DUALITY OF DRINFELD MODULES AND $\wp$-ADIC PROPERTIES OF DRINFELD MODULAR FORMS

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Abstract. Let p be a rational prime and q a power of p. Let $\wp$ be a monic irreducible polynomial of degree d in $\mathbb{F}_q[t]$. In this paper, we define an analogue of the Hodge-Tate map which is suitable for the study of Drinfeld modules over $\mathbb{F}_q[t]$ and, using it, develop a geometric theory of $\wp$-adic Drinfeld modular forms similar to Katz's theory in the case of elliptic modular forms. In particular, we show that for Drinfeld modular forms with congruent Fourier coefficients at ∞ modulo $\wp^n$, their weights are also congruent modulo $(q^d-1)p^{\lceil \log_p(n) \rceil}$, and that Drinfeld modular forms of level $\Gamma_1(\mathfrak{n}) \cap \Gamma_0(\wp)$, weight k and type m are $\wp$-adic Drinfeld modular forms for any tame level $\mathfrak{n}$ with a prime factor of degree prime to $q-1$.

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1. Introduction

Let p be a rational prime and q a power of p. The theory of p-adic modular forms, which originated from the work of Serre [Ser], has been highly developed, and now we have various p-adic families of eigenforms which play important roles in modern number theory. At the early stage of its development, Katz [Kat] initiated a geometric treatment of p-adic modular forms, and from the work of Katz to recent works on geometric study of p-adic modular forms including [AIS, AIP, Pil], one of the key ingredients is the theory of canonical subgroups of abelian varieties and Hodge-Tate maps for finite locally free (commutative) group schemes.

Let us briefly recall the definition. For a finite locally free group scheme $\mathcal{G}$ over a scheme S, we denote by $\omega_{\mathcal{G}}$ the sheaf of invariant differentials of $\mathcal{G}$ and by $\mathrm{Car}(\mathcal{G})$ the Cartier dual of $\mathcal{G}$. Then the Hodge-Tate map for $\mathcal{G}$ is by definition

$$\mathrm{Car}(\mathcal{G}) = \mathscr{H}om_S(\mathcal{G}, \mathbb{G}_{\mathrm{m}}) \to \omega_{\mathcal{G}}, \quad x \mapsto x^*\left(\frac{dT}{T}\right).$$

It can be considered as a comparison map between the etale side and the de Rham side; in fact, for any abelian scheme $\mathcal{A}$ with ordinary reduction over a complete discrete valuation ring $\mathcal{O}$ of mixed characteristic $(0,p)$, the Cartier dual $\mathrm{Car}(\mathcal{A}[p^n]^0)$ of the unit component of $\mathcal{A}[p^n]^0$ is etale, and the Hodge-Tate map gives an isomorphism of $\mathcal{O}/(p^n)$-modules

$$\mathrm{Car}(\mathcal{A}[p^n]^0) \otimes_{\mathbb{Z}} \mathcal{O} \to \omega_{\mathcal{A}} \otimes_{\mathcal{O}} \mathrm{Spec}(\mathcal{O}/(p^n)).$$

Moreover, if $\mathcal{A}$ is close enough to having ordinary reduction, then there exists a canonical subgroup of $\mathcal{A}$ which has a similar comparison property via the Hodge-Tate map, instead of $\mathcal{A}[p^n]^0$.

On the other hand, an analogue of the theory of p-adic modular forms in the function field case—the theory of v-adic modular forms—has also been actively investigated in this decade (see for example [Gos2, Pet, Vin]). A Drinfeld modular form is a rigid analytic function on the Drinfeld upper half plane over $\mathbb{F}_q((1/t))$, and it can be viewed as a section of an automorphic line bundle over a Drinfeld modular curve. The latter is a moduli space over $\mathbb{F}_q(t)$ classifying Drinfeld modules (of rank two), which are analogues of elliptic curves. It is widely believed that, for each finite place v of $\mathbb{F}_q(t)$, Drinfeld modular forms have deep

v-adic structures comparable to the p-adic theory of modular forms. However, we do not fully understand what it is like yet.

What is lacking is a geometric description of v-adic modular forms as in [Kat]. For this, the problem is that the usual Cartier duality does not work in the Drinfeld case: Since Drinfeld modules are additive group schemes, the Cartier dual of any non-trivial finite locally free closed subgroup scheme of a Drinfeld module is never etale and we cannot obtain an etale-to-de Rham comparison isomorphism via the Hodge-Tate map.

In this paper, we resolve this and develop a geometric theory of v-adic Drinfeld modular forms. In particular, we show the following theorems.

Theorem 1.1 (Corollary to Theorem 4.15)**.** *Let $\mathfrak{n}$ be a monic polynomial in $A = \mathbb{F}_q[t]$ and $\wp$ a monic irreducible polynomial in A which is prime to $\mathfrak{n}$. For $i = 1, 2$, let f_i be a Drinfeld modular form of level $\Gamma_1(\mathfrak{n})$, weight k_i and type m_i. Suppose that their Fourier expansions $(f_i)_\infty(x)$ at ∞ in the sense of Gekeler* [Gek3] *have coefficients in the localization $A_{(\wp)}$ of A at $(\wp)$ and satisfy the congruences*

$$(f_1)_\infty(x) \equiv (f_2)_\infty(x) \bmod \wp^n, \quad (f_2)_\infty(x) \not\equiv 0 \bmod \wp.$$

Then we have

$$k_1 \equiv k_2 \bmod (q^d - 1)p^{l_p(n)}, \quad l_p(n) = \min\{N \in \mathbb{Z} \mid p^N \geqslant n\}.$$

Theorem 1.2 (Theorem 4.19)**.** *Suppose that $\mathfrak{n}$ has a prime factor of degree prime to $q - 1$. Let f be a Drinfeld modular form of level $\Gamma_1(\mathfrak{n}) \cap \Gamma_0(\wp)$, weight k and type m such that Gekeler's Fourier expansion $f_\infty(x)$ at ∞ has coefficients in $A_{(\wp)}$. Then f is a $\wp$-adic Drinfeld modular form. Namely, $f_\infty(x)$ is the $\wp$-adic limit of Fourier expansions of Drinfeld modular forms of level $\Gamma_1(\mathfrak{n})$, type m and some weights.*

As a corollary, we define a notion of weight for "$\wp$-adic Drinfeld modular forms in the sense of Serre" (Definition 4.17). Note that Theorem 1.1 generalizes [Gek3, Corollary (12.5)] of the case $n = 1$, and Theorem 1.2 is a variant of [Vin, Theorem 4.1] with non-trivial tame level $\mathfrak{n}$.

The novelty of this paper lies in the systematic use of the duality theory of Taguchi [Tag] for Drinfeld modules and a certain class of finite locally free group schemes called finite v-modules. Using Taguchi's duality, we define an analogue of the Hodge-Tate map, which we refer to as the Hodge-Tate-Taguchi map. For a Drinfeld module E with ordinary reduction, we construct canonical subgroups of E such that their Taguchi duals are etale and the Hodge-Tate-Taguchi maps for them give isomorphisms between the etale and de Rham sides similar to the case of elliptic curves. Moreover, a study of Taguchi's duality for Drinfeld modules, including the invariance of a Hodge height under the duality (Proposition 3.4), compensates the lack of autoduality for them and yields the vanishing of the higher cohomology groups for a Hodge bundle (Corollary 4.2) and an analogue of Igusa's theorem (Lemma 4.9). These enable us to prove the above theorems by almost verbatim arguments as in [Kat].

The organization of this paper is as follows. In §2, we review Taguchi's duality theory. Here we need a description of the duality for Drinfeld modules in terms of biderivations [Gek4], which is done by Papanikolas-Ramachandran [PR] in the case over fields. For this reason, we follow the exposition of [PR] and generalize their results to general bases.

In §3, we develop the theory of canonical subgroups of Drinfeld modules with ordinary reduction and Hodge-Tate-Taguchi maps. In our case, the role of μ_{p^n} for elliptic curves is played by the $\wp^n$-torsion part $C[\wp^n]$ of the Carlitz module C, where the dual of $C[\wp^n]$ in the sense of Taguchi is the constant A-module scheme $\underline{A/(\wp^n)}$.

Then in §4 we prove the main theorems in a similar way to [Kat, Chapter 4], the point being the fact that the Riemann-Hilbert correspondence of Katz over the truncated Witt ring $W_n(\mathbb{F}_q)$ [Kat, Proposition 4.1.1] can be suitably generalized to the case over $A/(\wp^n)$.

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2. Taguchi duality

In this section, we review the duality theory for Drinfeld modules of rank two and an analogue of Cartier duality for this context, which are both due to Taguchi [Tag]. Let p be a rational prime, q a p-power and $\mathbb{F}_q$ the finite field with q elements. We put $A = \mathbb{F}_q[t]$. For any scheme S over $\mathbb{F}_q$, we denote the q-th power Frobenius map on S by $F_S : S \to S$. For any S-scheme T and $\mathcal{O}_S$-module $\mathcal{L}$, we put $T^{(q)} = T \times_{S,F_S} S$ and $\mathcal{L}^{(q)} = F_S^*(\mathcal{L})$. Note that for any $\mathcal{O}_S$-algebra $\mathcal{A}$, the q-th power Frobenius map induces an $\mathcal{O}_S$-algebra homomorphism

$f_{\mathcal{A}} : \mathcal{A}^{(q)} \to \mathcal{A}$. For any A-scheme S, the image of $t \in A$ by the structure map $A \to \mathcal{O}_S(S)$ is denoted by θ.

2.1. **Line bundles and Drinfeld modules.** For any scheme S over $\mathbb{F}_q$ and any invertible $\mathcal{O}_S$-module $\mathcal{L}$, we write the associated covariant and contravariant line bundles to $\mathcal{L}$ as

$$\mathbb{V}_*(\mathcal{L}) = \mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{L}^{\otimes -1})), \quad \mathbb{V}^*(\mathcal{L}) = \mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{L}))$$

with $\mathcal{L}^{\otimes -1} := \mathcal{L}^\vee = \mathscr{H}om_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_S)$. Note that they represent functors over S defined by

$$T \mapsto \mathcal{L}|_T(T), \quad T \mapsto \mathcal{L}^{\otimes -1}|_T(T),$$

where $\mathcal{L}|_T$ and $\mathcal{L}^{\otimes -1}|_T$ denote the pull-backs to T. The additive group $\mathbb{G}_{\mathrm{a}}$ acts on the group schemes $\mathbb{V}_*(\mathcal{L})$ and $\mathbb{V}^*(\mathcal{L})$ through the natural actions of $\mathcal{O}_T(T)$ on $\mathcal{L}|_T(T)$ and $\mathcal{L}^{\otimes -1}|_T(T)$, respectively. We often identify $\mathcal{L}$ with $\mathbb{V}_*(\mathcal{L})$. We have the q-th power Frobenius map

$$\tau : \mathcal{L} \to \mathcal{L}^{\otimes q}, \quad l \mapsto l^{\otimes q}.$$

This map induces a homomorphism of group schemes over S

$$\tau : \mathbb{V}_*(\mathcal{L}) \to \mathbb{V}_*(\mathcal{L}^{\otimes q}).$$

Note that τ also induces an $\mathcal{O}_S$-linear isomorphism $\mathcal{L}^{(q)} \to \mathcal{L}^{\otimes q}$, by which we identify $\mathbb{V}_*(\mathcal{L}^{\otimes q})$ with $\mathbb{V}_*(\mathcal{L})^{(q)}$. Then the relative q-th Frobenius map $\mathbb{V}_*(\mathcal{L}) \to \mathbb{V}_*(\mathcal{L})^{(q)} = \mathbb{V}_*(\mathcal{L}^{\otimes q})$ is induced by the natural inclusion

$$\mathrm{Sym}(\mathcal{L}^{\otimes -q}) \to \mathrm{Sym}(\mathcal{L}^{\otimes -1}). \tag{2.1}$$

For $S = \mathrm{Spec}(B)$ and $\mathcal{L} = \mathcal{O}_S$, we have $\mathbb{V}_*(\mathcal{O}_S) = \mathbb{G}_{\mathrm{a}}$ and τ induces the endomorphism of $\mathbb{G}_{\mathrm{a}} = \mathrm{Spec}(B[X])$ over B defined by $X \mapsto X^q$. This gives the equality

$$\mathrm{End}_{\mathbb{F}_q,S}(\mathbb{G}_{\mathrm{a}}) = B\{\tau\},$$

where $B\{\tau\}$ is the skew polynomial ring over B whose multiplication is defined by $a\tau^i \cdot b\tau^j = ab^{q^i}\tau^{i+j}$ for any $a, b \in B$.

Definition 2.1 ([Lau], Remark (1.2.2))**.** Let S be a scheme over A and r a positive integer. A (standard) Drinfeld (A-)module of rank r over S is a pair $E = (\mathcal{L}, \Phi^E)$ of an invertible sheaf $\mathcal{L}$ on S and an $\mathbb{F}_q$-algebra homomorphism

$$\Phi^E : A \to \mathrm{End}_S(\mathbb{V}_*(\mathcal{L}))$$

satisfying the following conditions for any $a \in A \backslash \{0\}$:

- the image Φ^E_a of a by Φ^E is written as
$$\Phi^E_a = \sum_{i=0}^{r\deg(a)} \alpha_i(a)\tau^i, \quad \alpha_i(a) \in \mathcal{L}^{\otimes 1-q^i}(S)$$
with $\alpha_{r\deg(a)}(a)$ nowhere vanishing.
- $\alpha_0(a)$ is equal to the image of a by the structure map $A \to \mathcal{O}_S(S)$.

We often refer to the underlying A-module scheme $\mathbb{V}_*(\mathcal{L})$ as E. A morphism $(\mathcal{L},\Phi) \to (\mathcal{L}',\Phi')$ of Drinfeld modules over S is defined to be a morphism of A-module schemes $\mathbb{V}_*(\mathcal{L}) \to \mathbb{V}_*(\mathcal{L}')$ over S. The category of Drinfeld modules over S is denoted by DM_S.

We denote the Carlitz module over S by C: it is the Drinfeld module $(\mathcal{O}_S,\Phi^C)$ of rank one over S defined by $\Phi^C_t = \theta+\tau$. We identify the underlying group scheme of C with $\mathbb{G}_{\mathrm{a}} = \mathrm{Spec}_S(\mathcal{O}_S[Z])$ using $1 \in \mathcal{O}_S(S)$.

2.2. **φ-modules and v-modules.** Let S be a scheme over A. Let $\mathcal{G}$ be an $\mathbb{F}_q$-module scheme $\mathcal{G}$ over S whose structure map $\pi:\mathcal{G}\to S$ is affine. Note that the additive group $\mathbb{G}_{\mathrm{a}}$ over S is endowed with a natural action of $\mathbb{F}_q$. Put
$$\mathcal{E}_\mathcal{G} = \mathscr{H}om_{\mathbb{F}_q,S}(\mathcal{G},\mathbb{G}_{\mathrm{a}}),$$
the $\mathcal{O}_S$-module of $\mathbb{F}_q$-linear homomorphisms $\mathcal{G}\to\mathbb{G}_{\mathrm{a}}$ over S. The Zariski sheaf $\mathcal{E}_\mathcal{G}$ is naturally considered as an $\mathcal{O}_S$-submodule of $\pi_*(\mathcal{O}_\mathcal{G})$.

On the other hand, if the formation of $\mathcal{E}_\mathcal{G}$ commutes with any base change, then the relative q-th Frobenius map $F_{\mathcal{G}/S}:\mathcal{G}\to\mathcal{G}^{(q)}$ defines an $\mathcal{O}_S$-linear map
$$\varphi_\mathcal{G}:\mathcal{E}_{\mathcal{G}^{(q)}} = \mathcal{E}_\mathcal{G}^{(q)} \to \mathcal{E}_\mathcal{G}$$
which commutes with $\mathbb{F}_q$-actions.

Definition 2.2. We say an $\mathbb{F}_q$-module scheme $\mathcal{G}$ over S is a φ-module over S if the following conditions hold:

- the structure morphism $\pi:\mathcal{G}\to S$ is affine,
- the $\mathcal{O}_S$-module $\mathcal{E}_\mathcal{G}$ is locally free (not necessarily of finite rank) and its formation commutes with any base change,
- the induced $\mathbb{F}_q$-action on the sheaf of invariant differentials $\omega_\mathcal{G}$ agrees with the action via the structure map $\mathbb{F}_q\to\mathcal{O}_S(S)$,
- the natural $\mathcal{O}_S$-algebra homomorphism $\mathcal{S} := \mathrm{Sym}_{\mathcal{O}_S}(\mathcal{E}_\mathcal{G}) \to \pi_*(\mathcal{O}_\mathcal{G})$ induces an isomorphism
$$\mathcal{S}/((f_\mathcal{S}\otimes 1-\varphi_\mathcal{G})(\mathcal{E}_\mathcal{G}^{(q)})) \to \pi_*(\mathcal{O}_\mathcal{G}).$$

A morphism of φ-modules over S is defined as a morphism of $\mathbb{F}_q$-module schemes over S. The category of φ-modules over S is denoted by φ-Mod_S.

The last condition of Definition 2.2 yields a natural isomorphism

$$\mathrm{Coker}(\varphi_{\mathcal{G}}) \to \omega_{\mathcal{G}}.$$

We also note that for any φ-module $\mathcal{G}$ over S, the natural map $\mathrm{Sym}_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{G}}) \to \pi_*(\mathcal{O}_{\mathcal{G}})$ defines a closed immersion of $\mathbb{F}_q$-module schemes

$$i_{\mathcal{G}} : \mathcal{G} \to \mathbb{V}^*(\mathcal{E}_{\mathcal{G}}).$$

Definition 2.3. A φ-sheaf over S is a pair $(\mathcal{E}, \varphi_{\mathcal{E}})$ of a locally free $\mathcal{O}_S$-module $\mathcal{E}$ and an $\mathcal{O}_S$-linear homomorphism $\varphi_{\mathcal{E}} : \mathcal{E}^{(q)} \to \mathcal{E}$. We abusively denote the pair $(\mathcal{E}, \varphi_{\mathcal{E}})$ by $\mathcal{E}$. A morphism of φ-sheaves is defined as a morphism of $\mathcal{O}_S$-modules compatible with $\varphi_{\mathcal{E}}$'s. A sequence of φ-sheaves is said to be exact if the underlying sequence of $\mathcal{O}_S$-modules is exact. The exact category of φ-sheaves over S is denoted by φ-Shv_S.

We have a contravariant functor

$$\mathrm{Sh} : \varphi\text{-}\mathrm{Mod}_S \to \varphi\text{-}\mathrm{Shv}_S, \quad \mathcal{G} \mapsto (\mathcal{E}_{\mathcal{G}}, \varphi_{\mathcal{G}}).$$

On the other hand, for any object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of the category φ-Shv_S, put $\mathcal{S}_{\mathcal{E}} = \mathrm{Sym}_{\mathcal{O}_S}(\mathcal{E})$ and

$$\mathrm{Gr}(\mathcal{E}) = \mathrm{Spec}_S(\mathcal{S}_{\mathcal{E}}/((f_{\mathcal{S}_{\mathcal{E}}} \otimes 1 - \varphi_{\mathcal{E}})(\mathcal{E}^{(q)}))).$$

Then the diagonal map $\mathcal{E} \to \mathcal{E} \oplus \mathcal{E}$ and the natural $\mathbb{F}_q$-action on $\mathcal{E}$ define on $\mathrm{Gr}(\mathcal{E})$ a structure of an affine $\mathbb{F}_q$-module scheme over S. The formation of $\mathrm{Gr}(\mathcal{E})$ is compatible with any base change. We also have a natural identification

$$\mathrm{Gr}(\mathcal{E})(T) = \mathrm{Hom}_{\varphi, \mathcal{O}_S}(\mathcal{E}, \pi_*(\mathcal{O}_T)) \tag{2.2}$$

for any morphism $\pi : T \to S$, where we consider on $\pi_*(\mathcal{O}_T)$ the natural φ-structure induced by the q-th power Frobenius map [Tag, Proposition (1.8)]. Since we have a natural isomorphism $\mathcal{E} \to \mathcal{E}_{\mathrm{Gr}(\mathcal{E})}$, we obtain a contravariant functor

$$\mathrm{Gr} : \varphi\text{-}\mathrm{Shv}_S \to \varphi\text{-}\mathrm{Mod}_S, \quad \mathcal{E} \mapsto \mathrm{Gr}(\mathcal{E}),$$

which gives an anti-equivalence of categories with quasi-inverse Sh.

A sequence of φ-modules is said to be Shv-exact if the corresponding sequence in the category φ-Shv_S via the functor Sh is exact. We consider φ-Mod_S as an exact category by this notion of exactness. The author does not know if it is equivalent to the exactness as group schemes.

The commutativity of $\mathcal{E}_{\mathcal{G}}$ with any base change in Definition 2.2 holds in the case where $\mathcal{G}$ is a line bundle over S. From this we can show that any Drinfeld module is a φ-module. Another case it holds is that of finite φ-modules, which is defined as follows.

Definition 2.4 ([Tag], Definition (1.3))**.** We say an $\mathbb{F}_q$-module scheme $\mathcal{G}$ over S is a finite φ-module over S if the following conditions hold:

- the structure morphism $\pi : \mathcal{G} \to S$ is affine,
- the induced $\mathbb{F}_q$-action on $\omega_{\mathcal{G}}$ agrees with the action via the structure map $\mathbb{F}_q \to \mathcal{O}_S(S)$,
- the $\mathcal{O}_S$-modules $\pi_*(\mathcal{O}_{\mathcal{G}})$ and $\mathcal{E}_{\mathcal{G}}$ are locally free of finite rank with
$$\mathrm{rank}_{\mathcal{O}_S}(\pi_*(\mathcal{O}_{\mathcal{G}})) = q^{\mathrm{rank}_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{G}})},$$
- $\mathcal{E}_{\mathcal{G}}$ generates the $\mathcal{O}_S$-algebra $\pi_*(\mathcal{O}_{\mathcal{G}})$.

A morphism of finite φ-modules over S is defined as a morphism of $\mathbb{F}_q$-module schemes over S.

Definition 2.5. A finite φ-sheaf over S is a φ-sheaf such that its underlying $\mathcal{O}_S$-module is locally free of finite rank. The full subcategory of φ-Shv_S consisting of finite φ-sheaves is denoted by φ-Shv_S^f.

Let $\mathcal{G}$ be a finite φ-module over S. Then we also have the natural closed immersion $i_{\mathcal{G}} : \mathcal{G} \to \mathbb{V}^*(\mathcal{E}_{\mathcal{G}})$, which implies that the Cartier dual $\mathrm{Car}(\mathcal{G})$ of $\mathcal{G}$ is of height $\leqslant 1$ in the sense of [Gabr, §4.1.3]. Then, by [Gabr, Théorème 7.4, footnote], the sheaf of invariant differentials $\omega_{\mathrm{Car}(\mathcal{G})}$ is a locally free $\mathcal{O}_S$-module of finite rank, and thus the formation of the Lie algebra
$$\mathrm{Lie}(\mathrm{Car}(\mathcal{G})) \simeq \mathscr{H}\!om_S(\mathcal{G}, \mathbb{G}_{\mathrm{a}})$$
commutes with any base change. Since $q-1$ is invertible in $\mathcal{O}_S(S)$, the $\mathcal{O}_S$-module $\mathcal{E}_{\mathcal{G}}$ is the image of the projector
$$\mathrm{Lie}(\mathrm{Car}(\mathcal{G})) \to \mathrm{Lie}(\mathrm{Car}(\mathcal{G})), \quad x \mapsto \frac{1}{q-1} \sum_{a \in \mathbb{F}_q^\times} \alpha(a)^{-1} \psi_a(x),$$
where $\alpha : A \to \mathcal{O}_S(S)$ is the structure map and ψ_a is the action of a on $\mathrm{Lie}(\mathrm{Car}(\mathcal{G}))$ induced by the $\mathbb{F}_q$-action on $\mathcal{G}$. Since the formation of this projector commutes with any base change, so does that of $\mathcal{E}_{\mathcal{G}}$. From this we see that any finite φ-module is a φ-module. We denote by φ-Mod_S^f the full subcategory of φ-Mod_S consisting of finite φ-modules. Then the functor Gr gives an anti-equivalence of categories φ-$\mathrm{Shv}_S^f \to \varphi$-$\mathrm{Mod}_S^f$ with quasi-inverse given by Sh.

On the category φ-Mod_S^f, the Shv-exactness agrees with the usual exactness of group schemes. Indeed, from (2.2) and comparing ranks we see that the Shv-exactness implies the usual exactness, and the converse also follows by using the compatibility of Sh with any base change and reducing to the case over a field by Nakayama's lemma.

Lemma 2.6. *Let E be a line bundle over S. Let $\mathcal{G}$ be a finite locally free closed $\mathbb{F}_q$-submodule scheme of E over S. Suppose that the rank of $\mathcal{G}$ is a q-power. Then $\mathcal{G}$ is a finite φ-module.*

Proof. We may assume that $S = \mathrm{Spec}(B)$ is affine, the underlying invertible sheaf of E is trivial and $\mathcal{G} = \mathrm{Spec}(B_{\mathcal{G}})$ is free of rank q^n over S. We write as $E = \mathrm{Spec}(B[X])$. We have a surjection $B[X] \to B_{\mathcal{G}}$ of Hopf algebras over B. Let $P(X) \in B[X]$ be the characteristic polynomial of the action of X on $B_{\mathcal{G}}$. Since $\deg(P(X)) = q^n$, the Cayley-Hamilton theorem implies that this surjection induces an isomorphism $B[X]/(P(X)) \simeq B_{\mathcal{G}}$.

Since $P(X)$ is monic, we can see that $P(X)$ is an additive polynomial as in [Wat, §8, Exercise 7]. Since $\mathcal{G}$ is stable under the $\mathbb{F}_q$-action on $\mathbb{G}_{\mathrm{a}}$, we have the equality of ideals $(P(\lambda X)) = (P(X))$ of $B[X]$ for any $\lambda \in \mathbb{F}_q^\times$. From this we see that $P(X)$ is $\mathbb{F}_q$-linear and

$$\mathcal{E}_{\mathcal{G}} = \bigoplus_{i=0}^{n-1} BX^{q^i},$$

from which the lemma follows. □

Corollary 2.7. *Let $\pi : E \to F$ be an $\mathbb{F}_q$-linear isogeny of line bundles over S. Then the group scheme $\mathcal{G} = \mathrm{Ker}(\pi)$ is a finite φ-module over S, and we have a natural exact sequence of φ-sheaves*

$$0 \longrightarrow \mathcal{E}_F \longrightarrow \mathcal{E}_E \longrightarrow \mathcal{E}_{\mathcal{G}} \longrightarrow 0. \tag{2.3}$$

Proof. The first assertion follows from Lemma 2.6. For the second one, it is enough to show the surjectivity of the natural map $i^* : \mathcal{E}_E \to \mathcal{E}_{\mathcal{G}}$. By Nakayama's lemma, we may assume $S = \mathrm{Spec}(k)$ for some field k. Then π is defined by an $\mathbb{F}_q$-linear additive polynomial as

$$X \mapsto P(X) = a_0 X + a_1 X^q + \cdots + a_n X^{q^n}, \quad a_n \neq 0$$

and the map i^* is identified with the natural map

$$\mathcal{E}_E = \bigoplus_{i \in \mathbb{Z}_{\geqslant 0}} kX^{q^i} \to \mathcal{E}_{\mathcal{G}} = \bigoplus_{i=0}^{n-1} kX^{q^i}$$

of taking modulo $\bigoplus_{l \geqslant 0} kP(X)^{q^l}$. Hence i^* is surjective. □

Lemma 2.8. *(1) Let E be a line bundle over S. Let $\mathcal{G}$ and $\mathcal{H}$ be finite locally free closed $\mathbb{F}_q$-submodule schemes of E over S satisfying $\mathcal{H} \subseteq \mathcal{G}$. Suppose that the ranks of $\mathcal{G}$ and $\mathcal{H}$ are constant*

q-powers. Then $E/\mathcal{H}$ is a line bundle over S and $\mathcal{G}/\mathcal{H}$ is a finite φ-module over S.

(2) ([Hat], *Lemma 2.2 (2)) Let E be a Drinfeld module of rank r. Let $\mathcal{H}$ be a finite locally free closed A-submodule scheme of E of constant q-power rank over S. Suppose either*
- *$\mathcal{H}$ is etale over S, or*
- *S is reduced and for any maximal point η of S, the fiber $\mathcal{H}_\eta$ of $\mathcal{H}$ over η is etale.*

Then $E/\mathcal{H}$ is a Drinfeld module of rank r with the induced A-action.

Proof. For (1), the assertion that $E/\mathcal{H}$ is a line bundle over S is [Hat, Lemma 2.2 (1)]. Moreover, applying Lemma 2.6 to the natural closed immersion $\mathcal{G}/\mathcal{H} \to E/\mathcal{H}$, we see that $\mathcal{G}/\mathcal{H}$ is a finite φ-module over S. □

Definition 2.9 ([Tag], Definition (2.1))**.** We say an A-module scheme $\mathcal{G}$ over S is a t-module over S if the following conditions hold:
- the induced A-action on $\omega_{\mathcal{G}}$ agrees with the action via the structure map $A \to \mathcal{O}_S(S)$,
- the underlying $\mathbb{F}_q$-module scheme of $\mathcal{G}$ is a φ-module over S.

We say $\mathcal{G}$ is a finite t-module if in addition the underlying $\mathbb{F}_q$-module scheme of $\mathcal{G}$ is a finite φ-module over S.

Note that the former condition in Definition 2.9 is automatically satisfied if $\mathcal{G}$ is etale.

Lemma 2.10. *Let E be a line bundle over S. Let $\mathcal{G}$ and $\mathcal{H}$ be finite locally free closed $\mathbb{F}_q$-submodule schemes of E over S satisfying $\mathcal{H} \subseteq \mathcal{G}$. Suppose that $\mathcal{G}$ is endowed with a t-action which makes it a finite t-module, $\mathcal{H}$ is stable under the A-action on $\mathcal{G}$ and the ranks of $\mathcal{G}$ and $\mathcal{H}$ are constant q-powers.*

1. *The A-module scheme $\mathcal{H}$ is a finite t-module over S.*
2. *Suppose moreover that $a\mathcal{G} = 0$ for some $\mathcal{O}_S$-regular element $a \in A$. Then the A-module scheme $\mathcal{G}/\mathcal{H}$ is a finite t-module over S.*

Proof. From Lemma 2.6 and Lemma 2.8 (1), we see that $\mathcal{H}$ and $\mathcal{G}/\mathcal{H}$ are finite φ-modules. We have an exact sequence of $\mathcal{O}_S$-modules

$$\omega_{\mathcal{G}/\mathcal{H}} \xrightarrow{\pi^*} \omega_{\mathcal{G}} \longrightarrow \omega_{\mathcal{H}} \longrightarrow 0$$

which is compatible with A-actions. Since the t-action on $\omega_{\mathcal{G}}$ is equal to the multiplication by θ, so is that on $\omega_{\mathcal{H}}$ and (1) follows. For (2), using co-Lie complexes we can deduce from the assumption that the map π^* is injective. This yields (2). $\square$

Definition 2.11 ([Tag], Definition (3.1))**.** A v-module over S is a pair $(\mathcal{G}, v_{\mathcal{G}})$ of a t-module $\mathcal{G}$ and an $\mathcal{O}_S$-linear map $v_{\mathcal{G}} : \mathcal{E}_{\mathcal{G}} \to \mathcal{E}_{\mathcal{G}}^{(q)}$ such that the map $\psi_t^{\mathcal{G}} : \mathcal{E}_{\mathcal{G}} \to \mathcal{E}_{\mathcal{G}}$ induced by the t-action on $\mathcal{G}$ satisfies

$$\psi_t^{\mathcal{G}} = \theta + \varphi_{\mathcal{G}} \circ v_{\mathcal{G}}, \quad (\psi_t^{\mathcal{G}} \otimes 1) \circ v_{\mathcal{G}} = v_{\mathcal{G}} \circ \psi_t^{\mathcal{G}}.$$

We refer to such $v_{\mathcal{G}}$ as a v-structure on $\mathcal{G}$ and denote the pair $(\mathcal{G}, v_{\mathcal{G}})$ abusively by $\mathcal{G}$. A morphism $g : \mathcal{G} \to \mathcal{H}$ of v-modules over S is defined as a morphism of A-module schemes over S which commutes with v-structures, in the sense that the following diagram is commutative.

$$\begin{tikzcd} \mathcal{E}_{\mathcal{H}} \arrow[r, "v_{\mathcal{H}}"] \arrow[d, "g^*"'] & \mathcal{E}_{\mathcal{H}}^{(q)} \arrow[d, "g^* \otimes 1"] \\ \mathcal{E}_{\mathcal{G}} \arrow[r, "v_{\mathcal{G}}"'] & \mathcal{E}_{\mathcal{G}}^{(q)} \end{tikzcd}$$

A sequence of v-modules over S is said to be exact if the underlying sequence of φ-modules is Shv-exact. The category of v-modules over S is denoted by $v\text{-}\mathrm{Mod}_S$.

A v-module over S is said to be a finite v-module if the underlying φ-module is a finite φ-module. The full subcategory of $v\text{-}\mathrm{Mod}_S$ consisting of finite v-modules is denoted by $v\text{-}\mathrm{Mod}_S^f$.

Definition 2.12 ([Tag], Definition (3.2))**.** A v-sheaf over S is a quadruple $(\mathcal{E}, \varphi_{\mathcal{E}}, \psi_{\mathcal{E},t}, v_{\mathcal{E}})$, which we abusively write as $\mathcal{E}$, consisting of the following data:

- $(\mathcal{E}, \varphi_{\mathcal{E}})$ is a φ-sheaf over S,
- $\psi_{\mathcal{E},t} : \mathcal{E} \to \mathcal{E}$ is an $\mathcal{O}_S$-linear map which commutes with $\varphi_{\mathcal{E}}$,
- $v_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^{(q)}$ is an $\mathcal{O}_S$-linear map which commutes with $\psi_{\mathcal{E},t}$ and satisfies $\psi_{\mathcal{E},t} = \theta + \varphi_{\mathcal{E}} \circ v_{\mathcal{E}}$.

A morphism of v-sheaves is defined as a morphism of underlying $\mathcal{O}_S$-modules which is compatible with the other data, and we say that a sequence of v-sheaves is exact if the underlying sequence of $\mathcal{O}_S$-modules is exact. The exact category of v-sheaves over S is denoted by $v\text{-}\mathrm{Shv}_S$.

A v-sheaf is said to be a finite v-sheaf if the underlying $\mathcal{O}_S$-module is locally free of finite rank. The full subcategory of $v\text{-}\mathrm{Shv}_S$ consisting of finite v-sheaves is denoted by $v\text{-}\mathrm{Shv}_S^f$.

Then the functor Gr induces anti-equivalences of categories

$$v\text{-}\mathrm{Shv}_S \to v\text{-}\mathrm{Mod}_S, \quad v\text{-}\mathrm{Shv}_S^f \to v\text{-}\mathrm{Mod}_S^f$$

with quasi-inverses given by Sh.

Note that for any v-module (*resp.* finite v-module) $\mathcal{G}$ over S and any S-scheme T, the base change $\mathcal{G}|_T = \mathcal{G} \times_S T$ has a natural structure of a v-module (*resp.* finite v-module) over T. For any Drinfeld module E over S, the map $\varphi_E : \mathcal{E}_E^{(q)} \to \mathcal{E}_E$ is injective and $\mathrm{Coker}(\varphi_E)$ is killed by $\psi_t^E - \theta$. Then E has a unique v-structure

$$v_E = \varphi_E^{-1} \circ (\psi_t^E - \theta)$$

and any morphism of Drinfeld modules is compatible with the unique v-structures. Thus we may consider the category DM_S as a full subcategory of v-Mod_S. Moreover, for any isogeny $\pi : E \to F$ of Drinfeld modules over S, Corollary 2.7 implies that $\mathrm{Ker}(\pi)$ has a unique structure of a finite v-module such that the exact sequence (2.3) is compatible with v-structures. Note that a v-structure of $\mathrm{Ker}(\pi)$ is not necessarily unique without this compatibility condition. On the other hand, in some cases a finite t-module over S has a unique v-structure, as follows.

Lemma 2.13 ([Tag], Proposition 3.5)**.** *Let $\mathcal{G}$ be a finite t-module over S. Suppose either*

(1) $\mathcal{G}$ is etale over S, or
(2) S is reduced and for any maximal point η of S, the fiber $\mathcal{G}_\eta$ of $\mathcal{G}$ over η is etale.

Then the map $\varphi_{\mathcal{G}} : \mathcal{E}_{\mathcal{G}}^{(q)} \to \mathcal{E}_{\mathcal{G}}$ is injective. In particular, there exists a unique v-structure on $\mathcal{G}$, and for any v-module $\mathcal{H}$, any morphism $\mathcal{G} \to \mathcal{H}$ of t-modules over S is compatible with v-structures.

Corollary 2.14. *Let S be a reduced scheme which is flat over A and E a Drinfeld module of rank r over S. Let $a \in A$ be a non-zero element and $\mathcal{G}$ a finite locally free closed A-submodule scheme of the a-torsion part $E[a]$ of E over S of constant q-power rank. Then $E/\mathcal{G}$ has a natural structure of a Drinfeld module of rank r. Moreover, $\mathcal{G}$ has a unique structure of a finite v-module induced from that of E and, for any v-module $\mathcal{H}$, any morphism $\mathcal{G} \to \mathcal{H}$ of t-modules over S is compatible with v-structures.*

Proof. The going-down theorem implies that a is invertible in the residue field of every maximal point η of S, and thus $E[a]$ is etale over η. Then the first assertion follows from Lemma 2.8 (2). Moreover, Lemma 2.10 (1) implies that $\mathcal{G}$ is a finite t-module. Since $\mathcal{G}$ is the kernel of an isogeny of Drinfeld modules, the v-structure on E induces that on $\mathcal{G}$. The other assertions follow from Lemma 2.13 (2). □

Remark 2.15. The notation here is slightly different from the literature including [Tag]. Finite φ-sheaves are usually referred to as φ-sheaves. In [Tag], finite t-modules, finite v-modules and finite v-sheaves are assumed to be killed by some nonzero element of A.

2.3. **Duality for finite v-modules.** Let S be a scheme over A. We denote by C the Carlitz module over S, as before. We have

$$\mathcal{E}_C = \mathscr{H}om_{\mathbb{F}_q,S}(C,\mathbb{G}_{\mathrm{a}}) = \bigoplus_{i\in\mathbb{Z}_{\geqslant 0}} \mathcal{O}_S Z^{q^i}$$

with its unique v-structure given by

$$v_C : \mathcal{E}_C \to \mathcal{E}_C^{(q)}, \quad Z^{q^i} \mapsto Z^{q^{i-1}} \otimes (\theta^{q^i} - \theta) + Z^{q^i} \otimes 1.$$

Note that v_C is surjective. We have

$$\psi^C_{t^i}(Z) := (\psi^C_t)^i(Z) = \theta^i Z + \cdots + Z^{q^i}$$

and thus the set $\{\psi^C_{t^i}(Z)\}_{i\geqslant 0}$ forms a basis of $\mathcal{E}_C$. For any scheme T over S and any v-module $\mathcal{H}$ over T, we denote by $\mathrm{Hom}_{v,T}(\mathcal{H}, C|_T)$ the A-module of morphisms $\mathcal{H} \to C|_T$ in the category $v\text{-}\mathrm{Mod}_T$.

The following theorem, due to Taguchi, gives a duality for finite v-modules over S which is more suitable to analyze Drinfeld modules and Drinfeld modular forms than usual Cartier duality for finite locally free group schemes.

Theorem 2.16 ([Tag], §4)**.** *(1) Let $\mathcal{G}$ be a finite v-module over S. Then the big Zariski sheaf*

$$\mathscr{H}om_{v,S}(\mathcal{G}, C) : (S\textit{-schemes}) \to (A\textit{-modules})$$

given by $T \mapsto \mathrm{Hom}_{v,T}(\mathcal{G}|_T, C|_T)$ is represented by a finite v-module $\mathcal{G}^D$ over S. We refer to $\mathcal{G}^D$ as the Taguchi dual of $\mathcal{G}$.
(2) $\mathrm{rank}(\mathcal{G}) = \mathrm{rank}(\mathcal{G}^D)$.
(3) The functor

$$v\text{-}\mathrm{Mod}^f_S \to v\text{-}\mathrm{Mod}^f_S, \quad \mathcal{G} \mapsto \mathcal{G}^D$$

is exact (in the usual sense) and commutes with any base change.
(4) There exists a natural isomorphism of v-modules $\mathcal{G} \to (\mathcal{G}^D)^D$.

Proof. For the convenience of the reader, we give a simpler proof than in [Tag, §4]. Consider the linear dual

$$\mathcal{E}^\vee_{\mathcal{G}} = \mathscr{H}om_{\mathcal{O}_S}(\mathcal{E}_{\mathcal{G}}, \mathcal{O}_S)$$

and the dual maps

$$(\psi^{\mathcal{G}}_t)^\vee : \mathcal{E}^\vee_{\mathcal{G}} \to \mathcal{E}^\vee_{\mathcal{G}}, \quad v^\vee_{\mathcal{G}} : (\mathcal{E}^\vee_{\mathcal{G}})^{(q)} \to \mathcal{E}^\vee_{\mathcal{G}}, \quad \varphi^\vee_{\mathcal{G}} : \mathcal{E}^\vee_{\mathcal{G}} \to (\mathcal{E}^\vee_{\mathcal{G}})^{(q)}$$

of $\psi_t^{\mathcal{G}}$, $v_{\mathcal{G}}$ and $\varphi_{\mathcal{G}}$, respectively. We define a finite v-module $\mathcal{G}^D$ over S by $\mathcal{G}^D = \mathrm{Gr}(\mathcal{E}_{\mathcal{G}}^\vee, v_{\mathcal{G}}^\vee)$ with t-action $(\psi_t^{\mathcal{G}})^\vee$ and v-structure $\varphi_{\mathcal{G}}^\vee$.

To see that it represents the functor in the theorem, let $\pi : T \to S$ be any morphism. Since v_C is surjective, to give a map of v-sheaves $g : \mathcal{E}_C|_T \to \mathcal{E}_{\mathcal{G}}|_T$ is the same as to give an $\mathcal{O}_T$-linear map which is compatible with t-actions and v-structures. Since $\psi_{t^{i+1}}^C(Z) = \psi_t^C(\psi_{t^i}^C(Z))$, to give an $\mathcal{O}_T$-linear map $g : \mathcal{E}_C|_T \to \mathcal{E}_{\mathcal{G}}|_T$ compatible with t-actions is the same as to give an element $x = g(Z)$ of $\mathcal{E}_{\mathcal{G}}|_T(T)$. As for the compatibility with v-structures, we see that if g is compatible with t-actions, then the relation $(g \otimes 1)(v_C(\psi_{t^i}^C(Z))) = v_{\mathcal{G}}(g(\psi_{t^i}^C(Z)))$ implies a similar relation for $\psi_{t^{i+1}}^C(Z) = \psi_t^C(\psi_{t^i}^C(Z))$. Thus we only need to impose on x the condition for $i = 0$. Namely, we have

$$\mathrm{Hom}_{v,T}(\mathcal{G}|_T, C|_T) = \{x \in \mathcal{E}_{\mathcal{G}}|_T(T) \mid x \otimes 1 = v_{\mathcal{G}}(x)\}, \tag{2.4}$$

where $x \otimes 1 \in (\mathcal{E}_{\mathcal{G}}|_T)^{(q)}(T)$ is the pull-back of x by the Frobenius map F_T. On the other hand, by (2.2) the set $\mathrm{Gr}(\mathcal{E}_{\mathcal{G}}^\vee, v_{\mathcal{G}}^\vee)(T)$ can be identified with the set of $\mathcal{O}_T$-linear homomorphisms $\chi : \mathcal{E}_{\mathcal{G}}^\vee|_T \to \mathcal{O}_T$ satisfying

$$\chi \circ v_{\mathcal{G}}^\vee = f_{\mathcal{O}_T} \circ (F_T^*(\chi)).$$

Via the natural isomorphisms

$$\mathscr{H}om_{\mathcal{O}_T}(\mathcal{E}_{\mathcal{G}}^\vee|_T, \mathcal{O}_T) \simeq \mathcal{E}_{\mathcal{G}}|_T, \quad f_{\mathcal{O}_T} : \mathcal{O}_T^{(q)} = F_T^*(\mathcal{O}_T) \simeq \mathcal{O}_T,$$

we can easily show that it agrees with (2.4). Thus we obtain a natural isomorphism

$$\mathrm{Hom}_{v,T}(\mathcal{G}|_T, C|_T) \simeq \mathrm{Gr}(\mathcal{E}_{\mathcal{G}}^\vee, v_{\mathcal{G}}^\vee)(T)$$

and we can check that it is compatible with A-actions. The assertion on the exactness follows from the agreement of the exactness and the Shv-exactness for the category v-Mod_S^f. The other assertions follow from the construction. $\square$

Lemma 2.17. *Let S be any scheme over A. Let $a \in A$ be any monic polynomial. Consider the finite t-module $C[a]$ endowed with the natural v-structure as the kernel of the isogeny $a : C \to C$. Then the Taguchi dual $C[a]^D$ of $C[a]$ is isomorphic as a v-module to the constant A-module scheme $\underline{A/(a)}$ endowed with the unique v-structure of Lemma 2.13 (1).*

Proof. Let $\iota : C[a] \to C$ be the natural closed immersion. From the definition of the v-structure on $C[a]$, it is compatible with v-structures. Thus we have a morphism of t-modules over S

$$\underline{A/(a)} \to C[a]^D = \mathscr{H}om_{v,S}(C[a], C), \quad 1 \mapsto \iota.$$

We claim that it is a closed immersion. Indeed, by Nakayama's lemma we may assume $S = \mathrm{Spec}(k)$ for some field k. Suppose that $b\iota = 0$ for some $b \in A$. Write as $b = sa + r$ with $s, r \in A$ satisfying $\deg(r) < \deg(a)$. Then we have $\Phi^C_a(Z) \mid \Phi^C_r(Z)$, which is a contradiction unless $r = 0$. This implies that the kernel of the above morphism is zero and the claim follows. Since both sides have the same rank over S, it is an isomorphism. Since both sides are etale, it is compatible with unique v-structures. □

2.4. **Duality for Drinfeld modules of rank two.** Let S be a scheme over A. Recall that for any S-scheme T, both of the categories DM_T of Drinfeld modules over T and $v\text{-}\mathrm{Mod}^f_T$ of finite v-modules over T are full subcategories of $v\text{-}\mathrm{Mod}_T$, and $v\text{-}\mathrm{Mod}_T$ is anti-equivalent to $v\text{-}\mathrm{Shv}_T$. For any v-modules $\mathcal{H}, \mathcal{H}'$ over T, we denote by $\mathrm{Ext}^1_{v,T}(\mathcal{H}, \mathcal{H}')$ the A-module of isomorphism classes of Yoneda extensions of $\mathcal{H}$ by $\mathcal{H}'$ in the category $v\text{-}\mathrm{Mod}_T$ with Shv-exactness. We identify this A-module with the A-module $\mathrm{Ext}^1_{v,T}(\mathcal{E}_{\mathcal{H}'}, \mathcal{E}_{\mathcal{H}})$ of isomorphism classes of Yoneda extensions of $\mathcal{E}_{\mathcal{H}'} = \mathrm{Sh}(\mathcal{H}')$ by $\mathcal{E}_{\mathcal{H}} = \mathrm{Sh}(\mathcal{H})$ in the exact category $v\text{-}\mathrm{Shv}_T$. We also define a big Zariski sheaf $\mathscr{E}xt^1_{v,S}(\mathcal{H}, \mathcal{H}')$ as the sheafification of $T \mapsto \mathrm{Ext}^1_{v,T}(\mathcal{H}|_T, \mathcal{H}'|_T)$.

Let E be a Drinfeld module over S and put $G = \mathbb{G}_{\mathrm{a}}$ or C over S. We write as $G = \mathrm{Spec}_S(\mathcal{O}_S[Z])$. Let us describe the isomorphism class of any extension

$$0 \longrightarrow G \longrightarrow L \longrightarrow E \longrightarrow 0$$

in the category $v\text{-}\mathrm{Mod}_S$. Consider the associated exact sequence

$$0 \longrightarrow \mathcal{E}_E \longrightarrow \mathcal{E}_L \longrightarrow \mathcal{E}_G \longrightarrow 0$$

in the category $v\text{-}\mathrm{Shv}_S$. Since $\mathcal{E}_G$ is a free $\mathcal{O}_S$-module, this sequence splits as $\mathcal{O}_S$-modules if S is affine. In this case, using $\varphi_G(Z^{q^i}) = Z^{q^{i+1}}$, we can show that there exists a splitting $s : \mathcal{E}_G \to \mathcal{E}_L$ of the above sequence which is compatible with φ-structures.

We assume that S is affine and fix such a φ-compatible splitting s for a while. Then the a-action on L for any $a \in A$ is given by

$$\Phi^L_a = (\Phi^G_a, \Phi^E_a + \delta_a)$$

with some $\mathbb{F}_q$-linear homomorphism

$$\delta : A \to \mathrm{Hom}_{\mathbb{F}_q,S}(E, G),\ a \mapsto \delta_a.$$

Here δ_a is associated to the map $\psi^L_a \circ s - s \circ \psi^G_a : \mathcal{E}_G \to \mathcal{E}_E$ and satisfies

(i) $\delta_\lambda = 0$ for any $\lambda \in \mathbb{F}_q$,
(ii) $\delta_{ab} = \Phi^G_a \circ \delta_b + \delta_a \circ \Phi^E_b$ for any $a, b \in A$.

Since $\varphi_L : \mathcal{E}_L^{(q)} \to \mathcal{E}_L$ is injective and by Definition 2.9 the map $\psi_t^L - \theta$ kills $\mathrm{Coker}(\varphi_L)$, the v-structure on L is uniquely determined by the data δ_t.

Definition 2.18 ([Gek4], §3 and [PR], §2)**.** Let S be an affine scheme over A, E a Drinfeld module over S and $G = \mathbb{G}_\mathrm{a}$ or C as above.

(1) An (E,G)-biderivation is an $\mathbb{F}_q$-linear homomorphism $\delta : A \to \mathrm{Hom}_{\mathbb{F}_q,S}(E,G)$, $a \mapsto \delta_a$ satisfying the above conditions (i) and (ii). The module of (E,G)-biderivations is denoted by $\mathrm{Der}(E,G)$, which admits two natural A-module structures defined by

$$(\delta * c)_a = \delta_a \circ \Phi_c^E, \quad (c * \delta)_a = \Phi_c^G \circ \delta_a \text{ for any } c \in A.$$

Note that we have a natural isomorphism

$$\mathrm{ev}_t : \mathrm{Der}(E,G) \to \mathrm{Hom}_{\mathbb{F}_q,S}(E,G), \quad \delta \mapsto \delta_t. \tag{2.5}$$

(2) An (E,G)-biderivation δ is said to be inner if there exists $f \in \mathrm{Hom}_{\mathbb{F}_q,S}(E,G)$ satisfying $\delta = \delta_f$, where the (E,G)-biderivation δ_f is defined by

$$\delta_{f,a} = f \circ \Phi_a^E - \Phi_a^G \circ f \text{ for any } a \in A.$$

The submodule of $\mathrm{Der}(E,G)$ consisting of inner (E,G)-biderivations is denoted by $\mathrm{Der}_{\mathrm{in}}(E,G)$, which is stable under two natural A-actions.

(3) We denote by $\mathrm{Der}_0(E,G)$ the submodule of $\mathrm{Der}(E,G)$ consisting of (E,G)-biderivations δ such that the induced map on sheaves of invariant differentials

$$\mathrm{Cot}(\delta_t) : \omega_G \to \omega_E$$

is the zero map. We have $\mathrm{Der}_{\mathrm{in}}(E,G) \subseteq \mathrm{Der}_0(E,G)$.

(4) An inner (E,G)-biderivation δ_f is said to be strictly inner if $\mathrm{Cot}(f) = 0$. We denote by $\mathrm{Der}_{\mathrm{si}}(E,G)$ the submodule of $\mathrm{Der}(E,G)$ consisting of strictly inner (E,G)-biderivations.

Then the two natural A-actions on $\mathrm{Der}(E,G)$ agree with each other on the quotient $\mathrm{Der}(E,G)/\mathrm{Der}_{\mathrm{in}}(E,G)$ [PR, p. 412] and we have natural isomorphisms of A-modules

$$\begin{aligned}\mathrm{Ext}^1_{v,S}(E,G) &\to \mathrm{Der}(E,G)/\mathrm{Der}_{\mathrm{in}}(E,G)\\ &\xrightarrow{\mathrm{ev}_t} \mathrm{Hom}_{\mathbb{F}_q,S}(E,G)/\mathrm{ev}_t(\mathrm{Der}_{\mathrm{in}}(E,G)).\end{aligned} \tag{2.6}$$

We define an A-submodule

$$\mathrm{Ext}^1_{v,S}(E,G)^0$$

of $\mathrm{Ext}^1_{v,S}(E,G)$ as the inverse image of $\mathrm{Der}_0(E,G)/\mathrm{Der}_{\mathrm{in}}(E,G)$ by the above isomorphism. Since another choice of a φ-compatible splitting gives the same biderivation modulo inner ones, the first map of (2.6) is independent of the choice of a φ-compatible splitting, and so is the the A-submodule $\mathrm{Ext}^1_{v,S}(E,G)^0$.

Suppose that $E=\mathbb{V}_*(\mathcal{L})$ is a Drinfeld module of rank two over the affine scheme S. We have a natural isomorphism

$$\bigoplus_{m\geqslant 0}\mathcal{L}^{\otimes -q^m}\to \mathscr{H}om_{\mathbb{F}_q,S}(E,G),\quad b\mapsto (Z\mapsto b),$$

by which we identify both sides. Then $\mathrm{Der}(E,G)$, $\mathrm{Der}(E,G)^0$ and $\mathrm{Der}_{\mathrm{in}}(E,G)$ are locally free $\mathcal{O}_S(S)$-modules, and we can show that

$$T\mapsto \mathrm{Ext}^1_{v,T}(E|_T,G|_T),\quad T\mapsto \mathrm{Ext}^1_{v,T}(E|_T,G|_T)^0$$

satisfy the axiom of sheaves on affine open subsets of S. This implies that, for the case where S is not necessarily affine, we have a subsheaf of A-modules

$$\mathscr{E}xt^1_{v,S}(E,G)^0\subseteq \mathscr{E}xt^1_{v,S}(E,G)$$

such that, for any affine scheme T over S and $\bullet\in\{\varnothing,0\}$, we have

$$\mathscr{E}xt^1_{v,S}(E,G)^\bullet(T)=\mathrm{Ext}^1_{v,T}(E|_T,G|_T)^\bullet.$$

Moreover, we have a natural isomorphism of big Zariski sheaves

$$\mathcal{L}^{\otimes -q}\to \mathscr{E}xt^1_{v,S}(E,G)^0 \tag{2.7}$$

sending, for any affine scheme T over S, any element $b\in\mathcal{L}^{\otimes -q}(T)$ to the unique extension class such that, for the associated $(E|_T,G|_T)$-biderivation δ, the map $\delta_t:E|_T\to G|_T$ is given by

$$\delta_t^*:\mathcal{O}_T[Z]\to \mathrm{Sym}(\mathcal{L}^{\otimes -1}|_T),\quad Z\mapsto b.$$

Thus, taking $G=C$, we have the following theorem, which is due to Taguchi [Tag, §5]. The interpretation of his duality using biderivations obtained here is a generalization of [PR, Theorem 1.1] to general base schemes.

Theorem 2.19. *Let S be any scheme over A.*

(1) Let $E=(\mathcal{L},\Phi^E)$ be any Drinfeld module of rank two over S with

$$\Phi^E_t=\theta+a_1\tau+a_2\tau^2,\quad a_i\in\mathcal{L}^{\otimes 1-q^i}(S).$$

Then the functor

$$\mathscr{E}xt^1_{v,S}(E,C)^0:(S\text{-schemes})\to(A\text{-modules})$$

is represented by a Drinfeld module E^D of rank two over S defined by

$$E^D=\mathbb{V}_*(\mathcal{L}^{\otimes -q}),\quad \Phi^{E^D}_t=\theta-a_1\otimes a_2^{\otimes -1}\tau+a_2^{\otimes -q}\tau^2.$$

(2) *The formation of E^D commutes with any base change.*

(3) *Let $F = (\mathcal{M}, \Phi^F)$ be any Drinfeld module of rank two over S. Then any morphism $f : E \to F$ of the category* DM_S *induces a morphism $f^D : F^D \to E^D$ of this category. If f is induced by an $\mathcal{O}_S$-linear map $f : \mathcal{L} \to \mathcal{M}$, then the dual map $f^D : F^D \to E^D$ is given by the q-th tensor power $(f^\vee)^{\otimes q}$ of the linear dual $f^\vee : \mathcal{M}^\vee \to \mathcal{L}^\vee$.*

(4) *If f is an isogeny, then f^D is also an isogeny of the same degree as f,* $\mathrm{Ker}(f)$ *has a natural structure of a finite v-module over S and there exists a natural isomorphism of A-module schemes over S*

$$(\mathrm{Ker}(f))^D \to \mathrm{Ker}(f^D).$$

Proof. The assertions (1) and (2) follow easily from the construction. The assertion (3) follows from the functoriality of $\mathscr{E}xt^1_{v,S}(-, C)^0$ and the isomorphism (2.7).

Let us show the assertion (4). Put $\mathcal{G} = \mathrm{Ker}(f)$. Corollary 2.7 implies that the exact sequence of group schemes

$$0 \longrightarrow \mathcal{G} \longrightarrow E \xrightarrow{f} F \longrightarrow 0$$

is also Shv-exact and thus $\mathcal{G}$ has a natural structure of a finite v-module such that this sequence is compatible with v-structures. Since E and C have different ranks, the long exact sequence of $\mathscr{H}om_{v,S}$ yields an exact sequence

$$0 \longrightarrow \mathcal{G}^D \longrightarrow \mathscr{E}xt^1_{v,S}(F, C) \longrightarrow \mathscr{E}xt^1_{v,S}(E, C).$$

From a description of the connecting homomorphism using Yoneda extension, we can show that it factors through the subsheaf $\mathscr{E}xt^1_{v,S}(F, C)^0$. Therefore we have an exact sequence of A-module schemes over S

$$0 \longrightarrow \mathcal{G}^D \longrightarrow F^D \xrightarrow{f^D} E^D,$$

from which we obtain a natural isomorphism $\mathcal{G}^D \to \mathrm{Ker}(f^D)$. To see that f^D is faithfully flat, by a base change we may assume $S = \mathrm{Spec}(k)$ for some field k. Then the group schemes F^D and E^D are isomorphic to $\mathbb{G}_\mathrm{a}$ and f^D is defined by an additive polynomial. Since $\mathrm{Ker}(f^D) = \mathcal{G}^D$ is finite over S, this polynomial is non-zero and thus f^D is faithfully flat. Since the ranks of $\mathcal{G}$ and $\mathcal{G}^D$ are the same, the assertion on $\deg(f^D)$ also follows. □

Remark 2.20. Suppose that there exists a section $h \in \mathcal{L}^{\otimes -(q+1)}(S)$ satisfying $h^{\otimes q-1} = -a_2$. Then the map $h : \mathcal{L} \to \mathcal{L}^{\otimes -q}$ gives an autoduality for Drinfeld modules of rank two. In the classical setting on the

Drinfeld upper half plane, this is the case because of the existence of Gekeler's h-function [Gek3, Theorem 9.1 (c)]. In general, we only have a weaker version of autoduality: the map

$$\mathcal{L}^{\otimes q-1} \to \mathcal{L}^{\otimes -q(q-1)}, \quad l \mapsto l \otimes a_2$$

is an isomorphism of invertible sheaves. This is enough for our purpose.

For a Drinfeld module E over S, we have analogues of the first de Rham cohomology group and the Hodge filtration for an abelian variety [Gek4, §5]. First we show the following lemma.

Lemma 2.21. *For any Drinfeld module E of rank two over an affine scheme S, we have natural isomorphisms*

$$\mathrm{Lie}(E^D) \to \mathrm{Ext}^1_{v,S}(E, \mathbb{G}_{\mathrm{a}})^0, \quad \mathrm{Der}_{\mathrm{in}}(E, \mathbb{G}_{\mathrm{a}})/\mathrm{Der}_{\mathrm{si}}(E, \mathbb{G}_{\mathrm{a}}) \to \mathrm{Lie}(E)^\vee.$$

Proof. For the former one, we put $S_\varepsilon = \mathrm{Spec}_S(\mathcal{O}_S[\varepsilon]/(\varepsilon^2))$. Then we have

$$\mathrm{Lie}(E^D) = \mathrm{Ker}(E^D(S_\varepsilon) \to E^D(S)).$$

For any $\mathbb{F}_q$-linear homomorphism $\delta : A \to \mathrm{Hom}_{\mathbb{F}_q, S_\varepsilon}(E|_{S_\varepsilon}, C|_{S_\varepsilon})$, we can write as

$$\delta_a = \delta_a^0 + \varepsilon \delta_a^1, \quad \delta_a^i \in \mathrm{Hom}_{\mathbb{F}_q, S}(E, C).$$

Then $\delta \in \mathrm{Der}_0(E|_{S_\varepsilon}, C|_{S_\varepsilon})$ if and only if

$$\delta^0 \in \mathrm{Der}_0(E, C), \quad \delta^1 \in \mathrm{Der}_0(E, \mathbb{G}_{\mathrm{a}}).$$

On the other hand, for any $g = g^0 + \varepsilon g^1 \in \mathrm{Hom}_{\mathbb{F}_q, S_\varepsilon}(E|_{S_\varepsilon}, C|_{S_\varepsilon})$, the associated inner biderivation δ_g is written as

$$\delta_g = \delta_{g^0} + \varepsilon(g^1 \circ \Phi^E - \Phi^{\mathbb{G}_{\mathrm{a}}} \circ g^1).$$

From this, we see that the map sending δ to the class of δ^1 gives a natural isomorphism $\mathrm{Lie}(E^D) \to \mathrm{Ext}^1_{v,S}(E, \mathbb{G}_{\mathrm{a}})^0$. The latter one is given by the natural map

$$\mathrm{Der}_{\mathrm{in}}(E, \mathbb{G}_{\mathrm{a}}) \to \mathrm{Hom}_{\mathcal{O}_S}(\mathrm{Lie}(E), \mathrm{Lie}(\mathbb{G}_{\mathrm{a}})), \quad \delta_f \mapsto \mathrm{Lie}(f).$$

$\square$

For any Drinfeld module E over an affine scheme S, we put

$$\mathrm{DR}(E, \mathbb{G}_{\mathrm{a}}) = \mathrm{Der}_0(E, \mathbb{G}_{\mathrm{a}})/\mathrm{Der}_{\mathrm{si}}(E, \mathbb{G}_{\mathrm{a}}).$$

From the proof of [PR, p. 412], we see that the two natural A-actions on $\mathrm{Der}_0(E, \mathbb{G}_{\mathrm{a}})$ define the same A-action on $\mathrm{DR}(E, \mathbb{G}_{\mathrm{a}})$. If E is of rank two, then Lemma 2.21 yields an exact sequence of A-modules

$$(2.8) \qquad 0 \longrightarrow \mathrm{Lie}(E)^\vee \longrightarrow \mathrm{DR}(E, \mathbb{G}_{\mathrm{a}}) \longrightarrow \mathrm{Lie}(E^D) \longrightarrow 0,$$

which is functorial on E.

Finally, we recall the construction of the Kodaira-Spencer map for a Drinfeld module E over an A-scheme S [Gek4, §6]. We only treat the case where $S = \mathrm{Spec}(B)$ is affine and the underlying invertible sheaf of E is trivial. Write as $E = \mathrm{Spec}(B[X])$ so that we identify as $\mathrm{Hom}_{\mathbb{F}_q,S}(E, \mathbb{G}_\mathrm{a}) = B\{\tau\}$. We define an action of $D \in \mathrm{Der}_A(B)$ on $B\{\tau\}$ by acting on coefficients. Then, via the isomorphism (2.5), the derivation D induces a map $\nabla_D : \mathrm{Der}_0(E, \mathbb{G}_\mathrm{a}) \to \mathrm{Der}_0(E, \mathbb{G}_\mathrm{a})$, which in turn defines

$$\pi_D : \mathrm{Lie}(E)^\vee \to \mathrm{DR}(E, \mathbb{G}_\mathrm{a}) \overset{\nabla_D}{\to} \mathrm{DR}(E, \mathbb{G}_\mathrm{a}) \to \mathrm{Lie}(E^D),$$

where the first and the last arrows are those of (2.8). Then the Kodaira-Spencer map for E over S is by definition

$$\mathrm{KS} : \mathrm{Der}_A(B) \to \mathrm{Hom}_B(\mathrm{Lie}(E)^\vee, \mathrm{Lie}(E^D)), \quad D \mapsto \pi_D.$$

Hence we also have the dual map

$$\mathrm{KS}^\vee : \omega_E \otimes_{\mathcal{O}_S} \omega_{E^D} \to \Omega^1_{S/A}.$$

3. Canonical subgroups of ordinary Drinfeld modules

Let $\wp$ be a monic irreducible polynomial of degree d in $A = \mathbb{F}_q[t]$. We denote by $\mathcal{O}_K$ the complete local ring of A at the prime ideal $(\wp)$, which is a complete discrete valuation ring with uniformizer $\wp$. We consider $\mathcal{O}_K$ naturally as an A-algebra. The fraction field and the residue field of $\mathcal{O}_K$ are denoted by K and $k(\wp) = \mathbb{F}_{q^d}$, respectively. We denote by $v_\wp$ the $\wp$-adic (additive) valuation on K normalized as $v_\wp(\wp) = 1$. For any $\mathcal{O}_K$-algebra B and any scheme X over B, we put $\bar{B} = B/\wp B$ and $\bar{X} = X \times_B \mathrm{Spec}(\bar{B})$.

We say an $\mathcal{O}_K$-algebra B is a $\wp$-adic ring if it is complete with respect to the $\wp$-adic topology. A $\wp$-adic ring B is said to be flat if it is flat over $\mathcal{O}_K$.

3.1. **Ordinary Drinfeld modules.** Let $\bar{S}$ be an A-scheme of characteristic $\wp$. Let $\bar{E} = (\bar{\mathcal{L}}, \Phi^{\bar{E}})$ be a Drinfeld module of rank two over $\bar{S}$. By [Sha, Proposition 2.7], we can write as

$$\Phi^{\bar{E}}_\wp = (\alpha_d + \cdots + \alpha_{2d}\tau^d)\tau^d, \quad \alpha_i \in \bar{\mathcal{L}}^{\otimes 1-q^i}(\bar{S}). \tag{3.1}$$

We put

$$F_{d,\bar{E}} = \tau^d : \bar{E} \to \bar{E}^{(q^d)}, \quad V_{d,\bar{E}} = \alpha_d + \cdots + \alpha_{2d}\tau^d : \bar{E}^{(q^d)} \to \bar{E}.$$

We also denote them by F_d and V_d if no confusion may occur. We also define a homomorphism $F^n_d : \bar{E} \to \bar{E}^{(q^{dn})}$ by

$$F^1_d = F_d, \quad F^n_d = (F^{n-1}_d)^{(q^d)} \circ F_d.$$

We define $V_d^n : \bar{E}^{(q^{dn})} \to \bar{E}$ similarly. They are isogenies of Drinfeld modules satisfying $V_d^n \circ F_d^n = \Phi^{\bar{E}}_{\wp^n}$ and $F_d^n \circ V_d^n = \Phi^{\bar{E}^{(q^{dn})}}_{\wp^n}$ [Sha, §2.8]. We also have exact sequences of A-module schemes over $\bar{S}$

$$0 \longrightarrow \mathrm{Ker}(F_d^n) \longrightarrow \bar{E}[\wp^n] \longrightarrow \mathrm{Ker}(V_d^n) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Ker}(F_d) \longrightarrow \mathrm{Ker}(F_d^n) \longrightarrow \mathrm{Ker}(F_d^{n-1})^{(q^d)} \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Ker}(V_d^{n-1})^{(q^d)} \longrightarrow \mathrm{Ker}(V_d^n) \longrightarrow \mathrm{Ker}(V_d) \longrightarrow 0.$$

Definition 3.1. We say $\bar{E}$ is ordinary if $\alpha_d \in \bar{\mathcal{L}}^{\otimes 1-q^d}(\bar{S})$ of (3.1) is nowhere vanishing, and supersingular if $\alpha_d = 0$.

By [Sha, Proposition 2.14], $\bar{E}$ is ordinary if and only if $\mathrm{Ker}(V_d)$ is etale if and only if $\mathrm{Ker}(V_d^n)$ is etale for any n.

We need a relation of the isogenies F_d and V_d with duality. For this, we first prove the following lemma.

Lemma 3.2. *Let C be the Carlitz module over A. Then the polynomial $\Phi^C_\wp(Z)$ is a monic Eisenstein polynomial in $\mathcal{O}_K[Z]$. In particular, we have*

$$\Phi^C_\wp(Z) \equiv Z^{q^d} \bmod \wp. \tag{3.2}$$

Proof. Let L be a splitting field of the polynomial $\Phi^C_\wp(Z)$ over K. Since the ring A acts on $C[\wp](L)$ transitively, any non-zero root $\beta \in L$ of $\Phi^C_\wp(Z)$ satisfies $v_\wp(\beta) = 1/(q^d - 1)$ and thus the monic polynomial $\Phi^C_\wp(Z)$ is Eisenstein over $\mathcal{O}_K$. □

Lemma 3.3.

$$F^D_{d,\bar{E}} = V_{d,\bar{E}^D}, \quad V^D_{d,\bar{E}} = F_{d,\bar{E}^D}.$$

Proof. First we prove the former equality. Since $F_{d,\bar{E}^D}$ is an isogeny, it is enough to show $F^D_{d,\bar{E}} \circ F_{d,\bar{E}^D} = \Phi^{\bar{E}^D}_\wp$. Let $\bar{\mathcal{L}}$ be the underlying invertible sheaf of $\bar{E}$. Take any section l of $\bar{\mathcal{L}}^{\otimes -q}$. We have $F_{d,\bar{E}^D}(l) = l^{\otimes q^d}$. From (2.1), we see that the map $F^D_{d,\bar{E}}$ sends it to the class of the biderivation δ such that δ_t agrees with the homomorphism

$$\bar{E} \to C = \mathrm{Spec}_{\bar{S}}(\mathcal{O}_{\bar{S}}[Z]), \quad Z \mapsto l^{\otimes q^d} \in \mathrm{Sym}(\bar{\mathcal{L}}^{\otimes -1}).$$

By (3.2), this is equal to the class of $\wp \cdot (Z \mapsto l)$ with respect to the A-module structure of $\mathscr{E}xt^1_{v,\bar{S}}(\bar{E}, C)^0$. Since l is a section of $\bar{\mathcal{L}}^{\otimes -q}$, the isomorphism (2.7) implies the assertion.

For the latter equality, it is enough to show $V^D_{d,\bar{E}} \circ V_{d,\bar{E}^D} = \Phi_\wp^{(\bar{E}^D)^{(q^d)}}$. By the former equality of the lemma, we have

$$V^D_{d,\bar{E}} \circ V_{d,\bar{E}^D} = V^D_{d,\bar{E}} \circ F^D_{d,\bar{E}} = (F_{d,\bar{E}} \circ V_{d,\bar{E}})^D = (\Phi_\wp^{\bar{E}^{(q^d)}})^D.$$

By the definition of the A-module structure on $\mathscr{E}xt^1_{v,\bar{S}}(\bar{E}^{(q^d)}, C)$, it is equal to $\Phi_\wp^{(\bar{E}^D)^{(q^d)}}$ and we obtain the latter equality of the lemma. □

Proposition 3.4. *Let $\bar{S}$ be an A-scheme of characteristic $\wp$ and $\bar{E}$ a Drinfeld module of rank two over $\bar{S}$. Consider the maps*

$$\mathrm{Lie}(V_{d,\bar{E}}) : \mathrm{Lie}(\bar{E}^{(q^d)}) \to \mathrm{Lie}(\bar{E}), \quad \mathrm{Lie}(V_{d,\bar{E}^D}) : \mathrm{Lie}((\bar{E}^D)^{(q^d)}) \to \mathrm{Lie}(\bar{E}^D)$$

and the linear dual $\mathrm{Lie}(V_{d,\bar{E}})^\vee$ of the former map. Then we have a natural isomorphism of $\mathcal{O}_{\bar{S}}$-modules

$$\mathrm{Coker}(\mathrm{Lie}(V_{d,\bar{E}})^\vee) \simeq \mathrm{Coker}(\mathrm{Lie}(V_{d,\bar{E}^D})).$$

In particular, $\bar{E}$ is ordinary if and only if $\bar{E}^D$ is ordinary.

Proof. We follow the proof of [Con, Theorem 2.3.6]. By gluing, we may assume that $\bar{S}$ is affine. By the exact sequence (2.8), we have a commutative diagram of A-modules

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Lie}(\bar{E}^{(q^d)})^\vee \arrow[r] \arrow[d, "\mathrm{Lie}(F_{d,\bar{E}})^\vee"'] & \mathrm{DR}(\bar{E}^{(q^d)}, \mathbb{G}_{\mathrm{a}}) \arrow[r] \arrow[d, "F^*_{d,\bar{E}}"] & \mathrm{Lie}((\bar{E}^D)^{(q^d)}) \arrow[r] \arrow[d, "\mathrm{Lie}(F^D_{d,\bar{E}})"] & 0 \\
0 \arrow[r] & \mathrm{Lie}(\bar{E})^\vee \arrow[r] \arrow[d, "\mathrm{Lie}(V_{d,\bar{E}})^\vee"'] & \mathrm{DR}(\bar{E}, \mathbb{G}_{\mathrm{a}}) \arrow[r] \arrow[d, "V^*_{d,\bar{E}}"] & \mathrm{Lie}(\bar{E}^D) \arrow[r] \arrow[d, "\mathrm{Lie}(V^D_{d,\bar{E}})"] & 0 \\
0 \arrow[r] & \mathrm{Lie}(\bar{E}^{(q^d)})^\vee \arrow[r] & \mathrm{DR}(\bar{E}^{(q^d)}, \mathbb{G}_{\mathrm{a}}) \arrow[r] & \mathrm{Lie}((\bar{E}^D)^{(q^d)}) \arrow[r] & 0,
\end{tikzcd}$$

where rows are exact and columns are complexes. Since $\mathrm{Lie}(F_{d,\bar{E}}) = \mathrm{Lie}(F_{d,\bar{E}^D}) = 0$, Lemma 3.3 implies that the middle column of the diagram induces the complex

$$0 \longrightarrow \mathrm{Lie}((\bar{E}^D)^{(q^d)}) \xrightarrow{F^*_{d,\bar{E}}} \mathrm{DR}(\bar{E}, \mathbb{G}_{\mathrm{a}}) \xrightarrow{V^*_{d,\bar{E}}} \mathrm{Lie}(\bar{E}^{(q^d)})^\vee \longrightarrow 0.$$

If it is exact, then as in the proof of [Con, Theorem 2.3.6], by using [Con, Lemma 2.3.7] and Lemma 3.3 we obtain

$$\mathrm{Coker}(\mathrm{Lie}(V_{d,\bar{E}})^\vee) \simeq \mathrm{Coker}(\mathrm{Lie}(F^D_{d,\bar{E}})) = \mathrm{Coker}(\mathrm{Lie}(V_{d,\bar{E}^D})).$$

Let us show the exactness. Since it is a complex of locally free $\mathcal{O}_{\bar{S}}$-modules of finite rank and its formation commutes with any base change of affine schemes, we may assume $\bar{S} = \mathrm{Spec}(k)$ for some field k.

By comparing dimensions, it is enough to show that, for any Drinfeld module $\bar{E}$ of rank two over k, the maps

$$F^*_{d,\bar{E}} : \mathrm{Der}_0(\bar{E}^{(q^d)}, \mathbb{G}_{\mathrm{a}})/\mathrm{Der}_{\mathrm{in}}(\bar{E}^{(q^d)}, \mathbb{G}_{\mathrm{a}}) \to \mathrm{Der}_0(\bar{E}, \mathbb{G}_{\mathrm{a}})/\mathrm{Der}_{\mathrm{si}}(\bar{E}, \mathbb{G}_{\mathrm{a}})$$
$$V^*_{d,\bar{E}} : \mathrm{Der}_0(\bar{E}, \mathbb{G}_{\mathrm{a}})/\mathrm{Der}_{\mathrm{si}}(\bar{E}, \mathbb{G}_{\mathrm{a}}) \to \mathrm{Der}_{\mathrm{in}}(\bar{E}^{(q^d)}, \mathbb{G}_{\mathrm{a}})/\mathrm{Der}_{\mathrm{si}}(\bar{E}^{(q^d)}, \mathbb{G}_{\mathrm{a}})$$

are non-zero.

For the assertion on $F^*_{d,\bar{E}}$, we write as

$$\Phi^{\bar{E}}_t = \theta + a_1\tau + a_2\tau^2, \quad \Phi^{\bar{E}}_{\wp} = (\alpha_d + \cdots + \alpha_{2d}\tau^d)\tau^d$$

with $a_2, \alpha_{2d} \neq 0$. Let δ be the element of $\mathrm{Der}_0(\bar{E}^{(q^d)}, \mathbb{G}_{\mathrm{a}})$ satisfying $\delta_t = \tau$ and suppose that $F^*_{d,\bar{E}}(\delta)$ is an element of $\mathrm{Der}_{\mathrm{si}}(\bar{E}, \mathbb{G}_{\mathrm{a}})$. Namely, we have

$$\tau^{d+1} = f \circ \Phi^{\bar{E}}_t - \Phi^{\mathbb{G}_{\mathrm{a}}}_t \circ f \tag{3.3}$$

for some $f \in \mathrm{Hom}_{\mathbb{F}_q,k}(\bar{E}, \mathbb{G}_{\mathrm{a}})$ satisfying $\mathrm{Cot}(f) = 0$. We write f as $f = b_r\tau^r + \cdots + b_s\tau^s$ with some $b_i \in k$ and $1 \leqslant r \leqslant s$ satisfying $b_r, b_s \neq 0$. Then we have $s = d-1$ and the coefficient of τ^r in the right-hand side of (3.3) is $(\theta^{q^r} - \theta)b_r$. Since $1 \leqslant r \leqslant d-1$ and the element θ generates $k(\wp) = \mathbb{F}_{q^d}$ over $\mathbb{F}_q$, this term does not vanish and thus we have $r = d+1$, which is a contradiction.

Let us consider the assertion on $V^*_{d,\bar{E}}$. If $\alpha_d \neq 0$, then the map $\mathrm{Lie}(V_{d,\bar{E}})$ is an isomorphism and the claim follows from the above diagram. Otherwise, [Sha, Lemma 2.5] yields $\alpha_i = 0$ unless $i = 2d$. Let δ be the element of $\mathrm{Der}_0(\bar{E}, \mathbb{G}_{\mathrm{a}})$ satisfying $\delta_t = \tau$ and suppose that $V^*_{d,\bar{E}}(\delta)$ is an element of $\mathrm{Der}_{\mathrm{si}}(\bar{E}^{(q^d)}, \mathbb{G}_{\mathrm{a}})$. We have

$$\tau(\alpha_{2d}\tau^d) = g \circ \Phi^{\bar{E}^{(q^d)}}_t - \Phi^{\mathbb{G}_{\mathrm{a}}}_t \circ g$$

for some $g \in \mathrm{Hom}_{\mathbb{F}_q,k}(\bar{E}^{(q^d)}, \mathbb{G}_{\mathrm{a}})$ satisfying $\mathrm{Cot}(g) = 0$. Then we obtain a contradiction as in the above case. $\square$

3.2. **Canonical subgroups.** Let B be an $\mathcal{O}_K$-algebra and E a Drinfeld module of rank two over B. We say E has ordinary reduction if $\bar{E} = E \times_B \mathrm{Spec}(\bar{B})$ is ordinary.

Lemma 3.5. *Let B be a $\wp$-adic ring and E a Drinfeld module of rank two over B with ordinary reduction. Then, for any positive integer n, there exists a unique finite locally free closed A-submodule scheme $\mathcal{C}_n(E)$ of $E[\wp^n]$ over B satisfying $\overline{\mathcal{C}_n(E)} = \mathrm{Ker}(F^n_{d,\bar{E}})$. The formation of $\mathcal{C}_n(E)$ commutes with any base change of $\wp$-adic rings. We refer to it as the canonical subgroup of level n of the Drinfeld module E with ordinary reduction.*

Proof. First note that, since $(B, \wp B)$ is a Henselian pair, the functor $X \mapsto \bar{X}$ gives an equivalence between the categories of finite etale schemes over B and those over $\bar{B}$ [Gabb, §1].

Let us show the existence. The finite etale A-module scheme $\bar{\mathcal{H}} = \mathrm{Ker}(V^n_{d,\bar{E}})$ can be lifted to a finite etale A-module scheme $\mathcal{H}$ over B. By the etaleness and [Gro, Proposition (17.7.10)], we can lift the map $\bar{E}[\wp^n] \to \bar{\mathcal{H}}$ to a finite locally free morphism of A-module schemes $\pi : E[\wp^n] \to \mathcal{H}$ over B. Then $\mathcal{C}_n(E) = \mathrm{Ker}(\pi)$ is a lift of $\mathrm{Ker}(F^n_{d,\bar{E}})$.

For the uniqueness, suppose that we have two subgroup schemes $\mathcal{C}_{n,1}, \mathcal{C}_{n,2}$ of $E[\wp^n]$ as in the lemma. Put $\mathcal{H}_i = E[\wp^n]/\mathcal{C}_{n,i}$. Since they are lifts of $\bar{\mathcal{H}}$, there exists an isomorphism $\theta : \mathcal{H}_1 \to \mathcal{H}_2$ over B reducing to $\mathrm{id}_{\bar{\mathcal{H}}}$ over $\bar{B}$. Then the etaleness implies that θ is compatible with the quotient maps $E[\wp^n] \to \mathcal{H}_i$. Therefore, $\mathcal{C}_{n,1}$ and $\mathcal{C}_{n,2}$ agree as A-submodule schemes of $E[\wp^n]$. Since the formation of $\mathrm{Ker}(F^n_{d,\bar{E}})$ commutes with any base change, the commutativity of $\mathcal{C}_n(E)$ with any base change follows from its uniqueness. □

We refer to the natural isogeny

$$\pi_{E,n} : E \to E/\mathcal{C}_n(E)$$

as the canonical isogeny of level n for E. We have $\pi_{E,n} \bmod \wp = F^n_d$.

On the other hand, since $E[\wp^n]/\mathcal{C}_n(E)$ is etale both over $\bar{B}$ and $B \otimes_{\mathcal{O}_K} K$, it is etale over B and we have a natural isomorphism

$$\omega_{E[\wp^n]} \to \omega_{\mathcal{C}_n(E)}.$$

Moreover, the map

$$\rho_{E,n} : E/\mathcal{C}_n(E) \to (E/\mathcal{C}_n(E))/(E[\wp^n]/\mathcal{C}_n(E)) \overset{\wp^n}{\simeq} E$$

is an etale isogeny satisfying

$$\rho_{E,n} \circ \pi_{E,n} = \Phi^E_{\wp^n}, \quad \pi_{E,n} \circ \rho_{E,n} = \Phi^{E/\mathcal{C}_n(E)}_{\wp^n}.$$

In particular, we have $\rho_{E,n} \bmod \wp = V^n_d$. We refer to $\rho_{E,n}$ as the canonical etale isogeny of level n for E. The formation of $\pi_{E,n}$ and $\rho_{E,n}$ also commutes with any base change of $\wp$-adic rings.

Suppose that the $\wp$-adic ring B is reduced and flat. Then by Corollary 2.14 the quotient $E/\mathcal{C}_n(E)$ has a natural structure of a Drinfeld module of rank two. Moreover, Lemma 2.10 implies that $E[\wp^n]$, $\mathcal{C}_n(E)$ and $E[\wp^n]/\mathcal{C}_n(E)$ are finite t-modules, and by Lemma 2.13 (2) they have unique structures of finite v-modules, which make the natural exact sequence

$$0 \longrightarrow \mathcal{C}_n(E) \longrightarrow E[\wp^n] \longrightarrow E[\wp^n]/\mathcal{C}_n(E) \longrightarrow 0 \tag{3.4}$$

compatible with v-structures. We also see that the formation of the v-structure on $\mathcal{C}_n(E)$ also commutes with any base change of reduced flat $\wp$-adic rings.

Lemma 3.6. *Let B be a reduced flat $\wp$-adic ring. Let E be a Drinfeld module of rank two over B with ordinary reduction. Then the Taguchi dual $\mathcal{C}_n(E)^D$ of the canonical subgroup $\mathcal{C}_n(E)$ is etale over B. Moreover, it is etale locally isomorphic as a finite v-module to the constant A-module scheme $\underline{A/(\wp^n)}$ over B.*

Proof. By Proposition 3.4, the dual E^D also has ordinary reduction. We claim that $E^D[\wp^n]/\mathcal{C}_n(E^D)$ is not killed by $\wp^{n-1}$. Indeed, if it is killed by $\wp^{n-1}$, then we have $E^D[\wp] \subseteq \mathcal{C}_n(E^D)$, which contradicts the fact that $\bar{E}^D[\wp]$ has an etale quotient. Since $E^D[\wp^n]/\mathcal{C}_n(E^D)$ is etale, the claim implies that it is etale locally isomorphic to $\underline{A/(\wp^n)}$. Note that this identification is compatible with v-structures by Lemma 2.13 (1).

Since Taguchi duality is exact, the exact sequence (3.4) for E^D yields an exact sequence of finite v-modules over B

$$0 \longrightarrow (E^D[\wp^n]/\mathcal{C}_n(E^D))^D \longrightarrow E^D[\wp^n]^D \longrightarrow \mathcal{C}_n(E^D)^D \longrightarrow 0.$$

By Theorem 2.19 (4), we also have a natural isomorphism of A-module schemes $E[\wp^n] \simeq E^D[\wp^n]^D$, by which we identify both sides. Hence we reduce ourselves to showing the equality

$$\mathcal{C}_n(E) = (E^D[\wp^n]/\mathcal{C}_n(E^D))^D.$$

For this, by the uniqueness of the canonical subgroup it is enough to show that the reduction of $(E^D[\wp^n]/\mathcal{C}_n(E^D))^D$ is killed by F_d^n. Since it can be checked after passing to a finite etale cover of $\mathrm{Spec}(B)$, we reduce ourselves to showing that the Taguchi dual $(\underline{A/(\wp^n)})^D$ of the constant A-module scheme $\underline{A/(\wp^n)}$ over $\bar{B}$ is killed by F_d^n. This follows from Lemma 2.17 and (3.2). □

3.3. **Hodge-Tate-Taguchi maps.** For any positive integer n, any A-algebra B and any scheme X over A, we put $B_n = B/(\wp^n)$ and $X_n = X \times_A \mathrm{Spec}(A_n)$. We identify a quasi-coherent module on the big fppf site of X with a quasi-coherent $\mathcal{O}_X$-module by descent.

Let S be a scheme over A and $\mathcal{G}$ a finite v-module over S. For any scheme T over S, Taguchi duality gives a natural homomorphism of A-modules

$$\begin{aligned} \mathcal{G}^D(T) \simeq \mathrm{Hom}_{v,T}(\mathcal{G}|_T, C|_T) &\to \omega_{\mathcal{G}|_T}(T) \\ (g : \mathcal{G}|_T \to C|_T) &\mapsto g^*(dZ), \end{aligned}$$

which defines a natural homomorphism of big fppf sheaves of A-modules over S

$$\mathrm{HTT}_{\mathcal{G}}:\mathcal{G}^D\to\omega_{\mathcal{G}}.$$

We refer to it as the Hodge-Tate-Taguchi map for the finite v-module $\mathcal{G}$ over S, and also denote it by HTT if no confusion may occur. The formation of the Hodge-Tate-Taguchi map commutes with any base change.

Suppose that the A-module scheme $\mathcal{G}$ is killed by $\wp^n$. Then the Hodge-Tate-Taguchi map defines a natural A-linear homomorphism of big fppf sheaves on S_n

$$\mathrm{HTT}:\mathcal{G}^D|_{S_n}\otimes_{\underline{A_n}}\mathcal{O}_{S_n}\to\omega_{\mathcal{G}_n}.$$

Note that, if $\mathcal{G}^D$ is etale locally isomorphic to the constant A-module scheme $\underline{A_n}$ over S, then the $\mathcal{O}_{S_n}$-module $\mathcal{G}^D|_{S_n}\otimes_{\underline{A_n}}\mathcal{O}_{S_n}$ is invertible. By Lemma 3.6, this is the case if $\mathcal{G}=\mathcal{C}_n(E)$ for any Drinfeld module E of rank two over a reduced flat $\wp$-adic ring B with ordinary reduction.

Lemma 3.7. *Let S be any scheme over A. We give the finite t-module $C[\wp^n]$ the v-structure induced from that of C. Then the Hodge-Tate-Taguchi map for $C[\wp^n]$*

$$\mathrm{HTT}:\underline{A_n}\otimes_{\underline{A_n}}\mathcal{O}_{S_n}\simeq C[\wp^n]^D|_{S_n}\otimes_{\underline{A_n}}\mathcal{O}_{S_n}\to\omega_{C[\wp^n]_n}=\mathcal{O}_{S_n}dZ$$

is an isomorphism satisfying $\mathrm{HTT}(1)=dZ$.

Proof. Let $\iota:C[\wp^n]\to C$ be the natural closed immersion, as in the proof of Lemma 2.17. The definition of the Hodge-Tate-Taguchi map gives $\mathrm{HTT}(1)=\iota^*(dZ)$, which yields the lemma. □

Proposition 3.8. *Let B be a reduced flat $\wp$-adic ring. Let E be a Drinfeld module of rank two over B with ordinary reduction. Then the Hodge-Tate-Taguchi map*

$$\mathrm{HTT}:\mathcal{C}_n(E)^D|_{B_n}\otimes_{\underline{A_n}}\mathcal{O}_{\mathrm{Spec}(B_n)}\to\omega_{\mathcal{C}_n(E)}\otimes_B B_n=\omega_E\otimes_B B_n$$

is an isomorphism of invertible sheaves over B_n.

Proof. It is enough to show that HTT is an isomorphism after passing to a finite etale cover $\mathrm{Spec}(B')$ of $\mathrm{Spec}(B)$. We may assume that the A-module scheme $\mathcal{C}_n(E)^D|_{B'}=(\mathcal{C}_n(E)|_{B'})^D$ over B' is constant. In this case, the proposition follows from Lemma 3.7. □

4. $\wp$-adic properties of Drinfeld modular forms

4.1. **Drinfeld modular curves.** Here we review the theory of Drinfeld modular curves of level $\Gamma_1^\Delta(\mathfrak{n})$ and their compactifications given in [Hat]. Let $\mathfrak{n}$ be a non-constant monic polynomial in $A=\mathbb{F}_q[t]$ which is prime to $\wp$. Put $A_{\mathfrak{n}} = A[1/\mathfrak{n}]$. For any Drinfeld module E of rank two over an $A_{\mathfrak{n}}$-scheme S, a $\Gamma(\mathfrak{n})$-structure on E is an isomorphism of A-module schemes $\alpha : \underline{(A/(\mathfrak{n}))^2} \to E[\mathfrak{n}]$ over S. We know that the functor over $A_{\mathfrak{n}}$ sending S to the set of isomorphism classes of such pairs (E,α) over S is represented by a regular affine scheme $Y(\mathfrak{n})$ which is smooth of relative dimension one over $A_{\mathfrak{n}}$.

For any Drinfeld module E of rank two over an $A_{\mathfrak{n}}$-scheme S, we define a $\Gamma_1(\mathfrak{n})$-structure on E as a closed immersion of A-module schemes $\lambda : C[\mathfrak{n}] \to E$ over S. Then it is known that the functor over $A_{\mathfrak{n}}$, sending an $A_{\mathfrak{n}}$-scheme S to the set of isomorphism classes $[(E,\lambda)]$ of pairs (E,λ) consisting of a Drinfeld module E of rank two over S and a $\Gamma_1(\mathfrak{n})$-structure λ on E, is representable by an affine scheme $Y_1(\mathfrak{n})$ which is smooth over $A_{\mathfrak{n}}$ of relative dimension one. For any $\Gamma_1(\mathfrak{n})$-structure λ on E, the quotient $E[\mathfrak{n}]/\mathrm{Im}(\lambda)$ is a finite etale A-module scheme over S which is etale locally isomorphic to $\underline{A/(\mathfrak{n})}$, and thus the functor

$$\mathscr{I}som_{A,S}(\underline{A/(\mathfrak{n})}, E[\mathfrak{n}]/\mathrm{Im}(\lambda))$$

is represented by a finite etale $(A/(\mathfrak{n}))^\times$-torsor $I_{(E,\lambda)}$ over S.

Suppose that there exists a prime factor $\mathfrak{q}$ of $\mathfrak{n}$ of degree prime to $q-1$. Then we can choose a subgroup $\Delta \subseteq (A/(\mathfrak{n}))^\times$ which is a direct summand of $\mathbb{F}_q^\times \subseteq (A/(\mathfrak{n}))^\times$. Then a $\Gamma_1^\Delta(\mathfrak{n})$-structure on E is defined as a pair $(\lambda,[\mu])$ of a $\Gamma_1(\mathfrak{n})$-structure λ on E and an element $[\mu] \in (I_{(E,\lambda)}/\Delta)(S)$ [Hat, §3]. We have a fine moduli scheme $Y_1^\Delta(\mathfrak{n})$ of the isomorphism classes of triples $(E,\lambda,[\mu])$. The natural map $Y_1^\Delta(\mathfrak{n}) \to Y_1(\mathfrak{n})$ is finite and etale. The universal Drinfeld module over $Y_1^\Delta(\mathfrak{n})$ is denoted by $E_{\mathrm{un}}^\Delta = \mathbb{V}_*(\mathcal{L}_{\mathrm{un}}^\Delta)$ and put

$$\omega_{\mathrm{un}}^\Delta := \omega_{E_{\mathrm{un}}^\Delta} = (\mathcal{L}_{\mathrm{un}}^\Delta)^\vee.$$

For any Drinfeld module E over an $A_{\mathfrak{n}}$-scheme S, a $\Gamma_0(\wp)$-structure on E is a finite locally free closed A-submodule scheme $\mathcal{G}$ of $E[\wp]$ of rank q^d over S. Then we have a fine moduli scheme $Y_1^\Delta(\mathfrak{n},\wp)$ classifying tuples $(E,\lambda,[\mu],\mathcal{G})$ consisting of a Drinfeld module E of rank two over an $A_{\mathfrak{n}}$-scheme S, a $\Gamma_1^\Delta(\mathfrak{n})$-structure $(\lambda,[\mu])$ and a $\Gamma_0(\wp)$-structure $\mathcal{G}$ on E. The natural map $Y_1^\Delta(\mathfrak{n},\wp) \to Y_1^\Delta(\mathfrak{n})$ is finite, and it is etale over $A_{\mathfrak{n}}[1/\wp]$. For any $A_{\mathfrak{n}}$-algebra R, we write as $Y_1^\Delta(\mathfrak{n})_R = Y_1^\Delta(\mathfrak{n}) \times_{A_{\mathfrak{n}}} \mathrm{Spec}(R)$ and similarly for other Drinfeld modular curves.

For any $A_{\mathfrak{n}}$-algebra R_0 which is Noetherian, excellent and regular, we have a natural compactification $X_1^\Delta(\mathfrak{n})_{R_0}$ of $Y_1^\Delta(\mathfrak{n})_{R_0}$ which is proper and smooth with geometrically connected fibers over $A_{\mathfrak{n}}$. Similarly, we also have a compactification $X_1^\Delta(\mathfrak{n},\wp)_{R_0}$ of $Y_1^\Delta(\mathfrak{n},\wp)_{R_0}$ which is proper and smooth with geometrically connected fibers over $A_{\mathfrak{n}}[1/\wp]$. The maps

$$\langle a\rangle_{\mathfrak{n}} : [(E,\lambda,[\mu])] \mapsto [(E,a\lambda,[\mu])],\quad \langle c\rangle_{\Delta} : [(E,\lambda,[\mu])] \mapsto [(E,\lambda,c[\mu])]$$

induce actions of the groups $(A/(\mathfrak{n}))^\times$ and $(A/(\mathfrak{n}))^\times/\Delta = \mathbb{F}_q^\times$ on $X_1^\Delta(\mathfrak{n})_{R_0}$.

If R_0 is in addition a domain, then the sheaf $\omega_{\mathrm{un}}^\Delta$ and its pull-back to $Y_1^\Delta(\mathfrak{n},\wp)_{R_0}$ extend to natural invertible sheaves $\bar{\omega}_{\mathrm{un}}^\Delta$ and $\bar{\omega}_{\mathrm{un}}^{\Delta,\wp}$ on $X_1^\Delta(\mathfrak{n})_{R_0}$ and $X_1^\Delta(\mathfrak{n},\wp)_{R_0}$, respectively. In fact, the latter sheaf is the pull-back of the former one to $X_1^\Delta(\mathfrak{n},\wp)_{R_0}$. The natural action of $\mathbb{F}_q^\times$ on $\omega_{\mathrm{un}}^\Delta$ via $\langle -\rangle_\Delta$ also extends to an action on $\bar{\omega}_{\mathrm{un}}^\Delta$ covering its action on $X_1^\Delta(\mathfrak{n})_{R_0}$, and similarly for $\bar{\omega}_{\mathrm{un}}^{\Delta,\wp}$ [Hat, Theorem 5.3 and §7].

Suppose that R_0 is a flat $A_{\mathfrak{n}}$-algebra which is an excellent regular domain. Let $W_{\mathfrak{n}}(X)$ be the unique monic prime factor of $\Phi_{\mathfrak{n}}^C(X)$ in $A[X]$ which does not divide $\Phi_{\mathfrak{m}}^C(X)$ for any non-trivial divisor $\mathfrak{m}$ of $\mathfrak{n}$ [Car, §3]. Let $R_{\mathfrak{n}}$ be the affine ring of a connected component of $I_{R_0} = \mathrm{Spec}(R_0[X]/(W_{\mathfrak{n}}(X)))$, which is a finite etale domain over R_0.

Then the formal completion of $X_1^\Delta(\mathfrak{n})_{R_{\mathfrak{n}}}$ along cusps is studied in [Hat, §6], by using Tate-Drinfeld modules. For this, we follow the notation in [Hat, §4]. In particular, for any non-zero element $f \in A$, put

$$f\Lambda = \left\{ \Phi_{fa}^C\left(\frac{1}{x}\right) \;\middle|\; a \in A \right\} \subseteq R_0((x)),\quad e_{f\Lambda}(X) = X \prod_{\alpha\neq 0\in f\Lambda}\left(1-\frac{X}{\alpha}\right).$$

We set

$$\Phi_a^{f\Lambda}(X) = e_{f\Lambda}(\Phi_a^C(e_{f\Lambda}^{-1}(X))) \in R_0[[x]][X]. \tag{4.1}$$

Then the additive group $\mathrm{Spec}(R_0((x))[X])$ is endowed with a structure of a Drinfeld module of rank two over $R_0((x))$ such that its a-multiplication map is given by $\Phi_a^{f\Lambda}(X)$ for any $a \in A$. We refer to it as the Tate-Drinfeld module and denote it by $\mathrm{TD}(f\Lambda)$.

Lemma 4.1. *Let X be the parameter of $\mathrm{TD}(\Lambda)$ as above. We trivialize the underlying invertible sheaf $\omega_{\mathrm{TD}(\Lambda)}^{\otimes q}$ of the dual $\mathrm{TD}(\Lambda)^D$ by $(dX)^{\otimes q}$, and we denote the corresponding parameter of $\mathrm{TD}(\Lambda)^D$ by Y. Put $\Phi_t^{\mathrm{TD}(\Lambda)} = \theta + a_1\tau + a_2\tau^2$. Then the dual of the Kodaira-Spencer map*

$$\mathrm{KS}^\vee : \omega_{\mathrm{TD}(\Lambda)} \otimes \omega_{\mathrm{TD}(\Lambda)^D} \to \Omega^1_{R_0((x))/R_0}$$

satisfies $\mathrm{KS}^\vee(dX\otimes dY) = l(x)dx$ with

$$l(x) = \frac{da_1}{dx} - \frac{a_1}{a_2}\frac{da_2}{dx} \equiv \frac{1}{x} \bmod R_0[[x]].$$

Proof. We want to compute $\nabla_{\frac{d}{dx}}(dX)$. For this, first note that the inner biderivation $\delta_{\mathrm{id}} \in \mathrm{Der}_{\mathrm{in}}(\mathrm{TD}(\Lambda), \mathbb{G}_{\mathrm{a}})$ gives dX via the second isomorphism of Lemma 2.21. Then we have

$$\delta_{\mathrm{id},t} = \mathrm{id} \circ \Phi_t^{\mathrm{TD}(\Lambda)} - \Phi_t^{\mathbb{G}_{\mathrm{a}}} \circ \mathrm{id} = a_1\tau + a_2\tau^2$$

and $\nabla_{\frac{d}{dx}}(dX)$ corresponds to the class of $\delta \in \mathrm{Der}_0(\mathrm{TD}(\Lambda), \mathbb{G}_{\mathrm{a}})$ satisfying $\delta_t = \frac{da_1}{dx}\tau + \frac{da_2}{dx}\tau^2$. Subtracting the inner biderivation δ_β for $\beta = a_2^{-1}\frac{da_2}{dx}$, we may assume $\delta_t = l(x)\tau$. Hence, the element $\pi_{\frac{d}{dx}}(dX) \in \mathrm{Lie}(\mathrm{TD}(\Lambda)^D)$ is given by the biderivation $\delta' \in \mathrm{Der}_0(\mathrm{TD}(\Lambda)|_{T_{0,\varepsilon}}, C|_{T_{0,\varepsilon}})$ satisfying $\delta'_t = \varepsilon l(x)\tau$, where we put

$$T_0 = \mathrm{Spec}(R_0((x))), \quad T_{0,\varepsilon} = \mathrm{Spec}_{T_0}(\mathcal{O}_{T_0}[\varepsilon]/(\varepsilon^2)).$$

The map δ'_t is an element of $\mathrm{Hom}_{\mathbb{F}_q, T_{0,\varepsilon}}(\mathrm{TD}(\Lambda)|_{T_{0,\varepsilon}}, C|_{T_{0,\varepsilon}})$ defined by $Z \mapsto \varepsilon l(x)X^q$. Let $\mathcal{L} = R_0((x))\frac{d}{dX}$ be the underlying invertible sheaf of $\mathrm{TD}(\Lambda)$. Via the identification (2.7), the above homomorphism corresponds to the element

$$\varepsilon l(x)(dX)^{\otimes q} \in \mathrm{Ker}(\mathbb{V}_*(\mathcal{L}^{\otimes -q})(T_{0,\varepsilon}) \to \mathbb{V}_*(\mathcal{L}^{\otimes -q})(T_0))$$

and, with the parameter Y of $\mathbb{V}_*(\mathcal{L}^{\otimes -q})$ in the lemma, it corresponds to $l(x)\frac{d}{dY}$. This concludes the proof. □

Let $\mathfrak{m} \in A$ be any monic polynomial. By [Hat, Lemma 4.2], the map $X \mapsto e_{f\Lambda}(Z)$ defines a natural A-linear closed immersion

$$\lambda_{\infty,\mathfrak{m}}^{f\Lambda} : C[\mathfrak{m}] \to \mathrm{TD}(f\Lambda).$$

Moreover, by [Hat, Lemma 4.4], we have a natural A-linear isomorphism

$$\mu_{\infty,\mathfrak{m}}^{f\Lambda} : \underline{A/(\mathfrak{m})} \to \mathcal{H}_{\infty,\mathfrak{m}}^{f\Lambda} := \mathrm{TD}(f\Lambda)[\mathfrak{m}]/\mathrm{Im}(\lambda_{\infty,\mathfrak{m}}^{f\Lambda}),$$

which is defined as follows. Put

$$B_{0,\mathfrak{m}}^{f\Lambda} = R_0((x))[\eta]/(\Phi_{\mathfrak{m}}^C(\eta) - \Phi_f^C(1/x)). \tag{4.2}$$

Then $\mu_{\infty,\mathfrak{m}}^{f\Lambda}$ is the unique map such that the image of the element

$$\mu_{\infty,\mathfrak{m}}^{f\Lambda}(1) \in (\mathrm{TD}(f\Lambda)[\mathfrak{m}]/\mathrm{Im}(\lambda_{\infty,\mathfrak{m}}^{f\Lambda}))(R_0((x)))$$

in $(\mathrm{TD}(f\Lambda)[\mathfrak{m}]/\mathrm{Im}(\lambda_{\infty,\mathfrak{m}}^{f\Lambda}))(B_{0,\mathfrak{m}}^{f\Lambda})$ agrees with the image of the element $e_{f\Lambda}(\eta) \in \mathrm{TD}(f\Lambda)[\mathfrak{m}](B_{0,\mathfrak{m}}^{f\Lambda})$.

The pair $(\lambda_{\infty,\mathfrak{n}}^{\Lambda}, [\mu_{\infty,\mathfrak{n}}^{\Lambda}])$ defines a $\Gamma_1^{\Delta}(\mathfrak{n})$-structure on $\mathrm{TD}(\Lambda)$. The corresponding map $\mathrm{Spec}(R_0((x))) \to Y_1^{\Delta}(\mathfrak{n})_{R_0}$ extends to a map

$$x_\infty^{\Delta} : \mathrm{Spec}(R_0[[x]]) \to X_1^{\Delta}(\mathfrak{n})_{R_0}.$$

We refer to the image of the point (x) by this map as the ∞-cusp. Via the map x_∞^Δ, the complete local ring of $X_1^\Delta(\mathfrak{n})_{R_0}$ at the ∞-cusp is identified with $R_0[[x]]$ [Hat, Theorem 5.3].

Now [Hat, Theorem 6.3] is summarized as follows. The set of cusps defines an effective Cartier divisor $\mathrm{Cusps}_{R_0}^\Delta$ of $X_1^\Delta(\mathfrak{n})_{R_0}$ over R_0. Any cusp of $X_1^\Delta(\mathfrak{n})_{R_\mathfrak{n}}$ is labeled by an element of

$$\mathscr{H} = \{(a,b) \in (A/(\mathfrak{n}))^2 \mid (a,b) = (1)\}.$$

For any element $(a,b) \in \mathscr{H}$, let f_b be the monic generator of the ideal $\mathrm{Ann}_A(b(A/(\mathfrak{n})))$. Then the formal completion of $X_1^\Delta(\mathfrak{n})_{R_\mathfrak{n}}$ at the cusp labeled by (a,b) is isomorphic to $\mathrm{Spec}(R_\mathfrak{n}[[w]])$ in such a way that the pull-back of the universal Drinfeld module E_{un}^Δ to $\mathrm{Spec}(R_\mathfrak{n}((w)))$ is isomorphic to $\mathrm{TD}(f_b\Lambda)$, which is the pull-back of $\mathrm{TD}(\Lambda)$ by the map

$$R_0((x)) \to R_\mathfrak{n}((w)), \quad x \mapsto \Phi_{f_b}^C(1/w)^{-1}.$$

Corollary 4.2. *Suppose that R_0 is a flat $A_\mathfrak{n}$-algebra which is an excellent regular domain. Let g be the common genus of the fibers of $X_1^\Delta(\mathfrak{n})_{R_0}$ over R_0. Then, on each fiber, the invertible sheaf $(\bar\omega_{\mathrm{un}}^\Delta)^{\otimes 2}$ has degree no less than $2g$.*

Proof. Since the map $Y(\mathfrak{n}) \to Y_1^\Delta(\mathfrak{n})$ is etale, [Gek4, Theorem 6.11] implies that the dual of the Kodaira-Spencer map for the universal Drinfeld module E_{un}^Δ over $Y_1^\Delta(\mathfrak{n})_{R_0}$

$$\mathrm{KS}^\vee : \omega_{E_{\mathrm{un}}^\Delta} \otimes \omega_{(E_{\mathrm{un}}^\Delta)^D} \to \Omega^1_{Y_1^\Delta(\mathfrak{n})_{R_0}/R_0}$$

is an isomorphism. We write as $\Phi_t^{E_{\mathrm{un}}^\Delta} = \theta + A_1\tau + A_2\tau^2$. Since we have the isomorphism

$$\omega_{E_{\mathrm{un}}^\Delta}^{\otimes q-1} \to \omega_{(E_{\mathrm{un}}^\Delta)^D}^{\otimes q-1}, \quad l \mapsto l \otimes A_2^{\otimes -1}, \tag{4.3}$$

the map $(\mathrm{KS}^\vee)^{\otimes q-1}$ induces an isomorphism

$$\omega_{E_{\mathrm{un}}^\Delta}^{\otimes q-1} \otimes \omega_{E_{\mathrm{un}}^\Delta}^{\otimes q-1} \to (\Omega^1_{Y_1^\Delta(\mathfrak{n})_{R_0}/R_0})^{\otimes q-1}. \tag{4.4}$$

Consider the cusp labeled by $(a,b) \in \mathscr{H}$ and the pull-back of this map to $R_\mathfrak{n}((w))$, as in [Hat, Theorem 6.3]. Since $R_\mathfrak{n}((w))$ is a domain, the isomorphism $E_{\mathrm{un}}^\Delta|_{R_\mathfrak{n}((w))} \to \mathrm{TD}(f_b\Lambda)$ is $R_\mathfrak{n}((w))$-linear. Using Theorem 2.19 (3) and the functoriality of KS, we can show that the pull-back of (4.4) is identified with a similar map

$$\omega_{\mathrm{TD}(f_b\Lambda)}^{\otimes q-1} \otimes \omega_{\mathrm{TD}(f_b\Lambda)}^{\otimes q-1} \to (\Omega^1_{R_\mathfrak{n}((w))/R_\mathfrak{n}})^{\otimes q-1}$$

induced by $\mathrm{KS}^\vee$ for $\mathrm{TD}(f_b\Lambda)$ over $R_{\mathfrak{n}}((w))$. Put $\Phi_t^{\mathrm{TD}(\Lambda)} = \theta + a_1\tau + a_2\tau^2$ with $a_i \in R_0[[x]]$. By Lemma 4.1 and (4.3), this map is given by

$$(dX)^{\otimes q-1} \otimes (dX)^{\otimes q-1} \mapsto a_2^{-1}(\frac{da_1}{dx} - \frac{a_1}{a_2}\frac{da_2}{dx})^{q-1}(dx)^{\otimes q-1}.$$

Since the right-hand side is an element of $R_{\mathfrak{n}}[[w]]^\times(\frac{dx}{x^2})^{\otimes q-1}$, [Hat, Theorem 6.3 (3)] implies that the isomorphism (4.4) extends to an isomorphism

$$(\bar{\omega}_{\mathrm{un}}^{\Delta})^{\otimes q-1} \otimes (\bar{\omega}_{\mathrm{un}}^{\Delta})^{\otimes q-1} \to (\Omega^1_{X_1^\Delta(\mathfrak{n})_{R_0}/R_0}(2\mathrm{Cusps}^\Delta_{R_0}))^{\otimes q-1}.$$

Since $\mathrm{Cusps}^\Delta_{R_0}$ is non-empty, the corollary follows. □

4.2. **Canonical subgroups of Tate-Drinfeld modules.** In this subsection, we consider the case $R_0 = \mathcal{O}_K$. Thus we have the Tate-Drinfeld module $\mathrm{TD}(f\Lambda)$ over $\mathcal{O}_K((x))$. Put $d = \deg(\wp)$ as before. We denote the normalized $\wp$-adic valuation of $\mathcal{O}_K((x))$ by $v_\wp$.

Lemma 4.3. *The Tate-Drinfeld module* $\mathrm{TD}(f\Lambda)$ *over* $\mathcal{O}_K((x))$ *has ordinary reduction.*

Proof. Put $\bar{\Phi}_\wp^{f\Lambda}(X) = \Phi_\wp^{f\Lambda}(X) \bmod \wp$, which is an element of the ring $k(\wp)[[x]][X]$. From [Hat, (4.4)], we see that the coefficient of X^{q^d} in $\bar{\Phi}_\wp^{f\Lambda}(X)$ is an x-adic unit and those of larger degree have positive x-adic valuations. By [Hat, Lemma 4.1], the coefficient of $X^{q^{2d}}$ is non-zero. An inspection of the Newton polygon of $\bar{\Phi}_\wp^{f\Lambda}(X)$ shows that this polynomial has at least $q^{2d} - q^d$ non-zero roots in an algebraic closure of $k(\wp)((x))$. Thus the reduction of $\mathrm{TD}(f\Lambda)$ modulo $\wp$ is ordinary. □

The map $\lambda_{\infty,\wp^n}^{f\Lambda}$ identifies $C[\wp^n]$ with a closed A-submodule scheme of $\mathrm{TD}(f\Lambda)[\wp^n]$, which we denote by $\mathcal{C}_n^{f\Lambda}$. We refer to $\mathcal{C}_n^{f\Lambda}$ as the canonical subgroup of $\mathrm{TD}(f\Lambda)$ of level n. The reduction $\overline{\mathcal{C}_n^{f\Lambda}}$ modulo $\wp$ agrees with $\mathrm{Ker}(F_d^n)$ of the reduction of $\mathrm{TD}(f\Lambda)$. Thus the pull-backs of $\mathcal{C}_n^{f\Lambda}$ to $(\mathcal{O}_K/(\wp^m))((x))$ and the $\wp$-adic completion $\mathcal{O}_K((x))^\wedge$ agree with the canonical subgroups of level n of $\mathrm{TD}(f\Lambda)$ over them in the sense of Lemma 3.5.

We define the canonical and canonical etale isogenies of level one for $\mathrm{TD}(f\Lambda)$ as the natural maps

$$\pi^{f\Lambda} : \mathrm{TD}(f\Lambda) \to \mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}, \quad \rho^{f\Lambda} : \mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda} \to \mathrm{TD}(f\Lambda).$$

They satisfy

$$\rho^{f\Lambda} \circ \pi^{f\Lambda} = \Phi_\wp^{\mathrm{TD}(f\Lambda)}, \quad \pi^{f\Lambda} \circ \rho^{f\Lambda} = \Phi_\wp^{\mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}}. \tag{4.5}$$

By Lemma 2.8 (2), the quotient $\mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}$ has a natural structure of a Drinfeld module of rank two which makes these isogenies compatible with A-actions. The $\Gamma_1^{\Delta}(\mathfrak{n})$-structure $(\lambda_{\infty,\mathfrak{n}}^{f\Lambda}, [\mu_{\infty,\mathfrak{n}}^{f\Lambda}])$ on $\mathrm{TD}(f\Lambda)$ induces that on $\mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}$, which we denote by $(\bar{\lambda}_{\infty,\mathfrak{n}}^{f\Lambda}, [\bar{\mu}_{\infty,\mathfrak{n}}^{f\Lambda}])$.

Since the power series $e_{f\Lambda}(X) \in \mathcal{O}_K[[x]][[X]]$ is entire, any root $\beta \neq 0$ of $\Phi_\wp^C(Z)$ in its splitting field L over K defines an element $e_{f\Lambda}(\beta)$ of $\mathcal{O}_L[[x]]$. From Lemma 3.2 and the definition of $e_{f\Lambda}(X)$, we obtain

$$\Phi_\wp^C(\beta) = 0, \beta \neq 0 \Rightarrow e_{f\Lambda}(\beta) \in \beta(\mathcal{O}_L[[x]]^\times). \tag{4.6}$$

Then we put

$$\Psi_\wp^{f\Lambda}(X) = \wp X \prod_{\Phi_\wp^C(\beta)=0, \beta\neq 0} \left(1 - \frac{X}{e_{f\Lambda}(\beta)}\right) \in \mathcal{O}_K[[x]][X].$$

As in the proof of [Leh, Ch. 2, Lemma 1.2], we see that this is an $\mathbb{F}_q$-linear additive polynomial, and (4.6) implies that its leading coefficient is an element of $\mathcal{O}_K[[x]]^\times$. Hence $X \mapsto \Psi_\wp^{f\Lambda}(X)$ defines an isogeny of $\mathbb{F}_q$-module schemes over $\mathcal{O}_K((x))$

$$\pi_\wp^{f\Lambda} : \mathrm{TD}(f\Lambda) \to \mathrm{TD}(f\Lambda).$$

Lemma 4.4. $\mathrm{Ker}(\pi_\wp^{f\Lambda}) = \mathcal{C}_1^{f\Lambda}$.

Proof. We denote by $\mathcal{O}_K[[x]]\langle Z\rangle$ the x-adic completion of $\mathcal{O}_K[[x]][Z]$. By comparing ranks, it is enough to show that the composite $\pi_\wp^{f\Lambda} \circ \lambda_{\infty,\wp}^{f\Lambda}$ is zero. From the definition of the map $\lambda_{\infty,\wp}^{f\Lambda}$, this amounts to showing that the image of $\Psi_\wp^{f\Lambda}(e_{f\Lambda}(Z))$ in the ring $\mathcal{O}_K[[x]]\langle Z\rangle/(\Phi_\wp^C(Z))$ is zero. For this, note that we have the equality of entire series over $K((x))$

$$\Psi_\wp^{f\Lambda}(e_{f\Lambda}(Z)) = e_{\wp f\Lambda}(\Phi_\wp^C(Z)), \tag{4.7}$$

since they have the same linear term $\wp$ and divisor $f\Lambda + (\Phi_\wp^C)^{-1}(0)$. Thus the equality also holds in $\mathcal{O}_K[[x]]\langle Z\rangle$. Since the latter ring is Noetherian, the ideal $(\Phi_\wp^C(Z))$ is x-adically closed and thus it contains the element $e_{\wp f\Lambda}(\Phi_\wp^C(Z))$. $\square$

Thus the a-multiplication map of $\mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}$ for any $a \in A$ is given by a unique polynomial $\Phi'_a(X)$ satisfying

$$\Psi_\wp^{f\Lambda}(\Phi_a^{f\Lambda}(X)) = \Phi'_a(\Psi_\wp^{f\Lambda}(X)).$$

We define

$$F_\wp(x) = \frac{1}{\Phi_\wp^C\left(\frac{1}{x}\right)} \in x^{q^d}(1 + \wp x\mathcal{O}_K[[x]]),$$

which gives an $\mathcal{O}_K$-algebra homomorphism

$$\nu_\wp^\sharp : \mathcal{O}_K((x)) \to \mathcal{O}_K((x)), \quad x \mapsto F_\wp(x)$$

and the induced map $\nu_\wp : \mathrm{Spec}(\mathcal{O}_K((x))) \to \mathrm{Spec}(\mathcal{O}_K((x)))$. For any element $F(X) = \sum_{l \geqslant 0} a_l X^l \in \mathcal{O}_K((x))[[X]]$, we put $\nu_\wp^*(F)(X) = \sum_{l \geqslant 0} \nu_\wp^\sharp(a_l) X^l$. Then we have

$$\nu_\wp^*(e_{f\Lambda})(X) = e_{\wp f\Lambda}(X). \tag{4.8}$$

Thus (4.1) yields

$$\nu_\wp^*(\Phi_a^{f\Lambda})(e_{\wp f\Lambda}(X)) = e_{\wp f\Lambda}(\Phi_a^C(X)) \tag{4.9}$$

for any $a \in A$. On the other hand, (4.7) and (4.8) yield

$$\Psi_\wp^{f\Lambda}(e_{f\Lambda}(X)) = e_{\wp f\Lambda}(\Phi_\wp^C(X)) = \nu_\wp^*(e_{f\Lambda})(\Phi_\wp^C(X)). \tag{4.10}$$

Lemma 4.5.

$$(\mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}, \bar{\lambda}_{\infty,\mathfrak{n}}^{f\Lambda}, [\bar{\mu}_{\infty,\mathfrak{n}}^{f\Lambda}]) = \nu_\wp^*(\mathrm{TD}(f\Lambda), \wp\lambda_{\infty,\mathfrak{n}}^{f\Lambda}, [\mu_{\infty,\mathfrak{n}}^{f\Lambda}]).$$

Proof. First let us show the equality $\nu_\wp^*(\mathrm{TD}(f\Lambda)) = \mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}$. This amounts to showing

$$\Psi_\wp^{f\Lambda}(\Phi_a^{f\Lambda}(X)) = \nu_\wp^*(\Phi_a^{f\Lambda})(\Psi_\wp^{f\Lambda}(X))$$

for any $a \in A$. It is enough to show the equality in the ring $K((x))[[X]]$. For this, (4.1), (4.9) and (4.10) yield

$$\begin{aligned}\Psi_\wp^{f\Lambda}(\Phi_a^{f\Lambda}(e_{f\Lambda}(X))) &= \Psi_\wp^{f\Lambda}(e_{f\Lambda}(\Phi_a^C(X))) = e_{\wp f\Lambda}(\Phi_{\wp a}^C(X)) \\ &= \nu_\wp^*(\Phi_a^{f\Lambda})(e_{\wp f\Lambda}(\Phi_\wp^C(X))) = \nu_\wp^*(\Phi_a^{f\Lambda})(\Psi_\wp^{f\Lambda}(e_{f\Lambda}(X)))\end{aligned}$$

and the claim follows by plugging in $e_{f\Lambda}^{-1}(X)$. The $\Gamma_1(\mathfrak{n})$-structure $\bar{\lambda}_{\infty,\mathfrak{n}}^{f\Lambda}$ is given by $X \mapsto \Psi_\wp^{f\Lambda}(e_{f\Lambda}(Z))$. By (4.10), the latter element is equal to $\nu_\wp^*(e_{f\Lambda})(\Phi_\wp^C(Z))$, which means $\bar{\lambda}_{\infty,\mathfrak{n}}^{f\Lambda} = \nu_\wp^*(\wp\lambda_{\infty,\mathfrak{n}}^{f\Lambda})$.

For the assertion on $[\bar{\mu}_{\infty,\mathfrak{n}}^{f\Lambda}]$, consider the ring $B_{0,\mathfrak{n}}^{f\Lambda}$ of (4.2) and its base extension $\nu_\wp^*(B_{0,\mathfrak{n}}^{f\Lambda})$ by the map $\nu_\wp^\sharp$. These rings are free of rank $q^{\deg(\mathfrak{n})}$ over $\mathcal{O}_K((x))$. We have a homomorphism of $\mathcal{O}_K((x))$-algebras

$$\nu_\wp^*(B_{0,\mathfrak{n}}^{f\Lambda}) \simeq \mathcal{O}_K((x))[\eta]/(\Phi_\mathfrak{n}^C(\eta) - \Phi_{f\wp}^C(1/x)) \to B_{0,\mathfrak{n}}^{f\Lambda}$$

defined by $\eta \mapsto \Phi_\wp^C(\eta)$. Since $(\wp, \mathfrak{n}) = 1$, we have $\alpha\wp + \beta\mathfrak{n} = 1$ for some $\alpha, \beta \in A$ and this map sends $\Phi_\alpha^C(\eta) + \Phi_{f\beta}^C(1/x)$ to η. Hence it is surjective and thus these two rings are isomorphic as $\mathcal{O}_K((x))$-algebras.

Now a similar argument as above implies that, for the map $\bar{\mu}_{\infty,\mathfrak{n}}^{f\Lambda} : \underline{A/(\mathfrak{n})} \to \nu_\wp^*(\mathcal{H}_{\infty,\mathfrak{n}}^{f\Lambda})$, the restriction $\bar{\mu}_{\infty,\mathfrak{n}}^{f\Lambda}(1)|_{B_{0,\mathfrak{n}}^{f\Lambda}}$ is equal to the image of the element

$$\nu_\wp^*(e_{f\Lambda})(\Phi_\wp^C(\eta)) \in \nu_\wp^*(\mathrm{TD}(f\Lambda))(B_{0,\mathfrak{n}}^{f\Lambda}).$$

On the other hand, for the pull-back $\nu_\wp^*(\mu_{\infty,\mathfrak{n}}^{f\Lambda}) : \underline{A/(\mathfrak{n})} \to \nu_\wp^*(\mathcal{H}_{\infty,\mathfrak{n}}^{f\Lambda})$, the restriction $\nu_\wp^*(\mu_{\infty,\mathfrak{n}}^{f\Lambda})(1)|_{\nu_\wp^*(B_{0,\mathfrak{n}}^{f\Lambda})}$ is equal to the image of the element

$$e_{f\Lambda}(\eta) \otimes 1 = \nu_\wp^*(e_{f\Lambda})(\eta) \in \nu_\wp^*(\mathrm{TD}(f\Lambda))(\nu_\wp^*(B_{0,\mathfrak{n}}^{f\Lambda})).$$

Since they agree with each other in $\nu_\wp^*(\mathrm{TD}(f\Lambda))(B_{0,\mathfrak{n}}^{f\Lambda})$, we obtain $\bar{\mu}_{\infty,\mathfrak{n}}^{f\Lambda} = \nu_\wp^*(\mu_{\infty,\mathfrak{n}}^{f\Lambda})$. □

By Lemma 4.5, the canonical etale isogeny $\rho^{f\Lambda}$ induces an isomorphism of $\mathcal{O}_K((x))$-modules

$$((\rho^{f\Lambda})^*)^{-1} : \omega_{\mathrm{TD}(f\Lambda)} \otimes_{\mathcal{O}_K((x)),\nu_\wp^\sharp} \mathcal{O}_K((x)) = \omega_{\mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}} \to \omega_{\mathrm{TD}(f\Lambda)}.$$

Corollary 4.6.

$$((\rho^{f\Lambda})^*)^{-1}(dX \otimes 1) = dX.$$

Proof. Since we have shown that the canonical isogeny $\pi^{f\Lambda}$ of level one for $\mathrm{TD}(f\Lambda)$ is given by $X \mapsto \Psi_\wp^{f\Lambda}(X)$, we have $(\pi^{f\Lambda})^*(dX) = \wp dX$. From (4.5), we obtain $(\rho^{f\Lambda})^*(dX) = dX$ in $\omega_{\mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}}$, which is identified with $dX \otimes 1$ via $\nu_\wp^*(\mathrm{TD}(f\Lambda)) = \mathrm{TD}(f\Lambda)/\mathcal{C}_1^{f\Lambda}$. □

4.3. **Drinfeld modular forms.** Let k be an integer. Let M be an $A_\mathfrak{n}$-module. We define a Drinfeld modular form of level $\Gamma_1^\Delta(\mathfrak{n})$ and weight k with coefficients in M as an element of

$$M_k(\Gamma_1^\Delta(\mathfrak{n}))_M = H^0(X_1^\Delta(\mathfrak{n})_{A_\mathfrak{n}}, (\bar{\omega}_{\mathrm{un}}^\Delta)^{\otimes k} \otimes_{A_\mathfrak{n}} M).$$

By [Hat, Theorem 5.3 (4)], the group $\mathbb{F}_q^\times$ acts on the $A_\mathfrak{n}$-module $M_k(\Gamma_1^\Delta(\mathfrak{n}))_M$ via $c \mapsto \langle c \rangle_\Delta$. Since $q-1$ is invertible in $A_\mathfrak{n}$, we have the decomposition

$$M_k(\Gamma_1^\Delta(\mathfrak{n}))_M = \bigoplus_{m \in \mathbb{Z}/(q-1)\mathbb{Z}} M_{k,m}(\Gamma_1(\mathfrak{n}))_M,$$

where the direct summand $M_{k,m}(\Gamma_1(\mathfrak{n}))_M$ is the maximal submodule on which the operator $\langle c \rangle_\Delta$ acts by the multiplication by c^{-m} for any $c \in \mathbb{F}_q^\times$. We say $f \in M_k(\Gamma_1^\Delta(\mathfrak{n}))_M$ is of type m if $f \in M_{k,m}(\Gamma_1(\mathfrak{n}))_M$.

Consider the map $x_\infty^\Delta : \mathrm{Spec}(A_\mathfrak{n}[[x]]) \to X_1^\Delta(\mathfrak{n})_{A_\mathfrak{n}}$ as in [Hat, Theorem 5.3]. For any $f \in M_k(\Gamma_1^\Delta(\mathfrak{n}))_M$, we define the x-expansion of f at the ∞-cusp as the unique power series $f_\infty(x) \in A_\mathfrak{n}[[x]] \otimes_{A_\mathfrak{n}} M$ satisfying

$$(x_\infty^\Delta)^*(f) = f_\infty(x)(dX)^{\otimes k}.$$

We also have a variant $M_k(\Gamma_1^\Delta(\mathfrak{n}, \wp))_M$ of level $\Gamma_1^\Delta(\mathfrak{n}, \wp)$, using $X_1^\Delta(\mathfrak{n}, \wp)$, the sheaf $\bar{\omega}_{\mathrm{un}}^{\Delta,\wp}$ and the ∞-cusp $x_\infty^{\Delta,\wp}$ of [Hat, §7].

Proposition 4.7. *(1) (x-expansion principle) For any $A_{\mathfrak{n}}$-module M and $f \in M_k(\Gamma_1^\Delta(\mathfrak{n}))_M$, if $f_\infty(x) = 0$ then $f = 0$. Moreover, for any A_n-modules $N \subseteq M$ and any $f \in M_k(\Gamma_1^\Delta(\mathfrak{n}))_M$, we have $f_\infty(x) \in A_{\mathfrak{n}}[[x]] \otimes_{A_{\mathfrak{n}}} N$ if and only if $f \in M_k(\Gamma_1^\Delta(\mathfrak{n}))_N$. The same assertions hold for the case of level $\Gamma_1^\Delta(\mathfrak{n},\wp)$ if M is an $A_{\mathfrak{n}}[1/\wp]$-module.*

(2) For any $k \geqslant 2$ and any $A_{\mathfrak{n}}$-module M, the natural map

$$M_k(\Gamma_1^\Delta(\mathfrak{n}))_{A_{\mathfrak{n}}} \otimes_{A_{\mathfrak{n}}} M \to M_k(\Gamma_1^\Delta(\mathfrak{n}))_M$$

is an isomorphism.

Proof. Since $X_1^\Delta(\mathfrak{n})_{A_{\mathfrak{n}}}$ and $X_1^\Delta(\mathfrak{n},\wp)_{A_{\mathfrak{n}}[1/\wp]}$ are smooth and geometrically connected, Krull's intersection theorem and [Hat, Theorem 5.3] (and for the case of level $\Gamma_1^\Delta(\mathfrak{n},\wp)$, the corresponding statements in [Hat, §7]) imply the assertion (1), as in the proof of [Kat, §1.6]. The assertion (2) follows from Corollary 4.2, similarly to the proof of [Kat, Theorem 1.7.1]. □

Let $\mathbb{C}_\infty$ be the $(1/t)$-adic completion of an algebraic closure of $\mathbb{F}_q((1/t))$. Put

$$\Gamma_1^\Delta(\mathfrak{n}) = \left\{ g \in GL_2(A) \;\middle|\; g \bmod (\mathfrak{n}) \in \begin{pmatrix} \Delta & * \\ 0 & 1 \end{pmatrix} \right\},$$
$$\Gamma(\mathfrak{n}) = \left\{ g \in GL_2(A) \;\middle|\; g \bmod (\mathfrak{n}) \in \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

where the former group is independent of the choice of Δ.

Note that our definition of Drinfeld modular forms is compatible with the classical one over $\mathbb{C}_\infty$ as in [Gek2, Gek3]; over the compactification $X(\mathfrak{n})_{\mathbb{C}_\infty}$ of $Y(\mathfrak{n})_{\mathbb{C}_\infty}$ this follows from [Gos1, Theorem 1.79], and the spaces of Drinfeld modular forms of level $\Gamma_1^\Delta(\mathfrak{n})$ and weight k in both definitions are the fixed parts of the natural action of $\left\{\begin{pmatrix} \Delta & * \\ 0 & 1 \end{pmatrix}\right\}$ on them. We can also show that our x-expansion $f_\infty(x)$ of f at the ∞-cusp agrees with Gekeler's t-expansion at ∞ (see [Gek2, Ch. V, §2], while the normalization we adopt is as in [Gek3, §5]) of the associated classical Drinfeld modular form to f.

By [Gek3, Proposition (6.11)] and Proposition 4.7 (1), Gekeler's lift g_d of the Hasse invariant is an element of $M_{q^d-1,0}(\Gamma_1(\mathfrak{n}))_{A_{\mathfrak{n}}}$ satisfying

$$(g_d)_\infty(x) \equiv 1 \bmod \wp. \tag{4.11}$$

4.4. **Ordinary loci.** In the rest of the paper, we write as $Y_{\mathrm{un}} = Y_1^{\Delta}(\mathfrak{n})_{\mathcal{O}_K}$ and $X_{\mathrm{un}} = X_1^{\Delta}(\mathfrak{n})_{\mathcal{O}_K}$. For any positive integer m, the pull-back of any scheme T over $\mathcal{O}_K$ to $\mathcal{O}_{K,m} = \mathcal{O}_K/(\wp^m)$ is denoted by T_m.

Since we know that $X_{\mathrm{un},1}$ has a supersingular point [Gek1, Satz (5.9)], the ordinary loci $X_{\mathrm{un},m}^{\mathrm{ord}}$ in $X_{\mathrm{un},m}$ and $Y_{\mathrm{un},m}^{\mathrm{ord}}$ in $Y_{\mathrm{un},m}$ are affine open subschemes of finite type over $\mathcal{O}_{K,m}$. We put

$$B_{\mathrm{un},m}^{\mathrm{ord}} = \mathcal{O}(Y_{\mathrm{un},m}^{\mathrm{ord}}).$$

This is a flat $\mathcal{O}_{K,m}$-algebra of finite type, and the collection $\{B_{\mathrm{un},m}^{\mathrm{ord}}\}_m$ forms a projective system of $\mathcal{O}_K$-algebras with surjective transition maps. We define

$$\hat{B}_{\mathrm{un}}^{\mathrm{ord}} = \varprojlim_n B_{\mathrm{un},m}^{\mathrm{ord}}, \quad Y_{\mathrm{un}}^{\mathrm{ord}} = \mathrm{Spec}(\hat{B}_{\mathrm{un}}^{\mathrm{ord}}).$$

Then we have $\hat{B}_{\mathrm{un}}^{\mathrm{ord}}/(\wp^m) = B_{\mathrm{un},m}^{\mathrm{ord}}$ and $\hat{B}_{\mathrm{un}}^{\mathrm{ord}}$ is flat over $\mathcal{O}_K$. This implies that $\hat{B}_{\mathrm{un}}^{\mathrm{ord}}$ is $\wp$-adically complete and topologically of finite type over $\mathcal{O}_K$. Moreover, since $B_{\mathrm{un},1}^{\mathrm{ord}}$ is a regular domain, the ring $\hat{B}_{\mathrm{un}}^{\mathrm{ord}}$ is reduced. Thus $\hat{B}_{\mathrm{un}}^{\mathrm{ord}}$ is a reduced flat $\wp$-adic ring. On the other hand, we have a map $Y_{\mathrm{un}}^{\mathrm{ord}} \to Y_{\mathrm{un}}$ and we denote by $\mathcal{E}_{\mathrm{un}}^{\mathrm{ord}}$ the pull-back of the universal Drinfeld module to $\hat{B}_{\mathrm{un}}^{\mathrm{ord}}$, which has ordinary reduction.

Now we can form the canonical subgroup $\mathcal{C}_n = \mathcal{C}_n(\mathcal{E}_{\mathrm{un}}^{\mathrm{ord}})$ of level n for $\mathcal{E}_{\mathrm{un}}^{\mathrm{ord}}$. As is seen in §3.2, it has the v-structure induced from that of $\mathcal{E}_{\mathrm{un}}^{\mathrm{ord}}$, which is unique by Lemma 2.13 (2). Lemma 3.6 implies that its Taguchi dual $\mathcal{C}_n^D$ is etale. We denote by $\mathcal{C}_{n,m}$ the pull-back of $\mathcal{C}_n$ to $Y_{\mathrm{un},m}^{\mathrm{ord}}$ endowed with the induced v-structure, and similarly for $(\mathcal{C}_n^D)_m$. Then the Taguchi dual $\mathcal{C}_{n,m}^D$ of $\mathcal{C}_{n,m}$ agrees with $(\mathcal{C}_n^D)_m$ as a finite v-module and they are finite and etale over $Y_{\mathrm{un},m}^{\mathrm{ord}}$.

Lemma 4.8. *The finite v-module $\mathcal{C}_{n,m}$ over $Y_{\mathrm{un},m}^{\mathrm{ord}}$ extends to a finite v-module $\bar{\mathcal{C}}_{n,m}$ over $X_{\mathrm{un},m}^{\mathrm{ord}}$ such that its Taguchi dual $\bar{\mathcal{C}}_{n,m}^D$ is etale locally isomorphic to $\underline{A/(\wp^n)}$.*

Proof. Let $K_{\mathfrak{n}}$ be a splitting field of $\Phi_{\mathfrak{n}}^C(X)$ over K. Note that $\mathrm{Spec}(\mathcal{O}_{K_{\mathfrak{n}}})$ is identified with a connected component of $I_{\mathcal{O}_K} = \mathrm{Spec}(\mathcal{O}_K[X]/(W_{\mathfrak{n}}(X)))$. Consider the formal completion of $X_{\mathrm{un}}|_{\mathcal{O}_{K_{\mathfrak{n}}}}$ at the cusp labeled by $(a,b) \in \mathscr{H}$, which is isomorphic to $\mathrm{Spec}(\mathcal{O}_{K_{\mathfrak{n}}}[[w]])$ by [Hat, Theorem 6.3]. Let f_b be the monic generator of $\mathrm{Ann}_A(b(A/(\mathfrak{n})))$

We denote the $\wp$-adic completion of $\mathcal{O}_{K_{\mathfrak{n}}}((w))$ by $\mathcal{O}$, which is a reduced flat $\wp$-adic ring. The pull-back of $\mathcal{E}_{\mathrm{un}}^{\mathrm{ord}}$ to $\mathcal{O}$ is isomorphic to that of the Tate-Drinfeld module $\mathrm{TD}(f_b\Lambda)$ over $\mathcal{O}_{K_{\mathfrak{n}}}((w))$ to $\mathcal{O}$. By the uniqueness of the canonical subgroup in Lemma 3.5, we have

$\mathcal{C}_n|_{\mathcal{O}} \simeq \mathcal{C}_n^{f_b\Lambda}|_{\mathcal{O}} = C[\wp^n]$. Lemma 2.13 (2) implies that this identification is compatible with v-structures, where we give $C[\wp^n]$ the induced v-structure from C. Taking modulo $\wp^m$, we obtain an isomorphism $\mathcal{C}_{n,m}|_{\mathcal{O}_{K_{\mathfrak{n}},m}((w))} \simeq C[\wp^n]$ of v-modules over $\mathcal{O}_{K_{\mathfrak{n}},m}((w))$.

This implies that, by an fpqc descent, the finite v-module $\mathcal{C}_{n,m}$ extends to a finite v-module $\bar{\mathcal{C}}_{n,m}$ over $X_{\mathrm{un},m}^{\mathrm{ord}}$ such that its restriction to the formal completion at each cusp is isomorphic to $C[\wp^n]$ with the induced v-structure from C. Then the Taguchi dual $\bar{\mathcal{C}}_{n,m}^D$ is finite etale and every geometric fiber is isomorphic to $\underline{A/(\wp^n)}$ as an A-module scheme. This yields the lemma. □

Lemma 4.9. *Let U be any non-empty open subscheme of $X_{\mathrm{un},m}^{\mathrm{ord}}$ and $\bar{\xi}$ any geometric point of U. Then the character of its etale fundamental group with base point $\bar{\xi}$*

$$r_{n,m} : \pi_1^{\mathrm{et}}(U) \to \pi_1^{\mathrm{et}}(X_{\mathrm{un},m}^{\mathrm{ord}}) \to (A/(\wp^n))^\times$$

defined by $\bar{\mathcal{C}}_{n,m}^D$ is surjective.

Proof. We may assume $m = 1$. Let L be the function field of $X_{\mathrm{un},1}$. As in [Kat, Theorem 4.3], it is enough to show that the restriction of $r_{n,1}$ to the inertia subgroup of $\mathrm{Gal}(L^{\mathrm{sep}}/L)$ at a supersingular point is surjective.

Take $\xi' \in X_{\mathrm{un},1}$ corresponding to a supersingular Drinfeld module over an algebraic closure k of $k(\wp)$. The complete local ring of $X_{\mathrm{un},1} \times_{k(\wp)} k$ at ξ' is isomorphic to $k[[u]]$. Let $\mathbb{E}$ be the restriction of $\mathcal{E}_{\mathrm{un}}$ to this complete local ring. By [Sha, Remark 3.15], we have $\mathrm{Lie}(V_{d,\mathbb{E}}) = -u$ and the restriction $\mathbb{E}|_{k((u))}$ to the generic fiber is ordinary. By Theorem 2.19 (4) and Lemma 3.3, we have $\mathcal{C}_n(\mathbb{E}|_{k((u))})^D = \mathrm{Ker}(V^n_{d,\mathbb{E}^D|_{k((u))}})$. Here $\mathbb{E}^D|_{k((u))}$ is the dual of $\mathbb{E}|_{k((u))}$, which is also ordinary by Proposition 3.4. Hence it suffices to show that the finite etale A-module scheme $\mathrm{Ker}(V^n_{d,\mathbb{E}^D|_{k((u))}})$ defines a totally ramified extension of $k((u))$ of degree $\sharp(A/(\wp^n))^\times$.

For this, Proposition 3.4 also implies that the map $\mathrm{Lie}(V_{d,\mathbb{E}^D})$ is the multiplication by an element of $k[[u]]$ with normalized u-adic valuation one. Let v_u be the normalized u-adic valuation on $k((u))$ and we extend it to its algebraic closure $k((u))^{\mathrm{alg}}$. Since the fiber of $\mathbb{E}^D$ at $u = 0$ is also supersingular, the map $V_{d,\mathbb{E}^D}$ can be written as

$$V_{d,\mathbb{E}^D}(X) = a_0X + \cdots + a_dX^{q^d}$$

with some $a_i \in k[[u]]$ satisfying $v_u(a_0) = 1$, $v_u(a_i) \geqslant 1$ for $1 \leqslant i < d$ and $v_u(a_d) = 0$. Then an inspection of the Newton polygon shows that any non-zero root z of $V_{d,\mathbb{E}^D}(X)$ satisfies $v_u(z) = 1/(q^d - 1)$ and there exists a root z' of $V^n_{d,\mathbb{E}^D}(X)$ with $v_u(z') = 1/((q^d-1)q^{d(n-1)}) = \sharp(A/(\wp^n))^\times$. This concludes the proof. $\square$

Consider the quotient $\mathcal{E}^{\mathrm{ord}}_{\mathrm{un}}/\mathcal{C}_1$ over $Y^{\mathrm{ord}}_{\mathrm{un}}$, which has a natural structure of a Drinfeld module of rank two by Lemma 2.8 (2). Since the universal $\Gamma_1^\Delta(\mathfrak{n})$-structure on $\mathcal{E}_{\mathrm{un}}$ induces that on $\mathcal{E}^{\mathrm{ord}}_{\mathrm{un}}/\mathcal{C}_1$, we have a corresponding map $\pi_d : Y^{\mathrm{ord}}_{\mathrm{un}} \to Y_{\mathrm{un}}$. Since $\mathcal{E}^{\mathrm{ord}}_{\mathrm{un}}/\mathcal{C}_1$ has ordinary reduction, the induced map $Y^{\mathrm{ord}}_{\mathrm{un},m} \to Y_{\mathrm{un},m}$ factors through $Y^{\mathrm{ord}}_{\mathrm{un},m}$. Hence π_d also factors as $\pi_d : Y^{\mathrm{ord}}_{\mathrm{un}} \to Y^{\mathrm{ord}}_{\mathrm{un}}$. On the other hand, the endomorphism $\langle \wp^{-1}\rangle_{\mathfrak{n}}$ of X_{un} defines endomorphisms of $X^{\mathrm{ord}}_{\mathrm{un}}$ and $Y^{\mathrm{ord}}_{\mathrm{un}}$, which we also denote by $\langle \wp^{-1}\rangle_{\mathfrak{n}}$. Put

$$\varphi_d = \langle \wp^{-1}\rangle_{\mathfrak{n}} \circ \pi_d.$$

This gives the cartesian diagram

$$\begin{tikzcd}
\mathcal{E}^{\mathrm{ord}}_{\mathrm{un}}/\mathcal{C}_1 \arrow[r] \arrow[d] & \mathcal{E}^{\mathrm{ord}}_{\mathrm{un}} \arrow[r] \arrow[d] & \mathcal{E}^{\mathrm{ord}}_{\mathrm{un}} \arrow[d] \\
Y^{\mathrm{ord}}_{\mathrm{un}} \arrow[r, "\pi_d"'] & Y^{\mathrm{ord}}_{\mathrm{un}} \arrow[r, "\langle \wp^{-1}\rangle_{\mathfrak{n}}"'] & Y^{\mathrm{ord}}_{\mathrm{un}}.
\end{tikzcd}$$

Lemma 4.10. *For any positive integer m, the induced map $\varphi_d : Y^{\mathrm{ord}}_{\mathrm{un},m} \to Y^{\mathrm{ord}}_{\mathrm{un},m}$ extends to $\bar{\varphi}_d : X^{\mathrm{ord}}_{\mathrm{un},m} \to X^{\mathrm{ord}}_{\mathrm{un},m}$ which is compatible with respect to m. Moreover, $\bar{\varphi}_d$ agrees with the q^d-th power Frobenius map on $X^{\mathrm{ord}}_{\mathrm{un},1}$.*

Proof. Let $K_{\mathfrak{n}}$ be a splitting field of $\Phi^C_{\mathfrak{n}}(X)$, as before. Put $\mathcal{O}_{K_{\mathfrak{n}},m} = \mathcal{O}_{K_{\mathfrak{n}}}/(\wp^m)$. By an fpqc descent, it suffices to show the existence of an extension as in the lemma around each cusp over $\mathcal{O}_{K_{\mathfrak{n}},m}$. For this, first note that the automorphism of $I_{\mathcal{O}_K} = \mathrm{Spec}(\mathcal{O}_K[X]/(W_{\mathfrak{n}}(X)))$ given by $X \mapsto \Phi^C_\wp(X)$ preserves its connected components, since so does its restriction over $k(\wp)$ by Lemma 3.2. Hence, for the image $\zeta \in \mathcal{O}_{K_{\mathfrak{n}}}$ of X, we have an automorphism of $\mathrm{Spec}(\mathcal{O}_{K_{\mathfrak{n}}})$ over $\mathcal{O}_K$ defined by $\zeta \mapsto \Phi^C_\wp(\zeta)$, which we denote by $\sigma_\wp$. We define an endomorphism $\tilde{\nu}_\wp$ of $\mathrm{Spec}(\mathcal{O}_{K_{\mathfrak{n}}}((w)))$ over $\mathcal{O}_K$ by $\tilde{\nu}_\wp = \sigma_\wp \otimes \nu_\wp$.

For any $(a,b) \in \mathscr{H}$, let f_b be the monic generator of $\mathrm{Ann}_A(b(A/(\mathfrak{n})))$, as before. Around the cusp labeled by (a,b), we have the Tate-Drinfeld module $\mathrm{TD}(f_b\Lambda)$ over $\mathcal{O}_{K_{\mathfrak{n}}}((w))$ endowed with a $\Gamma_1^\Delta(\mathfrak{n})$-structure $(\lambda,[\mu])$. As in the proof of Lemma 4.5, using [Hat, (6.3) and (6.4)] we see that the image of $(\lambda,[\mu])$ by the map $\mathrm{TD}(f_b\Lambda) \to \mathrm{TD}(f_b\Lambda)/\mathcal{C}_1^{f_b\Lambda}$ can be identified with $\tilde{\nu}^*_\wp(\wp\lambda,[\mu])$. We denote by

$$(\lambda,[\mu]) : \mathrm{Spec}(\mathcal{O}_{K_{\mathfrak{n}},m}((w))) \to Y^{\mathrm{ord}}_{\mathrm{un},m}$$

the map defined by the triple $(\mathrm{TD}(f_b\Lambda)|_{\mathcal{O}_{K_{\mathfrak{n}},m}((w))}, \lambda, [\mu])$. Then we have the commutative diagram

$$\begin{tikzcd} Y^{\mathrm{ord}}_{\mathrm{un},m} \arrow[r, "\varphi_d"] & Y^{\mathrm{ord}}_{\mathrm{un},m} \\ \mathrm{Spec}(\mathcal{O}_{K_{\mathfrak{n}},m}((w))) \arrow[u, "{(\lambda,[\mu])}"] \arrow[r, "\tilde{\nu}_\wp"'] & \mathrm{Spec}(\mathcal{O}_{K_{\mathfrak{n}},m}((w))), \arrow[u, "{(\lambda,[\mu])}"'] \end{tikzcd}$$

where the vertical arrows identify the lower term with the formal completion of $X^{\mathrm{ord}}_{\mathrm{un},m}$ at the cusp labeled by (a,b) with w inverted. Since we have $F_\wp(w) \in \mathcal{O}_{K_{\mathfrak{n}}}[[w]]$, we obtain an extension of φ_d to each cusp.

Since the canonical subgroup $\mathcal{C}_1$ is a lift of the Frobenius kernel, from (3.2) we see that the morphism $\varphi_d : Y^{\mathrm{ord}}_{\mathrm{un},1} \to Y^{\mathrm{ord}}_{\mathrm{un},1}$ agrees with the q^d-th power Frobenius map. Then the assertion on $X^{\mathrm{ord}}_{\mathrm{un},1}$ also follows, since it is integral and separated. □

We denote by $\omega^{\mathrm{ord}}_{\mathrm{un},m}$ and $\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},m}$ the pull-backs of the sheaf $\bar{\omega}^{\Delta}_{\mathrm{un}}$ to $Y^{\mathrm{ord}}_{\mathrm{un},m}$ and $X^{\mathrm{ord}}_{\mathrm{un},m}$, respectively.

Proposition 4.11. *Let $\rho_{\mathrm{un}} : \mathcal{E}^{\mathrm{ord}}_{\mathrm{un}}/\mathcal{C}_1 \to \mathcal{E}^{\mathrm{ord}}_{\mathrm{un}}$ be the canonical etale isogeny of $\mathcal{E}^{\mathrm{ord}}_{\mathrm{un}}$ over $Y^{\mathrm{ord}}_{\mathrm{un}}$. Then the isomorphism of $\mathcal{O}_{Y^{\mathrm{ord}}_{\mathrm{un},m}}$-modules*

$$F_{\omega^{\mathrm{ord}}_{\mathrm{un},m}} = (\rho^*_{\mathrm{un}})^{-1} : \varphi^*_d(\omega^{\mathrm{ord}}_{\mathrm{un},m}) \simeq \omega_{(\mathcal{E}^{\mathrm{ord}}_{\mathrm{un}}/\mathcal{C}_1)_m} \to \omega^{\mathrm{ord}}_{\mathrm{un},m}$$

extends to an isomorphism of $\mathcal{O}_{X^{\mathrm{ord}}_{\mathrm{un},m}}$-modules

$$F_{\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},m}} : \bar{\varphi}^*_d(\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},m}) \to \bar{\omega}^{\mathrm{ord}}_{\mathrm{un},m}.$$

Proof. As in the proof of Lemma 4.10, it is enough to extend $F_{\omega^{\mathrm{ord}}_{\mathrm{un},m}}$ to each cusp over $\mathcal{O}_{K_{\mathfrak{n}},m}$. This follows from Corollary 4.6 and the construction of $\bar{\omega}^{\Delta}_{\mathrm{un}}$. □

4.5. **Weight congruence.** First we give a version of the Riemann-Hilbert correspondence of Katz in our setting. Put $A_n = A/(\wp^n)$.

Lemma 4.12. *Let n be a positive integer. Let S_n be an affine scheme which is flat over A_n such that $S_1 = S_n \times_{A_n} \mathrm{Spec}(A_1)$ is normal and connected. Let $\varphi_d : S_n \to S_n$ be a morphism over A_n such that the induced map on S_1 agrees with the q^d-th power Frobenius map. We denote by $\pi_1^{\mathrm{et}}(S_n)$ the etale fundamental group for a geometric point of S_n. Then there exists an equivalence between the category $\mathrm{Rep}_{A_n}(S_n)$ of free A_n-modules of finite rank with continuous actions of $\pi_1^{\mathrm{et}}(S_n)$ and the category F-$\mathrm{Crys}^0(S_n)$ of pairs $(\mathcal{H}, F_{\mathcal{H}})$ consisting of a locally free $\mathcal{O}_{S_n}$-module $\mathcal{H}$ of finite rank and an isomorphism of $\mathcal{O}_{S_n}$-modules $F_{\mathcal{H}} : \varphi^*_d(\mathcal{H}) \to \mathcal{H}$.*

Proof. This follows by a verbatim argument as in the proof of [Kat, Proposition 4.1.1]. Here we sketch the argument for the convenience of the reader. For any object M of $\mathrm{Rep}_{A_n}(S_n)$, let T_n be a (connected) Galois covering of S_n such that $\pi_1^{\mathrm{et}}(S_n) \to \mathrm{Aut}(M)$ factors through the Galois group $G(T_n/S_n)$ of it. By the etaleness, we can uniquely lift the q^d-th power Frobenius map on T_1 to a φ_d-equivariant endomorphism φ_{T_n} of T_n over A_n.

We claim that the sequence

$$0 \longrightarrow A_n \longrightarrow \mathcal{O}(T_n) \xrightarrow{\varphi_{T_n}-1} \mathcal{O}(T_n) \tag{4.12}$$

is exact. Indeed, since T_n is flat over A_n we may assume $n = 1$, and in this case the claim follows since $\mathcal{O}(T_1)$ is an integral domain.

We have an endomorphism on $M \otimes_{A_n} \mathcal{O}(T_n)$ defined by $m \otimes f \mapsto m \otimes \varphi_{T_n}^*(f)$, and Galois descent yields an object $(\mathcal{H}(M), F_{\mathcal{H}(M)})$ of F-$\mathrm{Crys}^0(S_n)$. This defines a functor

$$\mathcal{H}(-) : \mathrm{Rep}_{A_n}(S_n) \to F\text{-}\mathrm{Crys}^0(S_n).$$

The exact sequence (4.12) implies $(\mathcal{H}(M)|_{T_n})^{\varphi_{T_n}-1} = M$ and thus the functor $\mathcal{H}(-)$ is fully faithful.

We prove the essential surjectivity by induction on n. For $n = 1$, it follows by applying the original result [Kat, Proposition 4.1.1] to the case where the extension $k/\mathbb{F}_q$ there is $\mathbb{F}_{q^d}/\mathbb{F}_{q^d}$. Suppose that the case of $n-1$ is valid. Let $(\mathcal{H}, F_{\mathcal{H}})$ be any object of F-$\mathrm{Crys}^0(S_n)$. By assumption, there exists a finite etale cover $T_{n-1} \to S_{n-1}$ such that $\mathcal{H}|_{T_{n-1}}$ has an $F_{\mathcal{H}}$-fixed basis $\bar{h}_1, \ldots, \bar{h}_r$. By Hensel's lemma, we can lift T_{n-1} to a finite etale cover $T_n \to S_n$. Take a lift h_i of $\bar{h}_i$ to $\mathcal{H}|_{T_n}$. We have

$$F_{\mathcal{H}}(h_1, \ldots, h_r) = (h_1, \ldots, h_r)(I + \wp^{n-1}N)$$

for some matrix $N \in M_r(\mathcal{O}(T_n))$. Then it is enough to solve the equation

$$F_{\mathcal{H}}((h_1, \ldots, h_r)(I + \wp^{n-1}N')) = (h_1, \ldots, h_r)(I + \wp^{n-1}N')$$

over some finite etale cover of T_n. Since $\mathcal{O}(T_n)$ is flat over A_n, the equation is equivalent to $N + \varphi_d(N') \equiv N' \bmod \wp$, from which the claim follows. $\square$

Corollary 4.13. *Let U be any non-empty affine open subscheme of S_n. Note that, since φ_d agrees with the q^d-th power Frobenius map on S_1, it induces a map $\varphi_d : U \to U$. Then the functor F-$\mathrm{Crys}^0(S_n) \to F$-$\mathrm{Crys}^0(U)$ defined by the restriction to U is fully faithful.*

Proof. It follows from the fact that, since S_1 is normal and connected, the restriction functor $\mathrm{Rep}_{A_n}(S_n) \to \mathrm{Rep}_{A_n}(U)$ is fully faithful. $\square$

By Lemma 4.10, $X^{\mathrm{ord}}_{\mathrm{un},m}$ satisfies the assumptions of Lemma 4.12.

Proposition 4.14. *By the equivalence of Lemma 4.12, the character*

$$r_{n,n} : \pi_1^{\mathrm{et}}(X^{\mathrm{ord}}_{\mathrm{un},n}) \to A_n^\times$$

of Lemma 4.9 associated to $\bar{\mathcal{C}}^D_{n,n}$ corresponds to the pair $(\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n}, F_{\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n}})$ of Proposition 4.11.

Proof. By Corollary 4.13, it is enough to show that the character of $\pi_1^{\mathrm{et}}(Y^{\mathrm{ord}}_{\mathrm{un},n})$ associated to $\mathcal{C}^D_{n,n}$ corresponds to the pair $(\omega^{\mathrm{ord}}_{\mathrm{un},n}, F_{\omega^{\mathrm{ord}}_{\mathrm{un},n}})$. By Proposition 3.8, the Hodge-Tate-Taguchi map yields an isomorphism of invertible $\mathcal{O}_{Y^{\mathrm{ord}}_{\mathrm{un},n}}$-modules

$$\mathrm{HTT} : \mathcal{C}^D_{n,n} \otimes_{\underline{A_n}} \mathcal{O}_{Y^{\mathrm{ord}}_{\mathrm{un},n}} \to \omega^{\mathrm{ord}}_{\mathrm{un},n}.$$

Note that, over any Galois covering $T_n \to Y^{\mathrm{ord}}_{\mathrm{un},n}$ trivializing $\mathcal{C}^D_{n,n}$, the map HTT is compatible with Galois actions. Hence it suffices to show that this map is also compatible with Frobenius structures, where we consider $1 \otimes \varphi_d$ on the left-hand side.

Since the natural map $B^{\mathrm{ord}}_{\mathrm{un},n} \to \mathcal{O}_{K,n}((x))$ is injective, we reduce ourselves to showing that at the ∞-cusp the Hodge-Tate-Taguchi map over $\mathcal{O}_{K,n}((x))$

$$\mathrm{HTT} : \mathcal{C}_n(\mathrm{TD}(\Lambda))^D_n \otimes_{\underline{A_n}} \mathcal{O}_{\mathrm{Spec}(\mathcal{O}_{K,n}((x)))} \to \omega_{\mathrm{TD}(\Lambda)} \otimes \mathcal{O}_{K,n}((x))$$

commutes with Frobenius structures. As is seen in the proof of Lemma 4.8, the induced v-structure on $\mathcal{C}_n(\mathrm{TD}(\Lambda))_n \simeq C[\wp^n]$ from $\mathcal{E}^{\mathrm{ord}}_{\mathrm{un}}$ agrees with that from C. By Lemma 2.17 we have $\mathcal{C}_n(\mathrm{TD}(\Lambda))^D_n \simeq \underline{A_n}$, and Lemma 3.7 implies that the isomorphism HTT is given by

$$\mathrm{HTT}(1) = ((\lambda^\Lambda_{\infty,\wp^n})^*)^{-1}(dZ) = dX. \tag{4.13}$$

Now the proposition follows from Corollary 4.6. $\square$

Theorem 4.15. *Let L/K be a finite extension. For $i = 1, 2$, let f_i be an element of $M_{k_i}(\Gamma_1^\Delta(\mathfrak{n}))_{\mathcal{O}_L}$. Suppose that their x-expansions at the ∞-cusp $(f_i)_\infty(x)$ satisfy the congruence*

$$(f_1)_\infty(x) \equiv (f_2)_\infty(x) \bmod \wp^n, \quad (f_2)_\infty(x) \not\equiv 0 \bmod \wp.$$

Then we have

$$k_1 \equiv k_2 \bmod (q^d - 1)p^{l_p(n)}, \quad l_p(n) = \min\{N \in \mathbb{Z} \mid p^N \geqslant n\}.$$

Proof. By choosing an isomorphism of $\mathcal{O}_K$-modules $\mathcal{O}_L \simeq \mathcal{O}_K^{\oplus[L:K]}$, we identify the $\mathcal{O}_{X_{\mathrm{un}}}$-module $(\bar{\omega}^\Delta_{\mathrm{un}})^{\otimes k} \otimes_{\mathcal{O}_K} \mathcal{O}_L$ with $((\bar{\omega}^\Delta_{\mathrm{un}})^{\otimes k})^{\oplus[L:K]}$ and $A_{\mathfrak{n}}[[x]] \otimes_{A_{\mathfrak{n}}} \mathcal{O}_L$ with $(A_{\mathfrak{n}}[[x]] \otimes_{A_{\mathfrak{n}}} \mathcal{O}_K)^{\oplus[L:K]}$, which are compatible with x-expansions. Thus we may assume $L = K$.

Let W be the maximal open subscheme of $Y^{\mathrm{ord}}_{\mathrm{un},n}$ on which f_1 and f_2 do not vanish. From the assumption, we see that W is non-empty. We have the section f_1/f_2 of $(\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n})^{\otimes k_1-k_2}$ on W. Since the ring $\mathcal{O}_{K,n}((x))$ is local with maximal ideal $\wp\mathcal{O}_{K,n}((x))$, the assumption also implies that $(f_i)_\infty(x)$ is invertible in $\mathcal{O}_{K,n}((x))$. Hence the natural map $\mathrm{Spec}(\mathcal{O}_{K,n}((x))) \to Y^{\mathrm{ord}}_{\mathrm{un},n}$ around the ∞-cusp factors through W, and the pull-back of f_1/f_2 by this map is equal to $(dX)^{k_1-k_2}$. Thus the section f_1/f_2 extends uniquely to a nowhere vanishing section of $(\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n})^{\otimes k_1-k_2}$ on an affine open subscheme U of $X^{\mathrm{ord}}_{\mathrm{un},n}$ containing the ∞-cusp such that its restriction to $\mathrm{Spec}(\mathcal{O}_{K,n}[[x]])$ around the ∞-cusp agrees with $(dX)^{k_1-k_2}$. We write it also as f_1/f_2.

By Corollary 4.6, $(dX)^{k_1-k_2}$ is fixed by the restriction of the Frobenius map of $(\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n})^{\otimes k_1-k_2}$ to $\mathrm{Spec}(\mathcal{O}_{K,n}((x)))$. Since the natural map $B^{\mathrm{ord}}_{\mathrm{un},n} \to \mathcal{O}_{K,n}((x))$ is injective, we see that the section f_1/f_2 on U itself is fixed by the Frobenius map. Hence the restriction of the pair $((\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n})^{\otimes k_1-k_2}, F^{\otimes k_1-k_2}_{\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n}})$ to U is trivial. Then Corollary 4.13 implies that the pair is trivial on $X^{\mathrm{ord}}_{\mathrm{un},n}$, and by Proposition 4.14 the (k_1-k_2)-nd tensor power of the character $r_{n,n}$ is trivial. Now Lemma 4.9 shows that k_1-k_2 is divisible by the exponent of the group $(A/(\wp^n))^\times$, which equals $(q^d-1)p^{l_p(n)}$. This concludes the proof. ∎

Then Theorem 1.1 follows by adding an auxiliary level of degree prime to $q-1$ and applying Theorem 4.15.

Following [Gos2, Definition 3], we define the $\wp$-adic weight set $\mathbb{S}$ as

$$\mathbb{S} = \mathbb{Z}/(q^d-1)\mathbb{Z} \times \mathbb{Z}_p$$

with the discrete topology on the first entry and the p-adic topology on the second entry. We embed $\mathbb{Z}$ into it diagonally.

Corollary 4.16. *Let L/K be a finite extension. Let $F_\infty(x)$ be a non-zero element of $\mathcal{O}_L[[x]][1/\wp]$. Suppose that there exists a sequence $\{f_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ satisfying $f_n \in M_{k_n}(\Gamma^\Delta_1(\mathfrak{n}))_L$ for some integer k_n and*

$$\lim_{n\to\infty}(f_n)_\infty(x) = F_\infty(x)$$

with respect to the $\wp$-adic topology defined by $\mathcal{O}_L[[x]]$. Then the sequence $\{k_n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ converges to some element $\chi \in \mathbb{S}$. Moreover, the element χ depends only on $F_\infty(x)$.

Proof. By Proposition 4.7 (1), we may assume $f_n \in M_{k_n}(\Gamma^\Delta_1(\mathfrak{n}))_{\mathcal{O}_L}$ and $F_\infty(x) \in \mathcal{O}_L[[x]]$. Note that for any positive integer c, the $\mathcal{O}_{K,c}$-module $\mathcal{O}_{L,c}$ is free of finite rank and thus the natural map $A_\mathfrak{n}[[x]]\otimes_{A_\mathfrak{n}}\mathcal{O}_{L,c} \to \mathcal{O}_{L,c}[[x]]$ is an isomorphism. Again by Proposition 4.7 (1), we may assume $F_\infty(x) \not\equiv 0 \bmod \wp$. By assumption, for any sufficiently large positive integer c, there exists an integer n_0 such that if $m, n \geqslant n_0$, then we have

$$(f_m)_\infty(x) \equiv (f_n)_\infty(x) \bmod \wp^c, \quad (f_n)_\infty(x) \not\equiv 0 \bmod \wp$$

in $A_{\mathfrak{n}}[[x]] \otimes_{A_{\mathfrak{n}}} \mathcal{O}_L$. Hence Theorem 4.15 implies

$$k_m \equiv k_n \bmod (q^d - 1)p^{l_p(c)}$$

and the sequence $\{k_n\}_{n \in \mathbb{Z}_{\geqslant 0}}$ converges in $\mathbb{S}$. For another such sequence $\{g_n\}_{n \in \mathbb{Z}_{\geqslant 0}}$ with $g_n \in M_{k'_n}(\Gamma_1^\Delta(\mathfrak{n}))_{\mathcal{O}_L}$, we also have $(f_n)_\infty(x) \equiv (g_n)_\infty(x) \bmod \wp^c$ and $(f_n)_\infty(x) \not\equiv 0 \bmod \wp$ for any sufficiently large n. Then Theorem 4.15 implies $k_n \equiv k'_n \bmod (q^d-1)p^{l_p(c)}$ and thus both converge to χ. □

We say any element $F_\infty(x) \in \mathcal{O}_L[[x]][1/\wp]$ as in Corollary 4.16 a $\wp$-adic Drinfeld modular form in the sense of Serre ([Gos2, Definition 5], [Vin, Definition 2.5]).

Definition 4.17. Let $F_\infty(x) \in \mathcal{O}_L[[x]][1/\wp]$ be a non-zero $\wp$-adic Drinfeld modular form in the sense of Serre. Take any sequence $\{f_n\}_{n \in \mathbb{Z}_{\geqslant 0}}$ satisfying $f_n \in M_{k_n}(\Gamma_1^\Delta(\mathfrak{n}))_L$ and $\lim_{n \to \infty}(f_n)_\infty(x) = F_\infty(x)$. Then we define the weight of $F_\infty(x)$ as the limit $\lim_{n\to\infty} k_n$ in $\mathbb{S}$, which is well-defined by Corollary 4.16.

4.6. **$\wp$-adic Drinfeld modular forms.** Let $\mathfrak{X}_{\mathrm{un}}$ be the $\wp$-adic completion of $X_{\mathrm{un}} = X_1^\Delta(\mathfrak{n})$ and $\mathfrak{X}_{\mathrm{un}}^{\mathrm{ord}}$ the formal open subscheme of $\mathfrak{X}_{\mathrm{un}}$ on which the Gekeler's lift g_d of the Hasse invariant is invertible. The latter is isomorphic to the $\wp$-adic completion of

$$\operatorname{Spec}_{X_{\mathrm{un}}}(\operatorname{Sym}((\bar{\omega}_{\mathrm{un}}^\Delta)^{\otimes q^d-1})/(g_d-1)). \tag{4.14}$$

Note that the reduction modulo $\wp^m$ of $\mathfrak{X}_{\mathrm{un}}^{\mathrm{ord}}$ is equal to $X_{\mathrm{un},m}^{\mathrm{ord}}$. We see that $\mathfrak{X}_{\mathrm{un}}^{\mathrm{ord}}$ is a Noetherian affine formal scheme by [Abb, Corollaire 2.1.37].

For any $\chi = (s_0, s_1) \in \mathbb{S}$, we have a continuous endomorphism of $\mathcal{O}_K^\times = \mathbb{F}_{q^d}^\times \times (1 + \wp\mathcal{O}_K)$ defined by

$$x = (x_0, x_1) \mapsto x^\chi = x_0^{s_0} x_1^{s_1}$$

which preserves the subgroup $1 + \wp^n\mathcal{O}_K$. Composing it with the character $r_{n,n} : \pi_1^{\mathrm{et}}(X_{\mathrm{un},n}^{\mathrm{ord}}) \to A_n^\times = \mathcal{O}_{K,n}^\times$, we obtain a character $r_{n,n}^\chi$. Let $\bar{\omega}_{\mathrm{un},n}^{\mathrm{ord},\chi}$ be the associated invertible sheaf on $X_{\mathrm{un},n}^{\mathrm{ord}}$ via the correspondence of Lemma 4.12. Since they form a projective system with surjective transition maps, they give an invertible sheaf $\bar{\omega}_{\mathrm{un}}^{\mathrm{ord},\chi}$ on $\mathfrak{X}_{\mathrm{un}}^{\mathrm{ord}}$ [Abb, Proposition 2.8.9].

For any finite extension L/K, we put

$$M_\chi(\Gamma_1^\Delta(\mathfrak{n}))_{\mathcal{O}_L} := H^0(\mathfrak{X}_{\mathrm{un}}^{\mathrm{ord}}|_{\mathcal{O}_L}, \bar{\omega}_{\mathrm{un}}^{\mathrm{ord},\chi}|_{\mathcal{O}_L}) = H^0(\mathfrak{X}_{\mathrm{un}}^{\mathrm{ord}}, \bar{\omega}_{\mathrm{un}}^{\mathrm{ord},\chi} \otimes_{\mathcal{O}_K} \mathcal{O}_L).$$

By [Abb, Proposition 2.7.2.9], we have

$$M_\chi(\Gamma_1^\Delta(\mathfrak{n}))_{\mathcal{O}_L} = \varprojlim_n H^0(X^{\mathrm{ord}}_{\mathrm{un},n}|_{\mathcal{O}_{L,n}}, \bar{\omega}^{\mathrm{ord},\chi}_{\mathrm{un},n}|_{\mathcal{O}_{L,n}})$$

and thus it is flat over $\mathcal{O}_L$. Put

$$M_\chi(\Gamma_1^\Delta(\mathfrak{n}))_L = M_\chi(\Gamma_1^\Delta(\mathfrak{n}))_{\mathcal{O}_L}[1/\wp].$$

We refer to any element of this module as a $\wp$-adic Drinfeld modular form of tame level $\mathfrak{n}$ and weight χ over L. Since the action of $\mathbb{F}_q^\times$ on $X_1^\Delta(\mathfrak{n})$ via $c \mapsto \langle c\rangle_\Delta$ induces an action on $H^0(X^{\mathrm{ord}}_{\mathrm{un},n}|_{\mathcal{O}_{L,n}}, \bar{\omega}^{\mathrm{ord},\chi}_{\mathrm{un},n}|_{\mathcal{O}_{L,n}})$, the module $M_\chi(\Gamma_1^\Delta(\mathfrak{n}))_L$ is decomposed as

$$M_\chi(\Gamma_1^\Delta(\mathfrak{n}))_L = \bigoplus_{m\in\mathbb{Z}/(q-1)\mathbb{Z}} M_{\chi,m}(\Gamma_1(\mathfrak{n}))_L,$$

where the space $M_{\chi,m}(\Gamma_1(\mathfrak{n}))_L$ of type m forms is the maximal subspace on which $\langle c\rangle_\Delta$ acts by c^{-m}.

For any $\chi \in \mathbb{S}$ and any positive integer n, we can find an integer k satisfying $\chi \equiv k \bmod (q^d-1)p^{l_p(n)}$. Then we have an isomorphism $\bar{\omega}^{\mathrm{ord},\chi}_{\mathrm{un},n} \simeq (\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n})^{\otimes k}$ compatible with Frobenius structures. Using this identification, we obtain a map of x-expansion

$$H^0(X^{\mathrm{ord}}_{\mathrm{un},n}|_{\mathcal{O}_{L,n}}, \bar{\omega}^{\mathrm{ord},\chi}_{\mathrm{un},n}|_{\mathcal{O}_{L,n}}) \to \mathcal{O}_{L,n}[[x]], \quad f_n \mapsto (f_n)_\infty(x). \tag{4.15}$$

For any such k and k', the correspondence of Lemma 4.12 gives an isomorphism $(\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n})^{\otimes k} \simeq (\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n})^{\otimes k'}$ compatible with Frobenius structures. Since (4.12) implies that such an isomorphism is unique up to the multiplication by an element of $A_n^\times$, by restricting to the ∞-cusp and using (4.13) we see that it agrees with the multiplication by $g_d^{(k'-k)/(q^d-1)}$. Since $(g_d)_\infty(x)^{p^{l_p(n)}} \equiv 1 \bmod \wp^n$, the map (4.15) is independent of the choice of k and induces

$$\begin{aligned} M_{\chi,m}(\Gamma_1(\mathfrak{n}))_L &\to \mathcal{O}_L[[x]][1/\wp] \\ f = (f_n)_n &\mapsto f_\infty(x) := \lim_{n\to\infty} (f_n)_\infty(x) \end{aligned} \tag{4.16}$$

which is an injection by Krull's intersection theorem. This map identifies our definition of $\wp$-adic Drinfeld modular forms with $\wp$-adic Drinfeld modular forms in the sense of Serre, by the following proposition.

Proposition 4.18. *The image of the injection (4.16) agrees with the space of power series $F_\infty(x) \in \mathcal{O}_L[[x]][1/\wp]$ which can be written as the $\wp$-adic limit of x-expansions $\{(h_n)_\infty(x)\}_n$, where h_n is an element of $M_{k_n,m}(\Gamma_1(\mathfrak{n}))_L$ for some integer k_n satisfying $\lim_{n\to\infty} k_n = \chi$ in $\mathbb{S}$.*

Proof. This can be shown as in the proof of [Kat, Theorem 4.5.1]. Indeed, let $f = (f_n)_n$ be an element of $M_{\chi,m}(\Gamma_1(\mathfrak{n}))_{\mathcal{O}_L}$. For any n we choose an integer $k_n \geqslant 2$ satisfying $\chi \equiv k_n \bmod (q^d-1)p^{l_p(n)}$. Note that, for any integer $k \geqslant 2$, the description (4.14) and Corollary 4.2 give an isomorphism

$$H^0(X_{\mathrm{un},n}^{\mathrm{ord}}|_{\mathcal{O}_{L,n}}, \bar{\omega}_{\mathrm{un},n}^{\mathrm{ord},k}|_{\mathcal{O}_{L,n}}) \to \left(\bigoplus_{j \geqslant 0} H^0(X_{\mathrm{un},n}|_{\mathcal{O}_{L,n}}, (\bar{\omega}_{\mathrm{un}}^{\Delta}|_{\mathcal{O}_{L,n}})^{\otimes k+j(q^d-1)})\right)/(g_d - 1).$$

Therefore, by Proposition 4.7 (2), for each f_n we can find an integer $k'_n \geqslant 2$ and an element $h_n \in M_{k'_n,m}(\Gamma_1(\mathfrak{n}))_{\mathcal{O}_L}$ satisfying $k'_n \equiv k_n \bmod (q^d-1)p^{l_p(n)}$ and $(f_n)_\infty(x) \equiv (h_n)_\infty(x) \bmod \wp^n$. This yields $\lim_{n\to\infty}(h_n)_\infty(x) = f_\infty(x)$ and $\lim_{n\to\infty} k'_n = \lim_{n\to\infty} k_n = \chi$.

Conversely, let $F_\infty(x) = \lim_{n\to\infty}(h_n)_\infty(x)$ be as in the proposition. We may assume $F_\infty(x) \neq 0$, and also $h_n \in M_{k_n,m}(\Gamma_1(\mathfrak{n}))_{\mathcal{O}_L}$ by Proposition 4.7 (1). Multiplying powers of g_d and dividing by $\wp$, without changing $\chi = \lim_{n\to\infty} k_n$ we may assume $k_{n+1} > k_n$ and

$$(h_{n+1})_\infty(x) \equiv (h_n)_\infty(x) \bmod \wp^n, \quad (h_n)_\infty(x) \not\equiv 0 \bmod \wp$$

for any n. Then Theorem 4.15 implies $k_{n+1} \equiv k_n \bmod (q^d-1)p^{l_p(n)}$. Now Proposition 4.7 (1) implies $h_{n+1} \equiv h_n g_d^{(k_{n+1}-k_n)/(q^d-1)} \bmod \wp^n$ and thus $(h_n)_n$ defines an element f of $M_{\chi,m}(\Gamma_1(\mathfrak{n}))_{\mathcal{O}_L}$ satisfying $f_\infty(x) = F_\infty(x)$. ☐

Theorem 4.19. *Let f be a Drinfeld modular form of level $\Gamma_1(\mathfrak{n}) \cap \Gamma_0(\wp)$, weight k and type m over $\mathbb{C}_\infty$ with x-expansion coefficients at ∞ in the localization $A_{(\wp)}$ of A at $(\wp)$. Then f is a $\wp$-adic Drinfeld modular form of tame level $\mathfrak{n}$, weight k and type m. Namely, the x-expansion $f_\infty(x)$ at the unramified cusp over the ∞-cusp* [Hat, §7] *is in the image of the map (4.16) for $\chi = k$.*

Proof. By Proposition 4.7 (1), we may assume $f \in M_k(\Gamma_1^\Delta(\mathfrak{n},\wp))_{\mathbb{F}_q(t)}$. By flat base change, we can find an element $g \in M_k(\Gamma_1^\Delta(\mathfrak{n},\wp))_{A_{(\wp)}}$ such that its image $M_k(\Gamma_1^\Delta(\mathfrak{n},\wp))_{\mathbb{F}_q(t)}$ agrees with the element $\wp^l f$ for some non-negative integer l.

For any integer $n > 0$, put $Y_{\mathrm{un},n}^{\wp} = Y_1^\Delta(\mathfrak{n},\wp) \times_{A_\mathfrak{n}} \mathrm{Spec}(\mathcal{O}_{K,n})$. The canonical subgroup $\mathcal{C}_{1,n}$ over $Y_{\mathrm{un},n}^{\mathrm{ord}}$ gives a section of the natural projection

$$\begin{tikzcd} & Y_{\mathrm{un},n}^{\wp} \arrow[d] \\ Y_{\mathrm{un},n}^{\mathrm{ord}} \arrow[ur] \arrow[r] & Y_{\mathrm{un},n}. \end{tikzcd}$$

Pulling back g by this section, we obtain an element g_n of the module $H^0(Y^{\mathrm{ord}}_{\mathrm{un},n}, (\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n})^{\otimes k})$. On each cusp Ξ labeled by $(a,b) \in \mathscr{H}$, the pull-back of g_n along this cusp agrees with the pull-back of g along the unramified cusp over Ξ [Hat, §7]. Hence $g_n \in H^0(X^{\mathrm{ord}}_{\mathrm{un},n}, (\bar{\omega}^{\mathrm{ord}}_{\mathrm{un},n})^{\otimes k})$. Since

$$\wp^l f_\infty(x) = g_\infty(x) = \lim_{n\to\infty} (g_n)_\infty(x),$$

this implies that f is a $\wp$-adic modular form of weight k. □

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