to W)$ in $\mathcal{G}/\mathcal{G}^{j+}$, we see that W extends to a finite flat group scheme $\mathcal{W}$ over $\mathcal{O}_K$ which is a quotient of $\mathcal{G}/\mathcal{G}^{j+}$. By Proposition 2.1, we see that the ramification filtration of $\mathcal{W}$ jumps at j. On the other hand, $\mathcal{W}$ is a Raynaud $\mathbb{F}_p$-vector space scheme with unramified generic fiber. Thus the proposition follows from [11, Theorem 4]. $\square$

For the rest of this section, we state some corollaries in the case where $\mathcal{G}$ is an $\mathbb{F}$-vector space scheme of rank one or two for a finite extension $\mathbb{F}$ over $\mathbb{F}_p$. In the case of rank one, Theorem 4.8 directly shows the claim below, while this can be shown also as a corollary of a determination of the conductor of a Raynaud $\mathbb{F}$-vector space scheme (Theorem 5.5).

Corollary 4.13 *Let $\mathcal{G}$ be an $\mathbb{F}$-vector space scheme of rank one over $\mathcal{O}_K$ and $c = c(\mathcal{G})$. Then the I_K-action on the $\mathbb{F}$-vector space $\mathcal{G}(\bar{K})$ of rank one is given by the fundamental character of level c.*

In the case of rank two, we have the following.

Corollary 4.14 *Let $\mathcal{G}$ be a finite flat $\mathbb{F}$-vector space scheme of rank two over $\mathcal{O}_K$ and $c = c(\mathcal{G})$. Then the I_K-module $\mathcal{G}(\bar{K}) \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p$ contains the fundamental character of level c. If the G_K-module $\mathcal{G}(\bar{K})$ is reducible, this holds true for $\mathcal{G}(\bar{K})$ itself.*

PROOF. The first assertion follows easily from Theorem 4.8 and the surjection $\mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p \to \mathcal{G}^c(\bar{K}) \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p$. Suppose the I_K-module $\mathcal{G}(\bar{K})$ is reducible. When $\mathcal{G}^c$ is of rank one, the assertion is clear from Theorem 4.8. If $\mathcal{G}^c = \mathcal{G}$, then $\mathcal{G}^c$ is reducible and the assertion follows also from Theorem 4.8. □

Unlike in the case of tame characters, the values of jumps of $\{\mathcal{G}^j\}_{j \in \mathbb{Q}_{>0}}$ usually carries no information about the extension structure of the I_K-module $\mathcal{G}(\bar{K})$, except in some extreme cases such as the corollary below.

Corollary 4.15 *Consider an exact sequence of finite flat $\mathbb{F}$-vector space schemes over $\mathcal{O}_K$*

$$0 \to \mathcal{G}_1 \to \mathcal{G} \to \mathcal{G}_2 \to 0$$

where $\mathcal{G}_1$ and $\mathcal{G}_2$ are connected of rank one. If $c(\mathcal{G}) = c(\mathcal{G}_2)$, then the I_K-module $\mathcal{G}(\bar{K})$ splits.

PROOF. Put $c = c(\mathcal{G})$. Take a formal resolution $(\mathcal{G} \to \Gamma)$ of dimension r and put $\Gamma_2 = \Gamma/\mathcal{G}_1$. Then we get a finite flat map of formal resolutions

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \mathcal{G}_2 & \longrightarrow & \Gamma_2. \end{array}$$

Therefore we have a finite flat homomorphism of rigid K-analytic groups $X^j_{\mathcal{G}} \to X^j_{\mathcal{G}_2}$ by Corollary 3.2. As in the proof of Lemma 3.4, we see that this map is finite etale.

Suppose $\mathcal{G}^c(\bar{K})$ is of rank one. If $\mathcal{G}^c(\bar{K}) \neq \mathcal{G}_1(\bar{K})$ as an $\mathbb{F}$-subspace of $\mathcal{G}(\bar{K})$, the I_K-module $\mathcal{G}(\bar{K})$ splits and the proposition follows. Suppose $\mathcal{G}^c(\bar{K}) = \mathcal{G}_1(\bar{K})$. The affinoid variety $X^c_{\mathcal{G}}$ decomposes to $\sharp\mathbb{F}$ components over some finite extension K' of K. Each component is a Zariski open and closed subset of $X^c_{\mathcal{G},K'}$. As the map $f : X^c_{\mathcal{G},K'} \to X^c_{\mathcal{G},K'}$ is finite etale and $X^c_{\mathcal{G}_2,K'}$ is connected, every component $X^{c,i}_{\mathcal{G},K'}$ maps surjectively to $X^c_{\mathcal{G}_2,K'}$. Take some $g_i \in \mathcal{G}(\bar{K}) \cap X^{c,i}_{\mathcal{G},K'}$. Using the group structure, we see that

$$\mathcal{G}(\bar{K}) \cap X^{c,i}_{\mathcal{G},K'} = g_i + \mathcal{G}^c(\bar{K}) = g_i + \mathcal{G}_1(\bar{K})$$

and $f(\mathcal{G}(\bar{K})\cap X^{c,i}_{\mathcal{G},K'}) = f(g_i)$. However, we have $f^{-1}(\mathcal{G}_2(\bar{K})) = \mathcal{G}(\bar{K})$ and thus $f(\mathcal{G}(\bar{K}) \cap X^{c,i}_{\mathcal{G},K'}) = \mathcal{G}_2(\bar{K})$. This is a contradiction. Therefore we may assume $\mathcal{G}^c(\bar{K}) = \mathcal{G}(\bar{K})$. In this case, the proposition follows from Theorem 4.7. $\square$

Remark 4.16 *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ and $\mathcal{G}^0$ be its unit component. Consider the canonical log structure $\mathbb{N} \to \mathcal{O}_K$ defined by $1 \mapsto \pi$ and a log structure on $\mathcal{G}$ over $\mathcal{O}_K$. From* [3, *Lemma 4.2*]*, we see that if the log structure of $\mathcal{G}$ is trivial on the generic fiber and $\mathcal{G}$ is log flat over $\mathcal{O}_K$, then $\mathcal{G}^0$ is strict over $\mathcal{O}_K$. Thus it makes no difference in the description of the tame characters whether we use the non-logarithmic or logarithmic ramification theory of* [2, 3]*.*

5 Example: rank one calculation

In this section, we calculate the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$. The point is that, as we can see from the bound of the conductor ([11, Theorem 7]), it is enough to consider the j-th tubular neighborhood only for $j \leq pe/(p-1)+\varepsilon$ with sufficiently small $\varepsilon > 0$. For such j, we can compute the tubular neighborhood easily by Lemma 5.4 below.

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$. We write e for its absolute ramification index. For $a \in \bar{K}$ and $j \in \mathbb{Q}_{>0}$, let $D(a, j)$ denote the closed disc $\{z \in \mathcal{O}_{\bar{K}} \mid v_K(z-a) \geq j\}$. This is the underlying subset of a $K(a)$-affinoid subdomain of the unit disc over $K(a)$.

For integers $0 \leq s_1, \ldots, s_r \leq e$, let $\mathcal{G}(s_1, \ldots, s_r)$ denote the Raynaud $\mathbb{F}_{p^r}$-vector space scheme ([14]) over $\mathcal{O}_K$ defined by the r equations

$$T_1^p = \pi^{s_1} T_2, T_2^p = \pi^{s_2} T_3, \ldots, T_r^p = \pi^{s_r} T_1.$$

We set $j_k = (ps_k + p^2 s_{k-1} + \ldots + p^k s_1 + p^{k+1} s_r + p^{k+2} s_{r-1} + \cdots + p^r s_{k+1})/(p^r - 1)$. Before the calculation of $c(\mathcal{G}(s_1, \ldots, s_r))$, we gather some elementary lemmas.

Lemma 5.1 *Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. Then the affinoid variety $X^j(\bar{K}) = \{x \in \mathcal{O}_{\bar{K}} \mid v_K(x^p - a) \geq j\}$ is equal to*

$$\begin{cases} D(a^{1/p}, j/p) & \text{if } j \leq s + pe/(p-1), \\ \coprod_{i=0}^{p-1} D(a^{1/p}\zeta_p^i, j - e - (p-1)s/p) & \text{if } j > s + pe/(p-1). \end{cases}$$

PROOF. We have $v_K(x^p - a) = \sum_i v_K(x - a^{1/p}\zeta_p^i)$. If $v_K(x - a^{1/p}\zeta_p^i) \geq$

$v_K(x - a^{1/p}\zeta_p^{i'})$ for any $i' \neq i$, then

$$v_K(x - a^{1/p}\zeta_p^{i'}) \leq v_K(a^{1/p}\zeta_p^{i'}(1 - \zeta_p^{i-i'})) = s/p + e/(p-1).$$

Thus we have $v_K(x - a^{1/p}\zeta_p^i) \geq \sup(j/p, j - (p-1)s/p - e)$ and

$$X^j(\bar{K}) \subseteq \bigcup_i D(a^{1/p}\zeta_p^i, \sup(j/p, j - (p-1)s/p - e)).$$

Suppose that $j/p \geq j - (p-1)s/p - e$. Then we have

$$v_K(a^{1/p}(1 - \zeta_p^i)) = s/p + e/(p-1) \geq j/p$$

for any i and thus

$$X^j(\bar{K}) = D(a^{1/p}, j/p).$$

When $j/p < j - (p-1)s/p - e$, we have

$$v_K(a^{1/p}(1 - \zeta_p^i)) = s/p + e/(p-1) < j - (p-1)s/p - e.$$

This shows that the discs $D(a^{1/p}\zeta_p^i, j - (p-1)s/p - e)$ are disjoint and

$$X^j(\bar{K}) = \coprod_i D(a^{1/p}\zeta_p^i, j - (p-1)s/p - e).$$

These are equalities of the underlying sets of affinoid subdomains of the unit disc over $K(a^{1/p}, \zeta_p)$. By the universality of an affinoid subdomain, this extends to an isomorphism of affinoid varieties. $\square$

We can prove the following lemma just in the same way.

Lemma 5.2 *The affinoid variety* $\{x \in \mathcal{O}_{\bar{K}} \mid v_K(x^{p^r} - ax) \geq j\}$ *is equal to*

$$\begin{cases} D(0, j/p^r) & \text{if } j \leq p^r v(a)/(p^r - 1), \\ \coprod_{i=0}^{p^r-1} D(\sigma_i, j - v(a)) & \text{if } j > p^r v(a)/(p^r - 1), \end{cases}$$

where σ_i*'s are the roots of* $X^{p^r} = aX$.

Lemma 5.3 *For* $g_1(Y_1, \ldots, Y_d)$, $g_2(Y_1, \ldots, Y_d) \in K[Y_1, \ldots, Y_d]$ *and* $j_1 \geq j_2$, *we have an equality of affinoid varieties*

$$\begin{aligned} &\{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}}^d \mid v_K(x - g_1(y)) \geq j_1, v_K(x - g_2(y)) \geq j_2\} \\ = &\{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}}^d \mid v_K(x - g_1(y)) \geq j_1, v_K(g_1(y) - g_2(y)) \geq j_2\}. \end{aligned}$$

PROOF. For fixed (x, y), these two conditions are equivalent. The universality of an affinoid subdomain proves the lemma. $\square$

Lemma 5.4 *Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. If $j \leq pe/(p-1) + s$, then the affinoid variety $X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_{\bar{K}}^2 \mid v_K(x^p - ay^{p^n}) \geq j\}$ is equal to $\{(x, y) \in \mathcal{O}_{\bar{K}}^2 \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\}$.*

PROOF. Lemma 5.1 shows that the fiber of the second projection $X^j(\bar{K}) \to \mathcal{O}_{\bar{K}}$ at y is equal to

$$\begin{cases} D(a^{1/p}y^{p^{n-1}}, j/p) & \text{if } j \leq s + p^{n-1}v_K(y) + pe/(p-1), \\ \coprod_{i=0}^{p-1} D(a^{1/p}\zeta_p^i y^{p^{n-1}}, j - e - (p-1)(s + p^{n-1}v_K(y))/p) & \text{otherwise.} \end{cases}$$

Thus we have

$$X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_{\bar{K}}^2 \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\}$$

for $j \leq pe/(p-1) + s$. This is the underlying set of a $K(a^{1/p})$-affinoid variety. Again this equality extends to an isomorphism over $K(a^{1/p})$. $\square$

Now we proceed to the proof of the main theorem of this section.

Theorem 5.5 $c(\mathcal{G}(s_1, \ldots, s_r)) = \sup_k j_k$.

PROOF. We may assume that j_r is the supremum of j_k's. If $j_r = 0$, then the group scheme is etale and the conductor is 0. Thus we may assume $j_r > 0$ and $\mathcal{G}(s_1, \ldots, s_r)$ is connected. Let A be the affine algebra of this group scheme and put $B = A[W]/(W^{p^{r-1}} - T_1)$. Consider the finite flat map of embeddings

$$\begin{array}{ccc} \mathcal{O}_K[[T_1, \ldots, T_r]] & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathcal{O}_K[[W, T_2, \ldots, T_r]] & \longrightarrow & B \end{array}$$

where the left vertical arrow is defined by $T_1 \mapsto W^{p^{r-1}}$. This induces a finite flat map of K-affinoid varieties

$$X_B^j(\bar{K}) \ni (w, t_2, \ldots, t_r) \mapsto (w^{p^{r-1}}, t_2, \ldots, t_r) \in X_A^j(\bar{K}),$$

where

$$\begin{aligned} X_A^j(\bar{K}) = \{(t_1, \ldots, t_r) \in \mathcal{O}_{\bar{K}}^r \mid v_K({t_1}^p - \pi^{s_1}t_2) \geq j, \ldots, \\ v_K({t_{r-1}}^p - \pi^{s_{r-1}}t_r) \geq j, v_K({t_r}^p - \pi^{s_r}t_1) \geq j\} \end{aligned}$$

and

$$X^j_B(\bar{K}) = \{(w, t_2, \ldots, t_r) \in \mathcal{O}^r_{\bar{K}} \mid v_K(w^{p^r} - \pi^{s_1} t_2) \geq j, \\ v_K({t_2}^p - \pi^{s_2} t_3) \geq j, \ldots, v_K({t_r}^p - \pi^{s_r} w^{p^{r-1}}) \geq j\}$$

be the j-th tubular neighborhoods of these two embeddings. We calculate a jump of $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$ at first.

Lemma 5.6 *If $j_r < pe/(p-1)$, then the first jump of $\{F^j(B)\}_{j\in\mathbb{Q}_{>0}}$ occurs at $j = j_r$ and $\sharp F^{j_r}(B) = p^r$.*

Note that the base change from K to a finite extension L multiplies s_i's, j_i's and e by the ramification index of L/K. Thus, to prove Lemma 5.6 and Theorem 5.5, we may assume that p^{r-1} divides s_i's and e.

PROOF. Consider the K-affinoid variety X^j_B for $j \leq pe/(p-1)$. Then the iterative use of Lemma 5.4 and Lemma 5.3 shows that the affinoid variety $X^j_B(\bar{K})$ is equal to

$$\{v_K(w^{p^r} - \pi^{(s_r+ps_{r-1}+\ldots+p^{r-1}s_1)/p^{r-1}} w) \geq pl_1(j),\ v_K(t_2 - g_2(w)) \geq u_2, \\ v_K(t_3 - g_3(t_2, w)) \geq u_3, \ldots,\ v_K(t_r - g_r(t_{r-1}, w)) \geq u_r\},$$

where $l_i(j)$, $g_i(t_{i-1}, w)$, $g_2(w)$ and u_i are defined as follows;

- $l_r(j) = j/p$,
- $l_{i-1}(j) = \inf(j, l_i(j) + s_{i-1})/p$,
- $g_i(t_{i-1}, w) = t^p_{i-1}/\pi^{s_{i-1}}$ and $u_i = j - s_{i-1}$ if $j \geq l_i(j) + s_{i-1}$,
- $g_i(t_{i-1}, w) = \pi^{(s_r+ps_{r-1}+\ldots+p^{r-i}s_i)/p^{r-i+1}} w^{p^{i-2}}$ and $u_i = l_i(j)$ if $j < l_i(j) + s_{i-1}$,
- $g_2(w) = g_2(w^{p^{r-1}}, w)$.

Note that $l_i(j)$ is a strictly monotone increasing function of j. This affinoid variety is isomorphic to the product of the affinoid variety

$$\{w \in \mathcal{O}_{\bar{K}} \mid v(w^{p^r} - \pi^{(s_r+ps_{r-1}+\ldots+p^{r-1}s_1)/p^{r-1}} w) \geq pl_1(j)\}$$

and discs. Therefore, from Lemma 5.2, we see that the first jump of $\{F^j(B)\}_{j\in\mathbb{Q}_{>0}}$ occurs at j such that $pl_1(j) = j_r$, provided this j satisfies $0 < j < pe/(p-1)$. Moreover, then we have $\sharp F^j(B) = p^r$. Thus the following lemma and the strict monotonicity of l_1 terminate the proof of Lemma 5.6. $\square$

Lemma 5.7 $l_1(j_r) = j_r/p$.

PROOF. Suppose that there is k such that $l_k(j_r) = j_r/p$ and $j_r \geq l_{k'}(j_r) +$

$s_{k'-1}$ for any $1 < k' \leq k$. Then we have

$$l_1(j_r) = \inf(j_r, (j_r + ps_{k-1} + p^2 s_{k-2} + \ldots + p^{k-1}s_1)/p^{k-1})/p$$

and the assumption $j_{k-1} \leq j_r$ implies $l_1(j_r) = j_r/p$. $\square$

On the other hand, let $s = (s_r + ps_{r-1} + \ldots + p^{r-1}s_1)/p^{r-1}$ and $\sigma_0, \ldots, \sigma_{p^r-1}$ be the roots of the equation $X^{p^r} - \pi^s X = 0$. Then we see that the images by $w \mapsto w^{p^{r-1}}$ of the discs $D(\sigma_i, pl_1(j) - s)$ are disjoint for $j > j_r$. Hence the surjection $\pi_0(X^j_{B,\bar{K}}) \to \pi_0(X^j_{A,\bar{K}})$ is bijective for $0 < j \leq pe/(p-1)$ and the first (and the last) jump of $\{F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ also occurs at j_r, provided $j_r < pe/(p-1)$.

When $j_r = pe/(p-1)$, we see that $s_k = e > 0$ for any k. Thus we can use Lemma 5.4 for $j < pe/(p-1) + \varepsilon$ with sufficiently small $\varepsilon > 0$. Then, by the same reasoning as above, we conclude that $c(A) = pe/(p-1)$. $\square$

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