HASSE-ARF THEOREM FOR $\mathbb{F}_p$-VECTOR SPACE SCHEMES OF RANK TWO

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1. Introduction

Let p be an odd prime and f be an elliptic modular form of level N prime to p and weight $k \le p+1$. Let us consider its associated mod p Galois representation $\bar{\rho}_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ and its restriction to the inertia subgroup I_p. The theorem of Deligne and Fontaine asserts that the tame characters appearing in $\bar{\rho}_f|_{I_p}$ are determined by k.

Theorem 1.1 (Deligne, Fontaine)**.**

$$\bar{\rho}_f|_{I_p} = \begin{cases} \begin{pmatrix} \chi_p^{k-1} & * \\ 0 & 1 \end{pmatrix} & \textit{if } f \textit{ is ordinary at } p, \\ \begin{pmatrix} \theta_{p^2-1}^{k-1} & 0 \\ 0 & \theta_{p^2-1}^{p(k-1)} \end{pmatrix} & \textit{if } f \textit{ is supersingular at } p, \end{cases}$$

where χ_p is the mod p cyclotomic character and θ_d is the fundamental character of level d in the sense of [14]*.*

This classification is the basis for the local analysis of $\bar{\rho}_f$, especially for the Serre conjecture of mod p modular forms ([14]). We have two proofs of this theorem for $k < p$: one uses Raynaud's full faithful theorem for finite flat representations ([9, section 6]) and the other uses p-adic Hodge theory and the Fontaine-Laffaille functor ([8, Proposition 4.1.1]). In both proofs, it is crucial that p is absolutely unramified, and this is the very obstacle to carry out a similar analysis on the weight of a modular form and its mod p Galois representation for a totally real number field F. In this note, we propose a new approach to tackle this problem which is applicable without any restriction to the ramification index e of F at p, at least in the case of parallel weight $(2,\ldots,2)$. Namely, we prove the following conjecture in the reducible case for $\mathbb{F} = \mathbb{F}_p$.

Conjecture 1.2. *Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with perfect residue field and I_K be its inertia subgroup. Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ and $\mathcal{G}$ be a finite flat $\mathbb{F}$-vector*

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space scheme of rank 2 over $\mathcal{O}_K$. Write $c = c(\mathcal{G})$ for the conductor of $\mathcal{G}$ ([2], [3]) and k/l for the prime-to-p part of c mod $\mathbb{Z}$. Then an I_K-module $\mathcal{G}(\bar{K}) \otimes_{\mathbb{F}} \bar{\mathbb{F}}_p$ contains θ_l^k as an I_K-submodule. Moreover, if $\mathcal{G}(\bar{K})$ is reducible, then we have $\theta_l^k \subseteq \mathcal{G}(\bar{K})$.

This conjecture can be regarded as the counterpart for finite flat group schemes of the Hasse-Arf theorem in the classical ramification theory. In fact, if the Galois group G_K acts trivially on $\mathcal{G}(\bar{K})$, this is equivalent to the assertion that, for a complete discrete valuation field M and an abelian extension L of M whose integer ring is a $\mathcal{G}$-torsor over $\mathcal{O}_M$, the denominator of the conductor $c(L/M)$ is p-power. In this case, the assertion follows easily from the theorem of Herbrand for finite flat group schemes ([1, Lemme 2.10]).

To prove the conjecture for $\mathbb{F} = \mathbb{F}_p$, we will firstly show the compatibility of the theory of Breuil ([5]) with respect to a base extension in $K_\infty = K(\pi^{p^{-\infty}})$. This makes us possible to describe a defining equation of $\mathcal{G}$ explicitly. By virtue of the full faithful theorem of Breuil ([6, Theorem 3.4.3]), such a base change is harmless to study finite flat representations. Next we gather some elementary lemmas for the calculation of the conductor. As a corollary, we determine the conductor of a Raynaud $\mathbb{F}$-vector space scheme, which is independent of the proof of the main theorem. Then we prove the main theorem by a lengthy calculation. In the forthcoming paper [12], we prove the conjecture in general, by a more geometrical method.

2. Base change property for a filtered ϕ_1-module of Breuil

In this section, we briefly recall the theory of a filtered ϕ_1-module of Breuil ([5]) and give a proof of its compatibility with the base change from K to K_∞.

Let K be a complete discrete valuation field of mixed characteristic $(0, p)$, k be its residue field which we suppose to be perfect in this section, e be its absolute ramification index, $W = W(k)$ and σ be the Frobenius of W. We fix once and for all an uniformizer π of K. Let $E(u) = u^e - pF(u)$ be the Eisenstein polynomial of π over W and Set $S = S_\pi = (W[u]^{\mathrm{PD}})^\wedge$, where the divided power envelope of $W[u]$ is taken with respect to an ideal $(E(u))$ and compatibility with the natural divided power structure on pW, and $\wedge$ means the π-adic completion. The ring S is endowed with a σ-semilinear map $\phi : u \mapsto u^p$, which we also call Frobenius, and the natural filtration induced by the divided power structure. We set $\phi_1 = 1/p.\phi|_{\mathrm{Fil}^1 S}$ and $c = \phi_1(E(u)) \in S^\times$. We define ϕ, ϕ_1 and a filtration on $S_n = S/p^n$ similarly.

In [5], following categories of filtered ϕ_1-modules are defined. Set $'\mathcal{M}$ be a category consisting of following data;

- an S-module M and its S-submodule $\mathrm{Fil}^1 M$ containing $\mathrm{Fil}^1 SM$,
- ϕ-semilinear map $\phi_1 : \mathrm{Fil}^1 M \to M$ satisfying $\phi_1(s_1 m) = \phi_1(s_1)\phi(m)$, where $s_1 \in \mathrm{Fil}^1 S$, $m \in M$ and $\phi(m) = c^{-1}\phi_1(E(u)m)$.

Let $\mathcal{M}_1$ be a full subcategory of $'\mathcal{M}$ consisting of M satisfying

- the S_1-module M is free of finite rank,
- $\phi_1(\mathrm{Fil}^1 M)$ generates M as an S-module.

and $\mathcal{M}$ be the minimal full subcategory of $'\mathcal{M}$ which contains $\mathcal{M}_1$ and stable under extension.

The category $\mathcal{M}$ is shown to be categorically anti-equivalent to the category $(p\text{-Gr}/\mathcal{O}_K)$ of the finite flat group schemes over $\mathcal{O}_K$ which is killed by some p-power ([5]). Let us recall the definition of this equivalence. Let $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$ be the category of the p-adic formal schemes of formally syntomic, endowed with the Grothendieck topology generated by the surjective families of formally syntomic morphisms. Write $(\mathrm{Ab}/\mathcal{O}_K)$ for the category of the abelian sheaves on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$. The sheaf $\mathcal{O}_{n,\pi}$ and $\mathcal{J}_{n,\pi}$ is defined by the formula

$$\mathcal{O}_{n,\pi}(\mathfrak{X}) = \mathrm{H}^0_{\mathrm{crys}}((\mathfrak{X}_n/S_n)_{\mathrm{crys}}, \mathcal{O}_{\mathfrak{X}_n/S_n})$$

and

$$\mathcal{J}_{n,\pi}(\mathfrak{X}) = \mathrm{H}^0_{\mathrm{crys}}((\mathfrak{X}_n/S_n)_{\mathrm{crys}}, \mathcal{J}_{\mathfrak{X}_n/S_n}),$$

where $\mathfrak{X}_n = \mathfrak{X}/p^n$. We also set $\mathcal{O}_{\infty,\pi} = \varinjlim \mathcal{O}_{n,\pi}$ and $\mathcal{J}_{\infty,\pi} = \varinjlim \mathcal{J}_{n,\pi}$. We denote by $\phi : \mathcal{O}_{n,\pi} \to \mathcal{O}_{n,\pi}$ the crystalline Frobenius map. We can define the natural morphism $\phi_1 : \mathcal{J}_{n,\pi} \to \mathcal{O}_{n,\pi}$ which makes the following diagram commutative

$$\begin{tikzcd}
\mathcal{J}_{n,\pi} \arrow[r, "\phi_1"] & \mathcal{O}_{n,\pi} \arrow[d, "\times p"] \\
\mathcal{J}_{n+1,\pi} \arrow[u] \arrow[r, "\phi"] & \mathcal{O}_{n+1,\pi}.
\end{tikzcd}$$

Let $\mathcal{G} \in (p\text{-Gr}/\mathcal{O}_K)$ and $M \in \mathcal{M}$. Define

$$\mathrm{Mod}_K(\mathcal{G}) = \mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_K)}(\mathcal{G}, \mathcal{O}_{\infty,\pi})$$

and

$$\mathrm{Gr}_K(M) = \mathrm{Hom}_{'\mathcal{M}}(M, \mathcal{O}_{\infty,\pi}).$$

Then the main theorem of [5] is the following.

Theorem 2.1 (Breuil)**.** *The functor* Gr_K *defines an anti-equivalence of categories* $\mathcal{M} \to (p\text{-Gr}/\mathcal{O}_K)$ *and its quasi-inverse is* Mod_K*.*

Next we consider the base change theorem of the functor Gr for an extension $K_1 = K(\pi^{1/p})$ over K. This extension is totally ramified of degree p. The minimal polynomial of $\pi_1 = \pi^{1/p}$ over W is $E_1(v) = E(v^p) = v^{ep} - pF(v^p)$. Set $S' = S_{\pi_1} = (W[v]^{PD})^\wedge$, where the divided power envelope is taken with respect to $(E_1(v))$ and compatibility with the natural divided power structure on pW. The ring S' has a σ-semilinear endomorphism $\phi : S' \to S'$ defined by $v \mapsto v^p$ and a ϕ-semilinear map $\mathrm{Fil}^1 S' \to S'$ satisfying $\phi\,|_{\mathrm{Fil}^1 S'} = p\phi_1$. We have a ring homomorphism $S \to S'$ which maps u to v^p. This respects the filtration and ϕ_1.

Lemma 2.2. *The S-module S' is free of finite rank.*

Proof. The $W[u]$-algebra $W[v]$ is free of finite rank. We have $(E(u))W[v] = (E_1(v))$. Therefore $W[v]^{PD} = W[u]^{PD} \otimes_{W[u]} W[v]$ from [4, Proposition 3.21] and $W[u]^{PD} \to W[v]^{PD}$ is also free of finite rank. Thus $(W[v]^{PD})^\wedge = (W[u]^{PD})^\wedge \otimes_{W[u]^{PD}} W[v]^{PD}$. This concludes the proof. □

Let us denote the category of filtered ϕ_1-modules over S' by ${}'\mathcal{M}'$ and $\mathcal{M}'$. From the lemma above, we can define a filtered ϕ_1-module structure on $M' = M \otimes_S S'$ for any $M \in {}'\mathcal{M}$ by $\mathrm{Fil}^1 M' = (\mathrm{Fil}^1 M) \otimes_S S'$ and $\phi_{1,M'} = \phi_1 \otimes \phi$. If $M \in \mathcal{M}$, then we have $M' \in \mathcal{M}'$.

For a presheaf $\mathcal{F}$ on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$, we denote by $\mathcal{F}|_{\mathcal{O}_{K_1}}$ the restriction of $\mathcal{F}$ to $\mathrm{Spf}(\mathcal{O}_{K_1})_{\mathrm{syn}}$. If $\mathcal{F}$ is a sheaf on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$, then $\mathcal{F}|_{\mathcal{O}_{K_1}}$ is also a sheaf on $\mathrm{Spf}(\mathcal{O}_{K_1})_{\mathrm{syn}}$.

Define a morphism $\Psi_M : \mathrm{Gr}(M)|_{\mathcal{O}_{K_1}} \to \mathrm{Gr}(M')$ of $(\mathrm{Ab}/\mathcal{O}_{K_1})$ as follows. For any $\mathfrak{X}'$, formally syntomic over $\mathrm{Spf}(\mathcal{O}_{K_1})$, we want to set $\Psi_{M,\mathfrak{X}'} : \mathrm{Hom}^S_{'\mathcal{M}}(M, \mathcal{O}_{n,\pi}(\mathfrak{X}')) \to \mathrm{Hom}^{S'}_{'\mathcal{M}'}(M \otimes_S S', \mathcal{O}_{n,\pi_1}(\mathfrak{X}'))$ by $f \mapsto (m \otimes s' \mapsto s'.\mathrm{pr}^*_{\mathfrak{X}'}(f(m)))$, where $\mathrm{pr}^*_{\mathfrak{X}'} : \mathcal{O}_{n,\pi}(\mathfrak{X}') = \mathrm{H}^0_{\mathrm{crys}}(\mathfrak{X}'_n/S_n) \to \mathrm{H}^0_{\mathrm{crys}}(\mathfrak{X}'_n/S'_n) = \mathcal{O}_{n,\pi_1}(\mathfrak{X}')$ is the natural pull-back. The map $\mathrm{pr}^*_{\mathfrak{X}'}$ respects the filtration. We have to show the compatibility with ϕ_1.

Consider $\mathfrak{X}' = \mathrm{Spf}(\mathfrak{A}')$. We can write $\mathfrak{A}' = \mathcal{O}_{K_1}\langle X'_1, \ldots, X'_r\rangle/(f_1, \ldots, f_s)$, where $\mathcal{O}_{K_1}\langle X'_1, \ldots, X'_r\rangle = \mathcal{O}_{K_1}[X'_1, \ldots, X'_r]^\wedge$ and $f_1, \ldots, f_s$ is a regular sequence in that ring ([5, Lemme 2.2.1]). Put $\mathfrak{A}'_i = \mathcal{O}_{K_1}\langle X_0'^{p^{-i}}, \ldots, X_r'^{p^{-i}}\rangle/(X'_0 - \pi_1, f_1, \ldots, f_s)$ and $\mathfrak{A}'_\infty = \varinjlim \mathfrak{A}'_i$. The W-algebra $\mathfrak{A}'_i$ is isomorphic to

$$\begin{aligned}
&\mathcal{O}_K[T]/(T^p - \pi)\langle X_0'^{p^{-i}}, \ldots, X_r'^{p^{-i}}\rangle/(X'_0 - T, f_1, \ldots, f_s)\\
&\qquad = W[u,T]/(E(u), T^p - u)\langle X_0'^{p^{-i}}, \ldots, X_r'^{p^{-i}}\rangle/(X'_0 - T, f_1, \ldots, f_s)\\
&\qquad = W\langle X_0'^{p^{-i}}, \ldots, X_r'^{p^{-i}}\rangle/(E(X_0'^p), f_1, \ldots, f_s).
\end{aligned}$$

We also set $A'_\infty = \mathfrak{A}'_\infty/p = k[X_0'^{p^{-\infty}}, \dots, X_r'^{p^{-\infty}}]/(X_0'^{ep}, \bar{f}_1, \dots, \bar{f}_s)$. Note that the formal scheme $\mathrm{Spf}(\mathfrak{A}'_i)$ is a covering of $\mathrm{Spf}(\mathfrak{A}')$ in $\mathrm{Spf}(\mathcal{O}_{K_1})_{\mathrm{syn}}$.

Lemma 2.3. *The following two sequences are exact;*

$$0 \to \mathcal{O}_{r,\pi}(\mathfrak{A}'_\infty) \overset{\times p^s}{\to} \mathcal{O}_{r+s,\pi}(\mathfrak{A}'_\infty) \to \mathcal{O}_{s,\pi}(\mathfrak{A}'_\infty) \to 0$$

$$0 \to \mathcal{J}_{r,\pi}(\mathfrak{A}'_\infty) \overset{\times p^s}{\to} \mathcal{J}_{r+s,\pi}(\mathfrak{A}'_\infty) \to \mathcal{J}_{s,\pi}(\mathfrak{A}'_\infty) \to 0.$$

In particular, there are exact sequences in $(\mathrm{Ab}/\mathcal{O}_{K_1})$

$$0 \to \mathcal{O}_{r,\pi}|_{\mathcal{O}_{K_1}} \overset{\times p^s}{\to} \mathcal{O}_{r+s,\pi}|_{\mathcal{O}_{K_1}} \to \mathcal{O}_{s,\pi}|_{\mathcal{O}_{K_1}} \to 0,$$

and

$$0 \to \mathcal{J}_{r,\pi}|_{\mathcal{O}_{K_1}} \overset{\times p^s}{\to} \mathcal{J}_{r+s,\pi}|_{\mathcal{O}_{K_1}} \to \mathcal{J}_{s,\pi}|_{\mathcal{O}_{K_1}} \to 0.$$

Proof. We repeat just the same argument as [5, Lemme 2.3.2].

Note that $\mathcal{O}_{n,\pi}(\mathfrak{A}'_\infty) = \mathrm{H}^0_{\mathrm{crys}}(\mathfrak{A}'_\infty/p^n/S_n)$ is isomorphic to $(W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{PD}$, where the divided power envelope in the right hand side is taken with respect to the kernel of a surjection $W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u] \to \mathfrak{A}'_\infty/p^n$ which sends $(x_0, \dots, x_{n-1}) \otimes u$ to $X_0'^p \sum_{k=0}^{n-1} p^k \hat{x}_k^{p^{n-k}}$, and compatibility with the natural divided power structure on pW. Here we denote an lifting of x_k in $\mathfrak{A}'_\infty/p^n$ by $\hat{x}_k$. In fact, this surjection induces a thickening $(W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{PD} \to \mathfrak{A}'_\infty/p^n$ of $\mathfrak{A}'_\infty/p^n$ over S_n and thus we have the natural projection $\mathrm{H}^0_{\mathrm{crys}}(\mathfrak{A}'_\infty/p^n/S_n) \to (W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{PD}$. Its inverse map $(W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{PD} \to \mathrm{H}^0_{\mathrm{crys}}(\mathfrak{A}'_\infty/p^n/S_n)$ is defined as follows. For any affine thickening $U \to T$ of $\mathfrak{A}'_\infty/p^n$ over S_n, we define a map $(W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{PD} \to \Gamma(U, \mathcal{O}_U)$ by $(x_0, \dots, x_{n-1}) \otimes u \mapsto u \sum_{k=0}^{n-1} p^k \hat{t}_k^{p^{n-k}}$, where $\hat{t}_k$ is a lifting of x_k in $\Gamma(T, \mathcal{O}_T)$. This is a well-defined ring homomorphism, patches in a non-affine case and induces the inverse map of the natural projection.

Let us consider a surjection $W_n[u][X_0'^{p^{-\infty}}, \dots, X_r'^{p^{-\infty}}] \to \mathfrak{A}'_\infty/p^n = W_n[X_0'^{p^{-\infty}}, \dots, X_r'^{p^{-\infty}}]/(E(X_0'^p), f_1, \dots, f_s)$ which sends u to $X_0'^p$ and $X_k'^{p^{-i}}$ to its image for any k, i. Let us denote its kernel $(u - X_0'^p, E(X_0'^p), f_1, \dots, f_s)$ by I. Taking its divided power envelope with respect to I and compatibility with the natural divided power structure on pW, we get a surjection $(W_n[u][X_0'^{p^{-\infty}}, \dots, X_r'^{p^{-\infty}}])^{PD} \to \mathfrak{A}'_\infty/p^n$. This map is S-linear, where $\mathfrak{A}'_\infty/p^n$ is considered as an S-algebra by $u \mapsto X_0'^p$. Thus this surjection defines a thickening of $\mathfrak{A}'_\infty/p^n$ over S_n and we get the natural projection $(W_n(A'_\infty) \otimes_{W_n,\sigma^n} W_n[u])^{PD} \to (W_n[u][X_0'^{p^{-\infty}}, \dots, X_r'^{p^{-\infty}}])^{PD}$.

Conversely, a surjection $W_n[u][X_0'^{p^{-\infty}},\dots,X_r'^{p^{-\infty}}]\to W_n(A'_\infty)\otimes_{W_n,\sigma^n} W_n[u]$ sending u to $1\otimes u$ and $X_k'^{p^{-i}}$ to $[X_k'^{p^{-i-n}}]$ makes the following diagram commutative;

$$\begin{array}{ccc} W_n[u][X_0'^{p^{-\infty}},\dots,X_r'^{p^{-\infty}}] & \longrightarrow & W_n(A'_\infty)\otimes_{W_n,\sigma^n} W_n[u] \\ \downarrow & & \downarrow \\ \mathfrak{A}'_\infty/p^n & = & \mathfrak{A}'_\infty/p^n. \end{array}$$

Therefore this surjection induces $(W_n[u][X_0'^{p^{-\infty}},\dots,X_r'^{p^{-\infty}}])^{PD}\to(W_n(A'_\infty)\otimes_{W_n,\sigma^n} W_n[u])^{PD}$. We see that this map is the inverse to the natural projection by the definition. Thus we get an identification $\mathcal{O}_{n,\pi}(\mathfrak{A}'_\infty)=(W_n[u][X_0'^{p^{-\infty}},\dots,X_r'^{p^{-\infty}}])^{PD}$ respecting the filtration and the Frobenius. Then [5, Lemme 2.3.2] and [4, 3.20, Remark 8] conclude the proof.

□

We insert here the next lemma for the sake of references.

Lemma 2.4. *Let $\psi_1,\dots,\psi_s\in k[X_0'^{p^{-\infty}},\dots,X_r'^{p^{-\infty}}]$ satisfying $\psi_k^p=\bar{f}_k$. Then $\mathcal{O}_{1,\pi}(\mathfrak{A}'_\infty)$ is isomorphic up to a σ-twist to*

$$\bigoplus_{m_0,\dots,m_{s+1}\in\mathbb{Z}_{\geq 0}} A'_\infty[u-X'_0]/(u-X'_0)^p\gamma_{pm_0}(X_0'^e)\gamma_{pm_1}(\psi_1)\cdots$$
$$\gamma_{pm_s}(\psi_s)\gamma_{pm_{s+1}}(u-X'_0).$$

Proof. The sequence $u-X_0'^p, X_0'^{ep},\bar{f}_1,\dots,\bar{f}_s$ is a regular in $k[u][X_0'^{p^{-\infty}},\dots,X_r'^{p^{-\infty}}]$. Their inverse images in $(A'_\infty\otimes_{k,\sigma}k[u])^{PD}$ are $u-X'_0, X_0'^e,\psi_1,\dots,\psi_s$, respectively. Thus the assertion follows from the proof of [10, Proposition 1.7].

□

From Lemma 2.3, we have a diagram

$$\begin{array}{ccccccc} \mathcal{J}_{n+1,\pi}|_{\mathcal{O}_{K_1}} & \longrightarrow & \mathcal{J}_{n,\pi}|_{\mathcal{O}_{K_1}} & \xrightarrow{\phi_1} & \mathcal{O}_{n,\pi}|_{\mathcal{O}_{K_1}} & \xrightarrow{\times p} & \mathcal{O}_{n+1,\pi}|_{\mathcal{O}_{K_1}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}_{n+1,\pi_1} & \longrightarrow & \mathcal{J}_{n,\pi_1} & \xrightarrow{\phi_1} & \mathcal{O}_{n,\pi_1} & \xrightarrow{\times p} & \mathcal{O}_{n+1,\pi_1}, \end{array}$$

where the vertical arrows are the pull-backs and the left and right squares are commutative. The compositions of the horizontal maps are ϕ. Thus we see that the middle square is also commutative. In other words, the map $\mathrm{pr}_{\mathfrak{X}'}^*$ is compatible with ϕ_1. Therefore, we get the morphism of $(\mathrm{Ab}/\mathcal{O}_{K_1})$

$$\Psi_M : \mathrm{Gr}(M)|_{\mathcal{O}_{K_1}} \to \mathrm{Gr}(M').$$

Theorem 2.5. *The canonical map Ψ_M is an isomorphism.*

Proof. The sheaves of both sides come from finite flat group schemes $\mathrm{Gr}(M)\times_{\mathcal{O}_K}\mathcal{O}_{K_1}$ and $\mathrm{Gr}(M')$. Thus the bijectivity can be checked after taking the functor Mod. In other words, it suffices to show that

$$\Phi_M : M \otimes_S S' \to \mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(\mathrm{Hom}^S_{(\mathrm{Ab}/\mathcal{O}_{K_1}),'\mathcal{M}}(M, \mathcal{O}_{1,\pi}|_{\mathcal{O}_{K_1}}), \mathcal{O}_{1,\pi_1}),$$

defined by $m\otimes s' \mapsto (f \mapsto s'.\mathrm{pr}^*(f(m)))$ is an isomorphism of $'\mathcal{M}'$. Here we denote by pr^* the pull-back map $\mathcal{O}_{1,\pi}|_{\mathcal{O}_{K_1}} \to \mathcal{O}_{1,\pi_1}$. We want by devissage to reduce this to the p-torsion case.

Lemma 2.6. *$\mathcal{E}xt^1_{'\mathcal{M}/S_1}(M, \mathcal{O}_{1,\pi}|_{\mathcal{O}_{K_1}}) = 0$ for any $M \in \mathcal{M}$ which is killed by p.*

Proof. Take some $\mathrm{Spf}(\mathfrak{A}) \in \mathrm{Spf}(\mathcal{O}_{K_1})_{\mathrm{syn}}$ and an extension

$$0 \to \mathcal{O}_{1,\pi}(\mathfrak{A}'_\infty) \to \mathcal{E} \to M \to 0.$$

We have to show that syntomic locally a splitting of $\mathcal{E}$ exists. Let $\{e_1,\dots,e_d\}$ be an adapted basis of M ([5, Proposition 2.1.2.5]) and $\hat{e}_1,\dots,\hat{e}_d$ be their lifts to $\mathcal{E}$. We mimic [5, Proposition 4.1.3] and seek for a splitting $e_i \mapsto \hat{e}_i$ by modifying $\hat{e}_i$'s.

Firstly, we modify $\hat{e}_i$'s to respect the filtration. Let r_j be the minimal natural number satisfying $u^{r_j}e_j \in \mathrm{Fil}^1 M$. There exists $\delta_j \in \mathcal{O}_{1,\pi}(\mathfrak{A}'_\infty)$ such that $u^{r_j}\hat{e}_j + \delta_j \in \mathrm{Fil}^1\mathcal{E}$. By Lemma 2.4, we can decompose δ_j as $\delta_j = \delta_{j,0} + \delta_{j,1}$, where $\delta_{j,0} \in A'_\infty$ and $\delta_{j,1} \in \mathcal{J}_{1,\pi}(\mathfrak{A}'_\infty)$. We have $u^e\hat{e}_j + u^{e-r_j}\delta_j \in \mathrm{Fil}^1\mathcal{E}$ and $u^{e-r_j}\delta_{j,0} \in \mathcal{J}_{1,\pi}(\mathfrak{A}'_\infty)$. As $\mathcal{J}_{1,\pi}(\mathfrak{A}'_\infty)$ contains $u - X'_0$, we get $X_0'^{e-r_j}\delta_{j,0} = 0$, and in particular $X_0'^{p(e-r_j)}\delta_{j,0}^p = 0$ in A'_∞. Take an lift $\hat{\delta}_{j,0}$ of $\delta_{j,0}$ in $\mathfrak{A}'_\infty$, where $X'_0 = \pi_1$ holds. Then we have $\pi^{e-r_j}\hat{\delta}^p_{j,0} = \pi^e x_j$ for some $x_j \in \mathfrak{A}'_\infty$. The ring $\mathfrak{A}'_\infty$ is π-torsion free and we have $\hat{\delta}^p_{j,0} = \pi^{r_j}x_j$. As $\mathfrak{A}'_\infty$ is perfect, we can take $y_j \in \mathfrak{A}'_\infty$ satisfying $y_j^p = x_j$. Then $(\delta_{j,0} - X_0'^{r_j}y_j)^p = 0$ in A'_∞. By the definition of the divided power structure on $(k[u][X_0'^{p^{-\infty}},\dots,X_r'^{p^{-\infty}}])^{PD}$, we see that $\delta_{j,0} - X_0'^{r_j}y_j \in \mathcal{J}_{1,\pi}(\mathfrak{A}'_\infty)$ and also $\delta_{j,0} - u^{r_j}y_j \in \mathcal{J}_{1,\pi}(\mathfrak{A}'_\infty)$.

Now we replace $\hat{e}_j$ by $\hat{e}_j + y_j$. Then, $u^{r_j}(\hat{e}_j + y_j) \equiv -\delta_j + \delta_{j,0} \equiv 0$ mod $\mathrm{Fil}^1\mathcal{E}$. Thus the map $e_j \mapsto \hat{e}_j + y_j$ respects the filtration.

Next we modify $\hat{e}_j$ to respect ϕ_1. Set $\begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix} = \phi_1 \begin{pmatrix} u^{r_1}\hat{e}_1 \\ \vdots \\ u^{r_d}\hat{e}_d \end{pmatrix} - \mathcal{G} \begin{pmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_d \end{pmatrix}$, where $\mathcal{G} \in \mathrm{GL}_d(S_1)$ satisfying $\phi_1 \begin{pmatrix} u^{r_1}e_1 \\ \vdots \\ u^{r_d}e_d \end{pmatrix} = \mathcal{G} \begin{pmatrix} e_1 \\ \vdots \\ e_d \end{pmatrix}$. We have to find $\begin{pmatrix} \delta_1 \\ \vdots \\ \delta_d \end{pmatrix} \in \mathcal{O}_{1,\pi}(\mathfrak{A}'_\infty)^{\oplus d}$ such that $\phi_1 \begin{pmatrix} u^{r_1}(\hat{e}_1+\delta_1) \\ \vdots \\ u^{r_d}(\hat{e}_d+\delta_d) \end{pmatrix} = \mathcal{G} \begin{pmatrix} \hat{e}_1+\delta_1 \\ \vdots \\ \hat{e}_d+\delta_d \end{pmatrix}$, or $\phi_1 \begin{pmatrix} u^{r_1}\delta_1 \\ \vdots \\ u^{r_d}\delta_d \end{pmatrix} = \mathcal{G} \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_d \end{pmatrix} - \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}$. Decompose $c_j = c_{j,0} + c_{j,1} + c_{j,2}$, where $c_{j,0} \in A'_\infty$, $c_{j,1} \in (u - X'_0)A'_\infty$ and $c_{j,2} \in \mathcal{J}^{[2]}_{1,\pi}(\mathfrak{A}'_\infty)$. By linearity, it suffices to find the solution for $\phi_1 \begin{pmatrix} u^{r_1}\delta_{1,k} \\ \vdots \\ u^{r_d}\delta_{d,k} \end{pmatrix} = \mathcal{G} \begin{pmatrix} \delta_{1,k} \\ \vdots \\ \delta_{d,k} \end{pmatrix} - \begin{pmatrix} c_{1,k} \\ \vdots \\ c_{d,k} \end{pmatrix}$ for $k = 0, 1, 2$. We can resolve these equations, taking an appropriate syntomic cover of $\mathfrak{A}'_\infty$ if necessary, just as the proof of [5, Proposition 4.1.3], if we replace Y_0 and X_0 there by X'_0 and $X_0'^p$, respectively.

□

Lemma 2.7. *$\mathcal{E}xt^1_{'\mathcal{M}/S_1}(M, \mathcal{O}_{\infty,\pi}|_{\mathcal{O}_{K_1}}) = 0$ for any $M \in \mathcal{M}$.*

Proof. By the Lemma 2.3, the same reasoning as the proof of [5, Lemme 4.1.2] works also in our case and shows that the lemma holds for any M killed by p. Then the definition of the category $\mathcal{M}$ and devissage conclude the proof.

□

Now consider an exact sequence in $\mathcal{M}$

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

From Lemma 2.7, we get an exact sequence

$$\begin{aligned} 0 \to &\mathrm{Hom}^{'\mathcal{M}/S}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(M_3, \mathcal{O}_{\infty,\pi}|_{\mathcal{O}_{K_1}}) \\ &\to \mathrm{Hom}^{'\mathcal{M}/S}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(M_2, \mathcal{O}_{\infty,\pi}|_{\mathcal{O}_{K_1}}) \to \mathrm{Hom}^{'\mathcal{M}/S}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(M_1, \mathcal{O}_{\infty,\pi}|_{\mathcal{O}_{K_1}}) \to 0. \end{aligned}$$

Here we know that $\mathrm{Hom}^{'\mathcal{M}/S}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(M_i, \mathcal{O}_{\infty,\pi}|_{\mathcal{O}_{K_1}}) = \mathrm{Gr}(M_i)|_{\mathcal{O}_{K_1}}$. Thus, from [5, Proposition 4.2.1.5], we have the following commutative diagram whose vertical sequences are exact;

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
M_1 \otimes_S S' & \xrightarrow{\Phi_{M_1}} & \mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(\mathrm{Hom}'^{\mathcal{M}/S}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(M_1, \mathcal{O}_{\infty,\pi}|_{\mathcal{O}_{K_1}}), \mathcal{O}_{\infty,\pi_1}) \\
\downarrow & & \downarrow \\
M_2 \otimes_S S' & \xrightarrow{\Phi_{M_2}} & \mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(\mathrm{Hom}'^{\mathcal{M}/S}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(M_2, \mathcal{O}_{\infty,\pi}|_{\mathcal{O}_{K_1}}), \mathcal{O}_{\infty,\pi_1}) \\
\downarrow & & \downarrow \\
M_3 \otimes_S S' & \xrightarrow{\Phi_{M_3}} & \mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(\mathrm{Hom}'^{\mathcal{M}/S}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(M_3, \mathcal{O}_{\infty,\pi}|_{\mathcal{O}_{K_1}}), \mathcal{O}_{\infty,\pi_1}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Thus, by devissage, to prove the theorem, we may assume that $pM = 0$. We have $\mathrm{rank}_{S'_1}(M \otimes_S S') = \mathrm{rank}_{S_1}(M)$ and

$$\begin{aligned}
&\mathrm{rank}_{S'_1}(\mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(\mathrm{Hom}'^{\mathcal{M}/S}_{(\mathrm{Ab}/\mathcal{O}_{K_1})}(M_1, \mathcal{O}_{\infty,\pi}|_{\mathcal{O}_{K_1}}), \mathcal{O}_{\infty,\pi_1})) \\
&\qquad = \mathrm{rank}_{S'_1}(\mathrm{Mod}_{K_1}(\mathrm{Gr}_K(M) \times_{\mathcal{O}_K} \mathcal{O}_{K_1})) = \mathrm{rank}_{S_1}(M).
\end{aligned}$$

By [5, Lemme 3.3.2], it suffices to show $\mathrm{Ker}(\Phi_M) \subseteq \mathrm{Fil}^p S'_1 M'$.

Take an adapted basis $\{e_1, \ldots, e_d\}$ as in the proof of Lemma 2.6. Let $m = \sum_{i=1}^d s'_i e_i$ be an element of $\mathrm{Ker}(\Phi_M)$. Consider the affine algebra R_M of $\mathrm{Gr}_K(M)$ and the element $f \in \mathrm{Hom}'^{\mathcal{M}}_S(M, \mathcal{O}_{1,\pi}(R_M)) \simeq \mathrm{Gr}_K(M)(R_M)$ corresponding to id_{R_M}. Then $f(e_i) \equiv \bar{X}_{i,0} + u\bar{X}_{i,1} + \cdots + u^{p-1}\bar{X}_{i,p-1} \mod \mathcal{J}^{[p]}_{1,\pi}(R_M)$, where $X_{i,0}, \ldots, X_{i,p-1}$ are the canonical generators of R_M and $\bar{X}_{i,k}$ its image in R_M/p. Here we regard $\bar{X}_{i,k}$ as an element of $\mathcal{O}_{1,\pi}(R_M)$ by the natural map $R_M/p \otimes_{k,\sigma} k[u] \to \mathcal{O}_{1,\pi}(R_M)$ (see the proof of [5, Proposition 3.1.1, Proposition 3.1.5]). Let us write f_1 for the image of f by the natural map $\mathrm{Hom}'^{\mathcal{M}}_S(M, \mathcal{O}_{1,\pi}(R_M)) \to \mathrm{Hom}'^{\mathcal{M}}_S(M, \mathcal{O}_{1,\pi}(R'_M))$, where $R'_M = R_M \otimes_{\mathcal{O}_K} \mathcal{O}_{K_1}$. As $m \in \mathrm{Ker}(\Phi_M)$, we have $\sum s'_i \mathrm{pr}^*_{R'_M}(f_1(e_i)) = 0$.

Let $\bar{X}'_{i,k}$ be the image of $\bar{X}_{i,k}$ by the natural map $R'_M/p \otimes_{k,\sigma} k[v] \to \mathcal{O}_{1,\pi_1}(R'_M)$. Now we claim that $\mathrm{pr}^*_{R'_M}(\bar{X}_{i,k}) = \bar{X}'_{i,k}$. It is sufficient to show this coincidence on an appropriate syntomic cover of R'_M. Thus we may consider $\mathrm{pr}^*_{R'_{M,\infty}} : \mathcal{O}_{1,\pi}(R'_{M,\infty}) \to \mathcal{O}_{1,\pi_1}(R'_{M,\infty})$, where $R'_{M,\infty}$ is the perfection of R'_M as before. Then the composition

$$(R'_{M,\infty}/p \otimes_{k,\sigma} k[u])^{PD} \overset{\mathrm{pr}^*}{\to} \mathrm{H}^0_{\mathrm{crys}}(R'_{M,\infty}/p/S'_1) \overset{\text{projection}}{\to} (R'_{M,\infty}/p \otimes_{k,\sigma} k[v])^{PD}$$

maps $1\otimes u$ to $1\otimes v^p$ and $r\otimes 1$ to $\hat{r}^p\otimes 1$, where $\hat{r}$ is a lifting of r by the canonical surjection $(R'_{M,\infty}/p\otimes_{k,\phi} k[v])^{PD}\to R'_{M,\infty}/p$. We may take $\hat{r}$ to be $r^{1/p}\otimes 1$. Thus the claim follows.

Therefore we have an equation $\sum_{i=1}^d s'_i(\bar{X}'_{i,0}+v^p\bar{X}'_{i,1}+\cdots+v^{p(p-1)}\bar{X}'_{i,p-1})=0$ in $\mathcal{O}_{1,\pi_1}(R'_M)/\mathcal{J}^{[p]}_{1,\pi_1}(R'_M)$. This equation also holds in $\mathcal{O}_{1,\pi_1}(R'_{M,\infty})/\mathcal{J}^{[p]}_{1,\pi_1}(R'_{M,\infty})$, and its subring $R'_{M,\infty}/p[v]/(v^p-X'_0)=R'_{M,\infty}/p[v]/(v^p-\pi_1)$ (see [5, Lemme 2.3.2]). As $R'_{M,\infty}$ is the direct limit of syntomic covers of R'_M, R'_M/p is a subring of $R'_{M,\infty}/p$. Thus the equation $\sum_{i=1}^d s'_i(\bar{X}'_{i,0}+v^p\bar{X}'_{i,1}+\cdots+v^{p(p-1)}\bar{X}'_{i,p-1})=0$ holds in $R'_M/p[v]/(v^p-\pi_1)$. Let us denote $s'_i \mod v\in k$ by $\bar{s}'_i$. Taking mod v, we have $\sum_{i=1}^d \bar{s}'_i\bar{X}'_{i,0}=0$ in $R'_M/p[v]/(v,v^p-\pi_1)=R'_M/\pi_1=R_M/\pi$. From the proof of [5, Proposition 3.1.1], we know that $X_{1,0},\ldots,X_{d,0}$ are linearly independent over k in R_M/π. Thus $\bar{s}'_i=0$ and $s'_i\in vS'_1+\mathrm{Fil}^pS'_1$ for all i. Take $s'^{(1)}_i\in S'_1$ satisfying $s'_i-vs'^{(1)}_i\in\mathrm{Fil}^pS'_1$. Then we have $v\sum_{i=1}^d s'^{(1)}_i(\bar{X}'_{i,0}+v^p\bar{X}'_{i,1}+\cdots+v^{p(p-1)}\bar{X}'_{i,p-1})=0$ in $R'_M/p[v]/(v^p-\pi_1)$. However, $R'_M/p\simeq(\mathcal{O}_{K_1}/p)^{\oplus N}\simeq(k[T]/(T^{ep}))^{\oplus N}$ for some N and $k[T]/(T^{ep})[v]/(v^p-T)\simeq k[v]/(v^{ep^2})$. Thus $R'_M/p[v]/(v^p-\pi_1)$ is finite flat over $k[v]/(v^{ep^2})$, and we have $\sum_{i=1}^d s'^{(1)}_i(\bar{X}'_{i,0}+v^p\bar{X}'_{i,1}+\cdots+v^{p(p-1)}\bar{X}'_{i,p-1})\in v^{ep^2-1}(R'_M/p[v]/(v^p-\pi_1))$. Taking mod v and repeating this procedure shows $s'_i\in v^{ep^2}S'_1+\mathrm{Fil}^pS'_1=\mathrm{Fil}^pS'_1$. In other words, $m\in\mathrm{Fil}^pS'_1M'$. This concludes the theorem.

□

Remark 2.8. In general, let L be a totally ramified extension over K of degree e' whose uniformizer we denote by π_L. When we define $S_L=S_{\pi_L}$ as above, there exists a morphism $S\to S_L$ respecting the filtration and ϕ_1 if and only if $\pi_L^{e'}=\pi\zeta_{p-1}^i$ for some i.

3. Rank one calculation

In this section, we calculate the conductor of a Raynaud $\mathbb{F}$-vector space scheme over $\mathcal{O}_K$. The point is that, as we can see from the bound of the conductor ([11, Theorem 7]), it is enough to consider the j-th tubular neighborhood only for $j\le pe/(p-1)+\varepsilon$ with sufficiently small $\varepsilon>0$. For such j, we can compute the tubular neighborhood easily by Lemma 3.4 below.

Let K be a complete discrete valuation field of mixed characteristic $(0,p)$. We write $\pi=\pi_K$ for its uniformizer and e for its absolute ramification index. We normalize a valuation v_K of K as $v_K(\pi)=1$ and extend it to the algebraic closure $\bar{K}$ of K. For $a\in\bar{K}$ and $j\in\mathbb{R}$, let $D(a,j)$ denote the closed disc $\{z\in\mathcal{O}_{\bar{K}}\mid v_K(z-a)\ge j\}$. This is the underlying subset of a $K(a)$-affinoid subdomain of the unit disc over $K(a)$.

For integers $0 \leq s_1, \ldots, s_r \leq e$, let $\mathcal{G}(s_1, \ldots, s_r)$ denote the Raynaud $\mathbb{F}_{p^r}$-vector space scheme over $\mathcal{O}_K$ defined by the r equations $T_1^p = \pi^{s_1} T_2, T_2^p = \pi^{s_1} T_3, \ldots, T_r^p = \pi^{s_r} T_1$ ([13]). We set $j_k = (ps_k + p^2 s_{k-1} + \ldots + p^k s_1 + p^{k+1} s_r + p^{k+2} s_{r-1} + \cdots + p^r s_{k+1})/(p^r - 1)$. Before the calculation of $c(\mathcal{G}(s_1, \ldots, s_r))$, we gather some elementary lemmas.

Lemma 3.1. *Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. Then the affinoid variety $X^j(\bar{K}) = \{x \in \mathcal{O}_{\bar{K}} \mid v_K(x^p - a) \geq j\}$ is equal to*

$$\begin{cases} D(a^{1/p}, j/p) & \text{if } j \leq s + pe/(p-1), \\ \coprod_{i=0}^{p-1} D(a^{1/p}\zeta_p^i, j - e - (p-1)s/p) & \text{if } j > s + pe/(p-1). \end{cases}$$

Proof. We have $v_K(x^p - a) = \sum_i v_K(x - a^{1/p}\zeta_p^i)$. If $v_K(x - a^{1/p}\zeta_p^i) \geq v_K(x - a^{1/p}\zeta_p^{i'})$ for any $i' \neq i$, then $v_K(x - a^{1/p}\zeta_p^{i'}) \leq v_K(a^{1/p}\zeta_p^{i'}(1 - \zeta_p^{i-i'})) = s/p + e/(p-1)$. Thus we have $v_K(x - a^{1/p}\zeta_p^i) \geq \sup(j/p, j - (p-1)s/p - e)$ and

$$X^j(\bar{K}) \subseteq \bigcup_i D(a^{1/p}\zeta_p^i, \sup(j/p, j - (p-1)s/p - e)).$$

Suppose that $j/p \geq j - (p-1)s/p - e$. Then we have $v_K(a^{1/p}(1 - \zeta_p^i)) = s/p + e/(p-1) \geq j/p$, $D(a^{1/p}, j/p) = D(a^{1/p}\zeta_p^i, j/p)$ for any i and thus

$$X^j(\bar{K}) = D(a^{1/p}, j/p).$$

When $j/p < j - (p-1)s/p - e$, we have $v_K(a^{1/p}(1 - \zeta_p^i)) = s/p + e/(p-1) < j - (p-1)s/p - e$. This means that if $w \in D(a^{1/p}\zeta_p^i, j - (p-1)s/p - e)$ for some i, then $v_K(w - a^{1/p}\zeta_p^{i'}) < j - (p-1)s/p - e$ for any other i'. Thus the discs $D(a^{1/p}\zeta_p^i, j - (p-1)s/p - e)$ are disjoint and

$$X^j(\bar{K}) = \coprod_i D(a^{1/p}\zeta_p^i, j - (p-1)s/p - e).$$

These are equalities of the underlying sets of affinoid subdomains of the unit disc over $K(a^{1/p}, \zeta_p)$. By the universality of an affinoid subdomain, this extends to an isomorphism of affinoid varieties.

□

We can prove the following lemma just in the same way.

Lemma 3.2. *The affinoid variety $\{x \in \mathcal{O}_{\bar{K}} \mid v_K(x^{p^r} - ax) \geq j\}$ is equal to*

$$\begin{cases} D(0, j/p^r) & \text{if } j \leq p^r v(a)/(p^r - 1), \\ \coprod_{i=0}^{p^r-1} D(\sigma_i, j - v(a)) & \text{if } j > p^r v(a)/(p^r - 1), \end{cases}$$

where σ_i's are the roots of $X^{p^r} = aX$.

Lemma 3.3. *For $g_1(Y_1, \ldots, Y_d), g_2(Y_1, \ldots, Y_d) \in K[Y_1, \ldots, Y_d]$ and $j_1 \geq j_2$, the affinoid variety $\{(x, y_1, \ldots, y_d) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}}^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(x - g_2(y_1, \ldots, y_d)) \geq j_2\}$ is equal to $\{(x, y_1, \ldots, y_d) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}}^d \mid v_K(x - g_1(y_1, \ldots, y_d)) \geq j_1, v_K(g_1(y_1, \ldots, y_d) - g_2(y_1, \ldots, y_d)) \geq j_2\}$.*

Proof. For fixed (x, y), these two conditions are equivalent. The universality of an affinoid subdomain proves the lemma. $\square$

Lemma 3.4. *Let $a \in \mathcal{O}_K$ and $s = v_K(a)$. If $j \leq pe/(p-1) + s$, then the affinoid variety $X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_K(x^p - ay^{p^n}) \geq j\}$ is equal to $\{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\}$.*

Proof. Lemma 3.1 shows that the fiber of the second projection $X^j(\bar{K}) \to \mathcal{O}_{\bar{K}}$ at y is equal to

$$\begin{cases} D(a^{1/p}y^{p^{n-1}}, j/p) & \text{if } j \leq s + p^{n-1}v_K(y) + pe/(p-1), \\ \coprod_{i=0}^{p-1} D(a^{1/p}\zeta_p^i y^{p^{n-1}}, j - e - (p-1)(s + p^{n-1}v_K(y))/p) & \text{otherwise.} \end{cases}$$

Thus we have $X^j(\bar{K}) = \{(x, y) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_K(x - a^{1/p}y^{p^{n-1}}) \geq j/p\}$ for $j \leq pe/(p-1) + s$. This is the underlying set of a $K(a^{1/p})$-affinoid variety. Again this equality extends to an isomorphism over $K(a^{1/p})$. $\square$

Now we proceed to the proof of the main theorem of this section.

Theorem 3.5. $c(\mathcal{G}(s_1, \ldots, s_r)) = \sup_k j_k$.

Proof. We may assume that j_r is the supremum of j_k's. If $j_r = 0$, then $\mathcal{G}(s_1, \ldots, s_r)$ is etale and $c(\mathcal{G}(s_1, \ldots, s_r)) = 0$. Thus we may assume $j_r > 0$. Consider the homomorphism of $\mathcal{O}_K$-algebras

$$\begin{aligned} A = \mathcal{O}_K[T_1, \ldots, T_r]/({T_1}^p - \pi^{s_1}T_2, \ldots, {T_r}^p - \pi^{s_r}T_1) \to \\ B = \mathcal{O}_K[W, T_2, \ldots, T_r]/(W^{p^r} - \pi^{s_1}T_2, {T_2}^p - \pi^{s_2}T_3, \ldots, \\ {T_{r-1}}^p - \pi^{s_{r-1}}T_r, {T_r}^p - \pi^{s_r}W^{p^{r-1}}), \end{aligned}$$

defined by $T_1 \mapsto W^{p^{r-1}}$. This induces a surjection of K-affinoid varieties

$$X_B^j(\bar{K}) \ni (w, t_2, \ldots, t_r) \mapsto (w^{p^{r-1}}, t_2, \ldots, t_r) \in X_A^j(\bar{K}),$$

where

$$\begin{aligned} X_A^j(\bar{K}) = \{(t_1, \ldots, t_r) \in \mathcal{O}_{\bar{K}}^r \mid v_K({t_1}^p - \pi^{s_1}t_2) \geq j, \ldots, \\ v_K({t_{r-1}}^p - \pi^{s_{r-1}}t_r) \geq j, v_K({t_r}^p - \pi^{s_r}t_1) \geq j\} \end{aligned}$$

and

$$X_B^j(\bar{K}) = \{(w, t_2, \dots, t_r) \in \mathcal{O}_{\bar{K}}^r \mid v_K(w^{p^r} - \pi^{s_1} t_2) \geq j,$$
$$v_K({t_2}^p - \pi^{s_2} t_3) \geq j, \dots, v_K({t_r}^p - \pi^{s_r} w^{p^{r-1}}) \geq j\}.$$

These are affinoid subdomains of the r-dimensional unit polydisc over K. We calculate a jump of $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$ at first.

Lemma 3.6. *If $j_r < pe/(p-1)$, then the first jump of $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$ occurs at $j = j_r$ and $\sharp F^{j_r}(B) = p^r$.*

Note that the base change from K to a finite extension L multiplies s_i's, j_i's and e by the ramification index of L/K. Thus, to prove Lemma 3.6 and Theorem 3.5, we may assume that p^{r-1} divides s_i's and e.

Proof. Consider the K-affinoid variety X_B^j for $j \leq pe/(p-1)$. Then the iterative use of Lemma 3.4 and Lemma 3.3 shows that the affinoid variety $X_B^j(\bar{K})$ is equal to

$$\{v_K(w^{p^r} - \pi^{(s_r + ps_{r-1} + \dots + p^{r-1} s_1)/p^{r-1}} w) \geq pl_1(j),\ v_K(t_2 - g_2(w)) \geq u_2,$$
$$v_K(t_3 - g_3(t_2, w)) \geq u_3, \dots,\ v_K(t_r - g_r(t_{r-1}, w)) \geq u_r\},$$

where $l_i(j)$, $g_i(t_{i-1}, w)$, $g_2(w)$ and u_i are defined as follows;

- $l_r(j) = j/p$,
- $l_{i-1}(j) = \inf(j, l_i(j) + s_{i-1})/p$,
- $g_i(t_{i-1}, w) = t_{i-1}^p / \pi^{s_{i-1}}$ and $u_i = j - s_{i-1}$ if $j \geq l_i(j) + s_{i-1}$,
- $g_i(t_{i-1}, w) = \pi^{s_r + ps_{r-1} + \dots + p^{r-i} s_i / p^{r-i+1}} w^{p^{i-2}}$ and $u_i = l_i(j)$ if $j < l_i(j) + s_{i-1}$,
- $g_2(w) = g_2(w^{p^{r-1}}, w)$.

Note that $l_i(j)$ is a strictly monotone increasing function of j. This affinoid variety is isomorphic to the product of the affinoid variety $\{w \in \mathcal{O}_{\bar{K}} \mid v(w^{p^r} - \pi^{(s_r + ps_{r-1} + \dots + p^{r-1} s_1)/p^{r-1}} w) \geq pl_1(j)\}$ and discs. Therefore, from Lemma 3.2, we see that the first jump of $\{F^j(B)\}_{j \in \mathbb{Q}_{>0}}$ occurs at j such that $pl_1(j) = j_r$, provided this j satisfies $0 < j < pe/(p-1)$. Moreover, then we have $\sharp F^j(B) = p^r$. Thus the following lemma and the strict monotonicity of l_1 terminate the proof of Lemma 3.6. $\square$

Lemma 3.7. $l_1(j_r) = j_r/p$.

Proof. Suppose that there is k such that $l_k(j_r) = j_r/p$ and $j_r \geq l_{k'}(j_r) + s_{k'}$ for any $1 < k' \leq k$. Then we have $l_1(j_r) = \inf(j_r, (j_r + ps_{k-1} + p^2 s_{k-2} + \dots + p^{k-1} s_1)/p^{k-1})/p$ and the assumption $j_{k-1} \leq j_r$ implies $l_1(j_r) = j_r/p$.

$\square$

On the other hand, let $s = (s_r + ps_{r-1} + \ldots + p^{r-1}s_1)/p^{r-1}$ and $\sigma_0, \ldots, \sigma_{p^r-1}$ be the roots of the equation $X^{p^r} - \pi^s X = 0$. Then we see that the images by $w \mapsto w^{p^{r-1}}$ of the discs $D(\sigma_i, pl_1(j) - s)$ are disjoint for $j > j_r$. Hence the surjection $\pi_0(X^j_B(\bar{K})) \to \pi_0(X^j_A(\bar{K}))$ is bijective for $0 < j \leq pe/(p-1)$ and the first (and the last) jump of $\{F^j(A)\}_{j \in \mathbb{Q}_{>0}}$ also occurs at j_r, provided $j_r < pe/(p-1)$.

When $j_r = pe/(p-1)$, we see that $s_k = e > 0$ for any k. Thus we can use Lemma 3.4 for $j < pe/(p-1)+\varepsilon$ with sufficiently small $\varepsilon > 0$. Then, by the same reasoning as above, we conclude that $c(A) = pe/(p-1)$. □

4. Hasse-Arf theorem for $\mathbb{F}_p$-rank two case

Let K be as in section 1. In this section, we prove Conjecture 1.2 in the case where $\mathbb{F} = \mathbb{F}_p$ and $\mathcal{G}(\bar{K})$ is reducible.

Theorem 4.1. *Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ of rank p^2 which is killed by p and reducible. Then the I_K-representation $\mathcal{G}(\bar{K})$ contains $\theta^k_{K,l}$, where $k/l \bmod \mathbb{Z}$ is the prime-to-p part of the conductor $c(\mathcal{G})$.*

We prove this theorem by a lengthy calculation of the conductor. The point is that, on the one hand, to check the assertion on a character, we may restrict to G_{K_∞} by the full faithful theorem of Breuil ([6, Theorem 3.4.3]) and on the other hand, we can describe a defining equation of R_M explicitly in terms of M after the base change to K_∞. By abuse of notation, we may write $F^j(M)$ for $F^j(\mathrm{Gr}(M))$ and $c(M)$ for $c(\mathrm{Gr}(M))$. We fix once and for all a $(p-1)$-st root $\pi_1^{1/(p-1)}$ of π_1 and set $\pi^{1/(p-1)} = \pi_1^{p/(p-1)}$.

Proof. It is sufficient to show the theorem in the case of $k = \bar{k}$. Let $M = \mathrm{Mod}_K(\mathcal{G})$ be the filtered ϕ_1-module of $\mathcal{G}$. By assumption, we have an exact sequence in $'\mathcal{M}$

$$0 \to M(s) \to M \to M(r) \to 0$$

for some integers $0 \leq r, s \leq e$, where $M(s)$ is the filtered ϕ_1-module defined by $M(s) = S_1 e$, $\mathrm{Fil}^1 M(s) = u^s S_1 e$ and $\phi_1(u^s e) = e$. We have $\mathrm{Gr}_K(M(s)) \simeq \mathcal{G}(e-s)$ by the notation of Section 3. By [7, Lemma 5.2.2], we may assume that $\tilde{M} = M/\mathrm{Fil}^p SM$ is of the following type;

- $\tilde{M} = \tilde{S}_1 e_0 \oplus \tilde{S}_1 e_1$, where $\tilde{S} = S/\mathrm{Fil}^p S = k[u]/(u^{ep})$,
- $\mathrm{Fil}^1 \tilde{M} = \langle u^s e_0, u^r e_1 + f e_0 \rangle$, where $f \in u^{\sup(0, r+s-e)} \tilde{S}_1$
- $\phi_1(u^s e_0) = e_0$ and $\phi_1(u^r e_1 + f e_0) = e_1$.

Put $m = v_u(f)$. Then we can take an adapted basis of $\tilde{M}$ as follows.

Case A: (e_0, s), $(e_1 + (f/u^r)e_0, r)$ if $m \geq r$
Case B: (e_0, s), (e_1, r) if $s \leq m < r$
Case C: $((f/u^m)e_0 + u^{r-m}e_1, m)$, $((u^m/f)e_1, r+s-m)$ if $m < r, s$

Before starting calculation of the conductor, we show the following lemma.

Lemma 4.2. *For any exact sequences of finite flat group schemes over $\mathcal{O}_K$*
$$0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0,$$
we have $c(\mathcal{G}_2) \geq c(\mathcal{G}_1), c(\mathcal{G}_3)$.

Proof. The inequality $c(\mathcal{G}_2) \geq c(\mathcal{G}_3)$ follows from [1, Lemme 2.10]. Let us show $c(\mathcal{G}_2) \geq c(\mathcal{G}_1)$. Taking a sufficiently large base change, we may assume that G_K acts trivially on $\mathcal{G}_3(\bar{K})$. Let $\mathcal{H}$ be the maximal prolongation of $\mathcal{G}_3$ ([13]) and $\mathcal{G}' = \mathcal{G}_2 \times_{\mathcal{G}_3} \mathcal{H}$. The group scheme $\mathcal{G}'$ is also finite flat over $\mathcal{O}_K$. We know $\mathcal{H}$ is constant. Thus we have $c(\mathcal{G}') = c(\mathcal{G}_1)$. However, the natural map $\mathcal{G}' \to \mathcal{G}_2$ is a prolongation. Therefore $c(\mathcal{G}') \leq c(\mathcal{G}_2)$ (see the proof of [11, Theorem 7]). $\square$

A. $m \geq r$. In this case, we have
$$\phi_1 \begin{pmatrix} u^s e_0 \\ u^r e_1 + f e_0 \end{pmatrix} = \begin{pmatrix} e_0 \\ e_1 \end{pmatrix} = \begin{pmatrix} u^s e_0 \\ (e_1 + (f/u^r)e_0) - (f/u^r)e_0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ -(f/u^r) & 1 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 + (f/u^r)e_0 \end{pmatrix}.$$
We calculate the conductor $c(R'_M) = pc(R_M)$. From Theorem 2.5, it is equal to $c(\mathrm{Gr}_{K_1}(M')) = c(M')$. We see that (e_0, ps), $(e_1 + (f(v^p)/v^{pr})e_0, pr)$ is an adapted basis of $\tilde{M}'$ and
$$\phi_1 \begin{pmatrix} v^{ps} e_0 \\ v^{pr} e_1 + f(v^p) e_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -f(v^p)/v^{pr} & 1 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 + (f(v^p)/v^{pr})e_0 \end{pmatrix}.$$
Consider the surjection $\mathcal{O}_{K_2} \to \mathcal{O}_{K_2}/p \otimes_{k,\phi} k \simeq k[v]/(v^{ep^2})$ where the last map is k-linear and maps π_1 to u. The matrix above lifts by this surjection to $\begin{pmatrix} 1 & 0 \\ f_0(\pi_1) & 1 \end{pmatrix}$, where $f_0(v) = -f^{\sigma^{-1}}(v)/v^r$. Then, from [5, Proposition 3.1.2], we see that
$$R'_M = \mathcal{O}_{K_1}[X_1, X_2]/(X_1^p + \pi^{e-s}F(\pi)^{-1}X_1, X_2^p + \pi^{e-r}F(\pi)^{-1}(X_2 + f_0(\pi_1)X_1)).$$
Let us calculate the affinoid variety
$$X^j_{M'}(\bar{K}) = \{ (x_1, x_2) \in \mathcal{O}_{\bar{K}_1} \times \mathcal{O}_{\bar{K}_1} \mid v_{K_1}(x_1^p + \pi^{e-s}F(\pi)^{-1}x_1) \geq j,$$
$$v_{K_1}(x_2^p + \pi^{e-r}F(\pi)^{-1}(x_2 + f_0(\pi_1)x_1)) \geq j \ \}.$$

Note that the second inequality is equivalent to

$$v_{K_1}(x_1+f_0(\pi_1)^{-1}x_2+F(\pi)(f_0(\pi_1)\pi^{e-r})^{-1}x_2^p)\geq j-p(e-r)-(m-r).$$

We have $c(M')\geq c_{K_1}(\mathcal{G}(e-s)\times_{\mathcal{O}_K}\mathcal{O}_{K_1})=p^2(e-s)/(p-1)$ from Lemma 4.2. Thus we may suppose $j>p^2(e-s)/(p-1)$ and that the affinoid variety defined by the first inequality in the definition of $X^j_{M'}(\bar{K})$ splits. Thus we have

$$\begin{aligned}X^j_{M'}(\bar{K})=\coprod_{l=0}^{p-1}\{\ (x_1,x_2)\in\mathcal{O}_{\bar{K}_1}\times\mathcal{O}_{\bar{K}_1}\mid v_{K_1}(x_1-\sigma_{s,l})\geq j-p(e-s),\\ v_{K_1}(x_2^p+\pi^{e-r}F(\pi)^{-1}x_2+\pi^{e-r}F(\pi)^{-1}f_0(\pi_1)x_1)\geq j\},\end{aligned}$$

where $\sigma_{s,l}=\begin{cases}0 & (l=0)\\ \pi^{(e-s)/(p-1)}\zeta_{p-1}^l & (l=1,\ldots,p-1).\end{cases}$ Let us denote its l-th component by $X^j_{M',l}$. We have a surjection of G_K-modules $F^j(M')\to F^j(M(s))$ ([1, Lemme 2.10]) and the inverse image of $\sigma_{s,l}\in F^j(M(s))$ by this surjection is equal to $\pi_0(X^j_{M',l})_{\bar{K}}$. Thus $X^j_{M'}$ splits if and only if $X^j_{M',0}$ splits.

If $s\geq r$, we have $v_{K_1}(\pi^{e-r}F(\pi)^{-1}f_0(\pi_1)x_1)\geq j$ and

$$\begin{aligned}X^j_{M',0}(\bar{K})=\{(x_1,x_2)\in\mathcal{O}_{\bar{K}_1}\times\mathcal{O}_{\bar{K}_1}\mid\\ v_{K_1}(x_1)\geq j-p(e-s),\ v_{K_1}(x_2^p+\pi^{e-r}F(\pi)^{-1}x_2)\geq j\ \}.\end{aligned}$$

Thus $c(M')=\sup(p^2(e-s)/(p-1),p^2(e-r)/(p-1))=p^2(e-r)/(p-1)$.

If $s<r$ and $m\geq p(r-s)+r$, Lemma 3.3 shows that $X^j_{M',0}(\bar{K})$ is the same as the case above and we have $c(M')=p^2(e-s)/(p-1)$. Suppose $s<r$ and $m<p(r-s)+r$. Then we have

$$\begin{aligned}X^j_{M',0}(\bar{K})=\{(x_1,x_2)\in\mathcal{O}_{\bar{K}_1}\times\mathcal{O}_{\bar{K}_1}\mid\\ v_{K_1}(x_2^p+\pi^{e-r}F(\pi)^{-1}(x_2+f_0(\pi_1)x_1)\geq j,\\ v_{K_1}(x_2^p+\pi^{e-r}F(\pi)^{-1}x_2)\geq j-(p(r-s)-(m-r))\ \}.\end{aligned}$$

This affinoid variety splits if and only if $j>p^2(e-r)/(p-1)+p(r-s)-(m-r)$. The conductor equals $\sup(p^2(e-r)/(p-1)+p(r-s)-(m-r),p^2(e-s)/(p-1))$. We see that $p^2(e-r)/(p-1)+p(r-s)-(m-r)\geq p^2(e-s)/(p-1)$ if and only if $(ps-r)/(p-1)\geq m$. This does not occur, since we have $s<r\leq m$ and $(ps-r)/(p-1)\leq s<m$. Thus we have $c(M')=p^2(e-s)/(p-1)$.

To terminate the proof of the theorem in Case A, we must show that $\mathcal{G}(\bar{K})$ contains $\theta^{e-s}_{K,p-1}$ if $c(M)\equiv p(e-s)/(p-1)\mod 1/p^\infty\mathbb{Z}$. Note that, if $p(e-r)/(p-1)\equiv p(e-s)/(p-1)\mod 1/p^\infty\mathbb{Z}$, then we have $p^M(e-r)\equiv p^M(e-s)\mod (p-1)\mathbb{Z}$ for some integer M and

$\theta^{e-r}_{K,p-1} = \theta^{e-s}_{K,p-1}$. Thus we may restrict our attention to the case $s < r$. By virtue of the full faithful theorem of Breuil, it suffices to show that the G_{K_1}-module $\mathrm{Gr}(M')(\bar{K})$ contains $\theta^{e-s}_{K_1,p-1}$.

We identify the finite G_{K_1}-set $\mathrm{Gr}(M')(\bar{K})$ with the solution (X, Y) of the equation

$$\begin{cases} X^p + \pi^{e-r}F(\pi)^{-1}X = 0 \\ Y^p + \pi^{e-r}F(\pi)^{-1}Y + \pi^{e-r}f_0(\pi_1)F(\pi)^{-1}X = 0. \end{cases}$$

Consider the equation $Y^p + \pi^{e-r}F(\pi)^{-1}Y + \pi^{e-r}f_0(\pi_1)F(\pi)^{-1}\sigma_{s,l} = 0$. Its Newton polygon has an internal vertex if and only if $(p-1)/p(m-r+p(e-r)+p(e-s)/(p-1)) - p(e-r) > 0$, which is equivalent to $r > s$. Thus there is one and only one root Y_l of this equation which satisfies $v_{K_1}(Y_l) = m - r + p(e-s)/(p-1)$ for each $l > 0$. Define $\alpha \in \mathcal{O}^\times_{\bar{K}}$ by $Y_l = \alpha^{-1}\pi_1^{m-r+p(e-s)/(p-1)}$. Then α is the root of the equation $T^{p-1} + aT^{p-2} + b = 0$, where $a, b \in \mathcal{O}_{K_1}$ with $v_{K_1}(a) = 0$ and $v_{K_1}(b) > 0$. Hensel's lemma shows that $\alpha \in K_1^{nr}$. Thus we have $g(Y_l) = Y_l g(\pi_1^{p(e-s)/(p-1)})/\pi_1^{p(e-s)/(p-1)}$ for any $g \in I_{K_1}$. Denote by P (resp. Q) an element $(X, Y) = (0, \sigma_{r,1})$ (resp. $(X, Y) = (\sigma_{s,1}, Y_1)$) of $V = \mathrm{Gr}(M')(\bar{K})$. From [5, Lemme 3.1.7], we see that the subspace $\mathcal{G}(p(e-r))(\bar{K}) \subseteq V$ can be identified with a subset $\{(0,0), (0,\sigma_{r,1}), \ldots, (0, \sigma_{r,p-1})\}$. Thus P and Q form a basis of V. For the Galois extension $L = K_1(\pi_1^{1/(p-1)})$ of degree $p-1$ over K_1, we see that I_L acts trivially on P and Q. This shows that the image of $I_{K_1} \to \mathrm{Aut}(V)$ has an order prime to p. Therefore we have $V = \theta^{e-s}_{K_1,p-1} \oplus \theta^{e-r}_{K_1,p-1}$ as an I_{K_1}-module. This concludes the proof in the case A.

B. $s \le m < r$**.** We have

$$\begin{aligned} \phi_1 \begin{pmatrix} u^s e_0 \\ u^r e_1 \end{pmatrix} &= \phi_1 \begin{pmatrix} u^s e_0 \\ (u^r e_1 + f e_0) - (f/u^s) u^s e_0 \end{pmatrix} \\ &= \begin{pmatrix} e_0 \\ e_1 - (f^\sigma(u^p)/u^{ps}) e_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -f^\sigma(u^p)/u^{ps} & 1 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \end{pmatrix}. \end{aligned}$$

Set $g_0(u) = -f(u)/u^s$. Then a matrix $\begin{pmatrix} 1 & 0 \\ g_0(\pi) & 1 \end{pmatrix}$ maps to the matrix above by the surjection $\mathcal{O}_{K_1} \to \mathcal{O}_{K_1} \otimes_{k,\sigma} k \simeq k[u]/(u^{ep})$. Then we see that

$$R_M = \mathcal{O}_K[X_1, X_2]/(X_1^p + \pi^{e-s}F(\pi)^{-1}X_1,\ X_2^p + \pi^{e-r}F(\pi)^{-1}(X_2 + g_0(\pi)X_1)).$$

An affinoid variety X_M^j we must calculate is

$$\{\ (x_1,x_2)\in\mathcal{O}_{\bar{K}}\times\mathcal{O}_{\bar{K}}\mid v_K(x_1^p+\pi^{e-s}F(\pi)^{-1}x_1)\geq j,$$
$$v_K(x_1+g_0(\pi)^{-1}x_2+F(\pi)(g_0(\pi)\pi^{e-r})^{-1}x_2^p)\geq j-(e-r)-(m-s)\ \}$$

Again it is sufficient to suppose $j>p(e-s)/(p-1)$ and consider an affinoid variety

$$\{\ (x_1,x_2)\in\mathcal{O}_{\bar{K}}\times\mathcal{O}_{\bar{K}}\mid v_K(x_1)\geq j-(e-s),$$
$$v_K(x_1+g_0(\pi)^{-1}x_2+F(\pi)(g_0(\pi)\pi^{e-r})^{-1}x_2^p)\geq j-(e-r)-(m-s)\ \}.$$

By the assumption $m<r$ and Lemma 3.3, this is equal to an affinoid variety

$$\{\ (x_1,x_2)\in\mathcal{O}_{\bar{K}}\times\mathcal{O}_{\bar{K}}\mid v_K(x_2^p+\pi^{e-r}F(\pi)^{-1}x_2)\geq j-(r-m),$$
$$v_K(x_1+g_0(\pi)^{-1}x_2+F(\pi)(g_0(\pi)\pi^{e-r})^{-1}x_2^p)\geq j-(e-r)-(m-s)\ \},$$

which splits if and only if $j>r-m+p(e-r)/(p-1)$. Therefore we get $c(M)=r-m+p(e-r)/(p-1)$ if $m\leq(ps-r)/(p-1)$ and $p(e-s)/(p-1)$ if $(ps-r)/(p-1)<m<r$. In the latter case, the verbatim arguments as in Case A shows that $\mathcal{G}(\bar{K})=\theta_{K,p-1}^{e-s}\oplus\theta_{K,p-1}^{e-r}$ as an I_K-module.

C. $r,s>m$**.** In this case,

$$\begin{aligned}\phi_1\begin{pmatrix}u^re_1+fe_0\\(u^{r+s}/f)e_1\end{pmatrix}&=\phi_1\begin{pmatrix}u^re_1+fe_0\\(u^s/f)(u^re_1+fe_0)-u^se_0\end{pmatrix}\\&=\begin{pmatrix}e_1\\(u^{ps}/f^\sigma(u^p))e_1-e_0\end{pmatrix}\\&=\begin{pmatrix}0&f/u^m\\-u^m/f&(f/u^m)(u^r/f+(u^s/f)^p)\end{pmatrix}\begin{pmatrix}(f/u^m)e_0+u^{r-m}e_1\\(u^m/f)e_1\end{pmatrix}.\end{aligned}$$

Again we consider $M'=M\otimes_S S'$. Then the last matrix is equal to $\begin{pmatrix}0&f(v^p)/v^{pm}\\-v^{pm}/f(v^p)&(f(v^p)/v^{pm})(v^{pr}/f(v^p)+(v^{ps}/f(v^p))^p)\end{pmatrix}$. We can take as a lifting of this matrix $\begin{pmatrix}0&c\\-1/c&c(\pi_1^{r-m}/c+(\pi_1^{s-m}/c)^p)\end{pmatrix}\in\mathrm{GL}_2(\mathcal{O}_{K_1})$ with $c=f^{\sigma^{-1}}(\pi_1)/\pi_1^m$. Thus we get

$$\begin{aligned}R'_M=\mathcal{O}_{K_1}[X_1,X_2]/(X_1^p&+c\pi^{e-m}F(\pi)^{-1}X_2,\\&X_2^p+\pi^{e+m-(r+s)}F(\pi)^{-1}(-c^{-1}X_1+dX_2)),\end{aligned}$$

where $d=c(\pi_1^{r-m}/c+(\pi_1^{s-m}/c)^p)$. As in Section 3, we firstly calculate the conductor of $\tilde{R}'_M=R'_M[W]/(W^p-X_2)$.

Consider an affinoid variety

$$Y^j_{M'}(\bar{K}) = \{\ (x,w) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_{K_1}(x^p + \pi^{e-m}F(\pi)^{-1}cw^p) \geq j,\\ v_{K_1}(w^{p^2} + \pi^{e+m-(r+s)}F(\pi)^{-1}(-c^{-1}x + dw^p)) \geq j\ \}.$$

From the bound of the conductor [11, Theorem 7], we may suppose $j \leq p^2e/(p-1) + \varepsilon$ with $\varepsilon > 0$ sufficiently small. Now we have $e > m$, and Lemma 3.4 shows that the first inequality in the definition of $Y^j_{M'}$ is equivalent to $v_{K_1}(x + \pi_1^{e-m}c^{1/p}F(\pi)^{-1/p}w) \geq j/p$. On the other hand, the second inequality can be written also as $v_{K_1}(x - cF(\pi)\pi^{r+s-e-m}w^{p^2} - cdw^p) \geq j - p(e+m-(r+s))$. Put $\lambda_0 = \pi_1^{(p+1)e+(p-1)m-p(r+s)}/(F(\pi)^{(p+1)/p}c^{(p-1)/p})$ and $\lambda_1 = d\pi_1^{p(e+m-(r+s))}/F(\pi)$. Using Lemma 3.3, we see that this affinoid variety is equal to

$$Y^j_{M'}(\bar{K}) = \{\ (x,w) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_{K_1}(x + \pi_1^{e-m}c^{1/p}F(\pi)^{-1/p}w) \geq j/p,\\ v_{K_1}(w^{p^2} + \lambda_1 w^p + \lambda_0 w) \geq j\ \}$$

if $j \leq p^2(e+m-(r+s))/(p^2-1)$ and

$$Y^j_{M'}(\bar{K}) = \{\ (x,w) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid\\ v_{K_1}(w^{p^2} + \pi^{e+m-(r+s)}F(\pi)^{-1}(-c^{-1}x + dw^p)) \geq j,\\ v_{K_1}(w^{p^2} + \lambda_1 w^p + \lambda_0 w) \geq j/p + p(e+m-(r+s))\ \}$$

if $j > p^2(e+m-(r+s))/(p^2-1)$.

Lemma 4.3. *Put $P(W) = W^{p^2} + \lambda_1 W^p + \lambda_0 W$. Then an affinoid variety $\{\ w \in \mathcal{O}_{\bar{K}} \mid v_{K_1}(P(w)) \geq j\ \}$ splits if and only if*

$$j > \begin{cases} p^2(e-r)/(p-1) - (p(s-m) - (r-m)) & \\ & \textit{if } m < (ps-r)/(p-1) \\ p^2(e-s)/(p-1) - p(r-m) & \textit{if } m \geq (ps-r)/(p-1). \end{cases}$$

Proof. Set $\mu_k = v_{K_1}(\lambda_k)$. Consider the Newton polygon of the polynomial $P(W)$. We have $\mu_0(p^2-p)/(p^2-1) - \mu_1 = p(r-m+s-m)/(p+1) - v_{K_1}(d)$ and

$$v_{K_1}(d) = \begin{cases} r-m & \text{if } m < (ps-r)/(p-1) \\ r-m+v_{K_1}(c+c^p) & \text{if } m = (ps-r)/(p-1) \\ p(s-m) & \text{if } m > (ps-r)/(p-1). \end{cases}$$

In the first and third case, $P(W)$ has $p-1$ roots of valuation $(\mu_0 - \mu_1)/(p-1)$ and p^2-p roots of valuation $\mu_1/(p^2-p)$. Let w be one of these roots and $V = W - w$. Then $P(V+w) = V^{p^2} + ({}_{p^2}\mathrm{C}_p w^{p^2-p} + \lambda_1)V^p + (p^2 w^{p^2-1} + p\lambda_1 w^{p-1} + \lambda_0)V + \sum_{k=2,\ldots,p-1,p+1,\ldots,p^2-1} {}_{p^2}\mathrm{C}_k w^{p^2-k}V^k + \sum_{k=2,\ldots,p-1} {}_p\mathrm{C}_k w^{p-k}V^k$. We see that $v_{K_1}({}_{p^2}\mathrm{C}_p w^{p^2-p}) = ep + (p^2 -$

$p)v_{K_1}(w) \geq ep + \mu_1 > \mu_1$, $v_{K_1}(p^2 w^{p^2-1}) - v_{K_1}(p\lambda_1 w^{p-1}) = ep + (p^2 - p)v_{K_1}(w) - \mu_0 > 0$ and $v_{K_1}(p\lambda_1 w^{p-1}) - \mu_0 = ep + \mu_1 + (p-1)v_{K_1}(w) - \mu_0 \geq ep - (\mu_0 - \mu_1(p+1)/p) > 0$. As for the former summation in the expansion above of $P(V+w)$, $v_{K_1}({}_{p^2}\mathrm{C}_k w^{p^2-k}) - (p^2-k)v(w) > 0$. As for the latter summation, the valuation of the coefficient of V^k is $v_{K_1}({}_p\mathrm{C}_k w^{p-k})$ and $v_{K_1}({}_p\mathrm{C}_k w^{p-k}) - (\mu_1 + (\mu_0 - \mu_1)(p-k)/(p-1)) \geq ep + \mu_1(p-k)/(p^2 - p) - (\mu_1 + (\mu_0 - \mu_1)(p-k)/(p-1)) = ep + (k(p\mu_0 - (p+1)\mu_1) + 2p\mu_1 - p^2\mu_0)/(p^2 - p) > ep + (2\mu_1 - p\mu_0)/(p-1) > 0$. Thus $P(V+w)$ has the same Newton polygon as $P(W)$. Then [11, Theorem 4] shows that the affinoid variety splits if and only if $j > (p\mu_0 - \mu_1)/(p-1)$ and the lemma follows.

In the second case, the Newton polygon of $P(W)$ has no internal vertex. Thus the nonzero roots of $P(W)$ has valuation $\mu_0/(p^2-1)$. Take one of these roots w and consider the polynomial $P(V+w)$ and its expansion as above. We have $v_{K_1}(p\lambda_1 w^{p-1}) - \mu_0 = ep + \mu_1 - p\mu_0/(p+1) > 0$ and $v_{K_1}(p^2 w^{p^2-1}) - \mu_0 = 2ep > 0$. Thus the valuation of the coefficient of V in $P(V+w)$ is μ_0. Let us show that the valuation of the coefficient of V^k is larger than $v_{K_1}((p^2-k)\mu_0/(p^2-1)) = (p^2-k)v_{K_1}(w)$ for any $k < p^2$. For $k = p$, we have $\mu_1 \geq (p^2-p)\mu_0/(p^2-1)$ and $v_{K_1}({}_{p^2}\mathrm{C}_p w^{p^2-p}) > (p^2-p)\mu_0/(p^2-1)$. As for the former summation in the expansion above, $v_{K_1}({}_{p^2}\mathrm{C}_k w^{p^2-k}) - (p^2-k)v(w) > 0$. As for the latter summaton, $v_{K_1}({}_p\mathrm{C}_k w^{p-k}) - (p^2-k)v_{K_1}(w) = ep - p\mu_0/(p+1) > 0$. Again we see that the Newton polygon of $P(V+w)$ is the same as that of $P(W)$ and the affinoid variety splits if and only if $j > p^2\mu_0/(p^2-1)$. This concludes the lemma.

□

From this lemma, we see that the affinoid variety $Y^j_{M'}$ does not split for $j \leq p^2(e+m-(r+s))/(p^2-1)$ and splits if and only if $j' = p(e+m-(r+s)) + j/p$ satisfies the inequality of the lemma. Thus we have

$$c(\tilde{R}'_M) = \begin{cases} p^2(e-r)/(p-1) + p(r-m) & \text{if } m < (ps-r)/(p-1) \\ p^2(e-s)/(p-1) & \text{if } m \geq (ps-r)/(p-1). \end{cases}$$

Now we consider the affinoid variety

$$X^j_{M'}(\bar{K}) = \{\ (x_1, x_2) \in \mathcal{O}_{\bar{K}} \times \mathcal{O}_{\bar{K}} \mid v_{K_1}(x_1^p + \pi^{e-m}F(\pi)^{-1}cx_2) \geq j,$$
$$v_{K_1}(x_2^{p^2} + \pi^{e+m-(r+s)}F(\pi)^{-1}(-c^{-1}x_1 + dx_2)) \geq j\ \}.$$

The map $R'_M \to \tilde{R}'_M$ induces the affinoid map $f : Y^j_{M'} \to X^j_{M'}$. Note that f sends (x, w) to (x, w^p) and is surjective. From the proof of [11, Theorem 4], we see that

$$Y^j_{M'}(\bar{K}) = \coprod_{k=0}^{p^2-1} \{ (x,w) \in \mathcal{O}_{\bar{K}} \times D(w_k, j' - \sum_{w_k \neq 0} v_{K_1}(w_k)) \mid$$
$$v_{K_1}(w^{p^2} + \pi^{e+m-(r+s)} F(\pi)^{-1}(-c^{-1}x + dw^p)) \geq j\}$$

for $j > c(\tilde{R}'_M)$, where $j' = p(e+m-(r+s)) + j/p$ and w_k's is the roots of the polynomial $P(W)$. Let us denote its k-th component by Y_k. We claim that $f(Y_k) \cap f(Y_l) = \emptyset$ for $k \neq l$. Suppose that $(x,w) \in Y_k$ and $(x, w\zeta_p^i) \in Y_l$. Then $v_{K_1}(w\zeta_p^i - w_l) = v_{K_1}((w-w_k)\zeta_p^i + (\zeta_p^i - 1)w_k + (w_k - w_l))$. Now we have $v_{K_1}((w-w_k)\zeta_p^i) \geq j' - \sum v_{K_1}(w_k) > \sup v_{K_1}(w_k)$ and thus $v_{K_1}((w-w_k)\zeta_p^i + (w_k - w_l)) = v_{K_1}(w_k - w_l)$. If $m = (ps - r)/(p-1)$, we have $v_{K_1}(w_k) = v_{K_1}(w_k - w_l)$ for any $k \neq l$ and therefore $v_{K_1}(w\zeta_p^i - w_l) = v_{K_1}(w_k) < j' - \sum v_{K_1}(w_k)$, which is a contradiction. Suppose $m \neq (ps-r)/(p-1)$. Then, by the notation in the previous lemma, we have $v_{K_1}(w_k - w_l) - v_{K_1}(w_k) \leq (\mu_0 - \mu_1)/(p-1) - \mu_1/(p^2 - p)$, which equals

$$\begin{cases} (p(s-m) - (r-m))/(p^2-p) & \text{if } m < (ps-r)/(p-1) \\ ((r-m) - p(s-m))/(p-1) & \text{if } m > (ps-r)/(p-1). \end{cases}$$

We see that these values are strictly smaller than $v_{K_1}(1 - \zeta_p^i) = pe/(p-1)$ and $v_{K_1}(w\zeta_p^i - w_l) = v_{K_1}(w_k - w_l) \leq \sup v_{K_1}(w_k) < j' - \sum v_{K_1}(w_k)$. Again this is a contradiction. Therefore we get $\sharp \pi_0(X^j_{M'})_{\bar{K}} = p^2$. For $j \leq c(\tilde{R}'_M)$, we have $\sharp\pi_0(X^j_{M'})_{\bar{K}} < p^2$ by the surjectivity of f. Thus $c(M') = c(\tilde{R}'_M)$.

Next we prove the assertion on a character. For $m = (ps-r)/(p-1)$, we have $s \equiv r \mod p-1$ and the I_K-module $V = \mathcal{G}(\bar{K})$ contains $\theta^{e-s}_{K,p-1} = \theta^{e-r}_{K,p-1}$. Thus we may suppose that $m > (ps-r)/(p-1)$. By the full faithful theorem of Breuil, it suffices to show that V contains $\theta^{e-s}_{K,p-1}$ as an I_{K_1}-module. The G_{K_1}-set V is identified with the roots of the polynomial $Q(X_2) = (X_2^p + \lambda_1 X_2)^p + \lambda_0^p X_2 \in \mathcal{O}_{K_1}[X_2]$. Consider the Newton polygon of $Q(X_2)$. For $1 \leq k \leq p-1$, the coefficient of $X_2^{p+(p-1)k}$ in $Q(X_2)$ is ${}_p\mathrm{C}_k \lambda_1^{p-k}$ and $v_{K_1}({}_p\mathrm{C}_k\lambda_1^{p-k}) - p\mu_1(p^2 - (p+(p-1)k))/(p^2-p) = ep > 0$. Thus $Q(X_2)$ has $p-1$ roots of valuation $p(\mu_0 - \mu_1)/(p-1) = p(e-s)/(p-1) - p(s-m)$. Put $X_2 = T^{-1}\pi_1^{p(e-s)/(p-1)-p(s-m)}$. Then $Q(X_2) = 0$ if and only if $T^{p^2-1} + a_0^{-1}(a_1 T^{p-1} + \pi_1^{p\mu_0 - (p+1)\mu_1})^p = 0$, where $a_k = \lambda_k / \pi_1^{v_{K_1}(\lambda_k)}$. By Hensel's lemma, there exists a polynomial $R(T) \in \mathcal{O}_{K_1}[T]$ of degree $p-1$, satisfying $R(T) \equiv T^{p-1} + t \mod \pi_1$ where $t \neq 0 \in k$ and with the property that $R(\alpha) = 0$ if and only if $\alpha^{-1}\pi_1^{p(e-s)/(p-1)-p(s-m)}$ is a root of $Q(X_2)$ with valuation $p(e-s)/(p-1)-p(s-m)$. Take such a root w. Then we see that $w/\pi_1^{p(e-s)/(p-1)-p(s-m)} \in K_1^{nr}$ and $g(w) = wg(\pi_1^{p(e-s)/(p-1)})/\pi_1^{p(e-s)/(p-1)}$ for any $g \in I_{K_1}$. By [5, Lemme 3.1.7], we can identify the subspace $\mathcal{G}(p(e-r))(\bar{K})$ of V with the set $\{0, c^{-1}\pi_1^{p(e-r)/(p-1)}, c^{-1}\pi_1^{p(e-r)/(p-1)}\zeta_{p-1}, \ldots, c^{-1}\pi_1^{p(e-r)/(p-1)}\zeta_{p-1}^{p-2}\}$. From the shape of the Newton polygon of $Q(X_2)$, we see that w is not contained in this subspace. Therefore the I_{K_1}-action on V is tame and thus $V = \theta_{K_1,p-1}^{e-s} \oplus \theta_{K_1,p-1}^{e-r}$.

□

In the proof of the theorem, we have shown the following.

Corollary 4.4. *If $s, m \geq r$, then $c(\mathcal{G}) = p(e-r)/(p-1)$. Otherwise,*

$$c(\mathcal{G}) = \begin{cases} \sup(p(e-r)/(p-1), p(e-s)/(p-1)) & \text{if } m \geq (ps-r)/(p-1), \\ p(e-r)/(p-1) + (r-m) & \text{if } m < (ps-r)/(p-1). \end{cases}$$

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