

CANONICAL SUBGROUPS VIA BREUIL-KISIN MODULES

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ABSTRACT. Let $p > 2$ be a rational prime and K/\mathbb{Q}_p be an extension of complete discrete valuation fields. Let \mathcal{G} be a truncated Barsotti-Tate group of level n , height h and dimension d over \mathcal{O}_K with $0 < d < h$. In this paper, we show that if the Hodge height of \mathcal{G} is less than $1/(p^{n-2}(p+1))$, then there exists a finite flat closed subgroup scheme of \mathcal{G} of order p^{nd} over \mathcal{O}_K with standard properties as the canonical subgroup.

1. INTRODUCTION

Let p be a rational prime and K/\mathbb{Q}_p be an extension of complete discrete valuation fields. Let k be the residue field of K and $e = e(K)$ be the absolute ramification index of K . Let \mathcal{O}_K denote the ring of integers of K and put $\tilde{\mathcal{O}}_K = \mathcal{O}_K/p\mathcal{O}_K$. We fix an algebraic closure \bar{K} of K and let v_p be the valuation of K which is normalized as $v_p(p) = 1$ and extended to \bar{K} . For a non-negative rational number i , we put $m_{\bar{K}}^{\geq i} = \{a \in \mathcal{O}_{\bar{K}} \mid v_p(a) \geq i\}$ and similarly for $m_K^{\geq i}$.

Let E be an elliptic curve over \mathcal{O}_K . By fixing a formal parameter of the formal completion of E along the zero section, we identify the group of p -torsion points $E[p](\mathcal{O}_{\bar{K}})$ as a subset of $m_{\bar{K}}$. If the group $E[p](\mathcal{O}_{\bar{K}})$ has a subgroup C of order p whose elements have valuations greater than those of the elements of $E[p](\mathcal{O}_{\bar{K}}) \setminus C$, then the subgroup C is called the (level one) canonical subgroup of E . It was shown that the canonical subgroup of E exists if the Hodge height of E , namely the p -adic valuation of the Hasse invariant of E , is less than $p/(p+1)$, and based on this result, Katz studied a spectral theory of the U_p operator for overconvergent elliptic modular forms ([14]).

For a similar investigation of p -adic modular forms for general reductive algebraic groups, we need a generalization of the existence theorem of the canonical subgroup to higher dimensional abelian schemes over \mathcal{O}_K . Such a generalization was first obtained by Abbes and Mokrane via a calculation of p -adic vanishing cycles of abelian schemes ([1]). Namely, for an abelian scheme A over \mathcal{O}_K with relative dimension g , they found a subgroup of $A[p](\mathcal{O}_{\bar{K}})$ of order p^g in the upper ramification filtration $\{A[p]^{j+}(\mathcal{O}_{\bar{K}})\}$ if the Hodge height of A ([1, Subsection 1.2]) is less than an explicit bound.

Date: October 3, 2012.

Supported by Grant-in-Aid for Young Scientists B-21740023.

Their work was followed by many improvements and generalizations with various methods, such as [2], [7], [10], [11], [17], [19], [21] and especially [8] and [22].

In this paper, we prove an existence theorem of level n canonical subgroups for $p > 2$, using a classification theory of finite flat (commutative) group schemes due to Breuil and Kisin ([4], [5], [15], [16]), and a ramification-theoretic technique developed by the author ([12]).

Before stating the main theorem, we fix some notation. For a finite flat group scheme \mathcal{G} over \mathcal{O}_K and its module of invariant differentials $\omega_{\mathcal{G}}$ over \mathcal{O}_K , write $\omega_{\mathcal{G}} \simeq \bigoplus_i \mathcal{O}_K/(a_i)$ with some $a_i \in \mathcal{O}_K$ and put $\deg(\mathcal{G}) = \sum_i v_p(a_i)$. We define the Hodge-Tate map

$$\mathrm{HT}_i : \mathcal{G}(\mathcal{O}_{\bar{K}}) \rightarrow \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i}$$

by $x \mapsto (x^\vee)^*(dt/t)$, where $x^\vee : \mathcal{G}^\vee \times \mathrm{Spec}(\mathcal{O}_{\bar{K}}) \rightarrow \mu_p$ is the dual map of $x \in \mathcal{G}(\mathcal{O}_{\bar{K}})$. Let \mathcal{G}^j and \mathcal{G}^{j+} (resp. \mathcal{G}_i and \mathcal{G}_{i+}) denote the upper (resp. lower) ramification subgroup schemes of \mathcal{G} (see [12]). Here we normalize the indices of the filtrations by multiplying the usual ones by $1/e(K)$, so that these ramification subgroup schemes are compatible with any base change of complete discrete valuation rings. In particular, \mathcal{G}_i is defined by

$$\mathcal{G}_i(\mathcal{O}_{\bar{K}}) = \mathrm{Ker}(\mathcal{G}(\mathcal{O}_{\bar{K}}) \rightarrow \mathcal{G}(\mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i})).$$

For a truncated Barsotti-Tate group \mathcal{G} of level n , height h and dimension $d < h$ over \mathcal{O}_K , let us consider its Cartier dual \mathcal{G}^\vee and the p -torsion subgroup scheme $\mathcal{G}^\vee[p]$. Then the Lie algebra $\mathrm{Lie}(\mathcal{G}^\vee[p] \times \mathrm{Spec}(\mathcal{O}_K))$ is a free \mathcal{O}_K -module of rank $h - d$. We define the Hodge height $\mathrm{Hdg}(\mathcal{G})$ of \mathcal{G} to be the truncated valuation $v_p(\det(V_{\mathcal{G}^\vee[p]})) \in [0, 1]$ of the determinant of the action of the Verschiebung $V_{\mathcal{G}^\vee[p]}$ on this \mathcal{O}_K -module. Note the equality $\mathrm{Hdg}(\mathcal{G}) = \mathrm{Hdg}(\mathcal{G}^\vee)$ (see for example [8, Proposition 2], where the Hodge height of \mathcal{G} is denoted by $\mathrm{Ha}(\mathcal{G})$ and referred as the Hasse invariant of \mathcal{G}). Then the main theorem of this paper is the following.

Theorem 1.1. *Let $p > 2$ be a rational prime and K/\mathbb{Q}_p be an extension of complete discrete valuation fields. Let \mathcal{G} be a truncated Barsotti-Tate group of level n , height h and dimension d over \mathcal{O}_K with $0 < d < h$ and Hodge height $w = \mathrm{Hdg}(\mathcal{G})$.*

- (1) *If $w < 1/(p^{n-2}(p+1))$, then there exists a finite flat closed subgroup scheme \mathcal{C}_n of \mathcal{G} of order p^{nd} over \mathcal{O}_K , which we call the level n canonical subgroup of \mathcal{G} , such that $\mathcal{C}_n \times \mathrm{Spec}(\mathcal{O}_K/m_{\bar{K}}^{\geq 1-p^{n-1}w})$ coincides with the kernel of the n -th iterated Frobenius homomorphism F^n of $\mathcal{G} \times \mathrm{Spec}(\mathcal{O}_K/m_{\bar{K}}^{\geq 1-p^{n-1}w})$. Moreover, the group scheme \mathcal{C}_n has the following properties:*
 - (a) $\deg(\mathcal{G}/\mathcal{C}_n) = w(p^n - 1)/(p - 1)$.
 - (b) *Put \mathcal{C}'_n to be the level n canonical subgroup of \mathcal{G}^\vee . Then we have the equality of subgroup schemes $\mathcal{C}'_n = (\mathcal{G}/\mathcal{C}_n)^\vee$, or equivalently*

$\mathcal{C}_n(\mathcal{O}_{\bar{K}}) = \mathcal{C}'_n(\mathcal{O}_{\bar{K}})^\perp$, where \perp means the orthogonal subgroup with respect to the Cartier pairing.

- (c) If $n = 1$, then $\mathcal{C}_1 = \mathcal{G}_{(1-w)/(p-1)} = \mathcal{G}^{pw/(p-1)+}$.
- (2) If $w < (p-1)/(p^n-1)$, then the subgroup scheme \mathcal{C}_n also satisfies the following:
- (d) the group $\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^d$.
- (e) The scheme-theoretic closure of $\mathcal{C}_n(\mathcal{O}_{\bar{K}})[p^i]$ in \mathcal{C}_n coincides with the subgroup scheme \mathcal{C}_i of $\mathcal{G}[p^i]$ for $1 \leq i \leq n-1$.
- (3) If $w < (p-1)/p^n$, then the subgroup $\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ coincides with the kernel of the Hodge-Tate map $\mathrm{HT}_{n-w(p^n-1)/(p-1)}$.
- (4) If $w < 1/(2p^{n-1})$, then the subgroup scheme \mathcal{C}_n coincides with the upper ramification subgroup scheme \mathcal{G}^{j+} for

$$pw(p^n-1)/(p-1)^2 \leq j < p(1-w)/(p-1).$$

Moreover, we show that such \mathcal{C}_n is unique if $w < p(p-1)/(p^{n+1}-1)$ (Proposition 4.4). Since the upper ramification subgroups can be patched into a family (see Lemma 4.5), we also have the following corollary.

Corollary 1.2. *Let K/\mathbb{Q}_p be an extension of complete discrete valuation fields and j be a positive rational number. Let \mathfrak{X} be an admissible formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$ which is quasi-compact and \mathfrak{G} be a truncated Barsotti-Tate group of level n over \mathfrak{X} of constant height h and dimension d with $0 < d < h$. We let X and G denote the Raynaud generic fibers of the formal schemes \mathfrak{X} and \mathfrak{G} , respectively. For a finite extension L/K and $x \in X(L)$, we put $\mathfrak{G}_x = \mathfrak{G} \times_{\mathfrak{X}, x} \mathrm{Spf}(\mathcal{O}_L)$, where we let x also denote the map $\mathrm{Spf}(\mathcal{O}_L) \rightarrow \mathfrak{X}$ obtained from x by taking the scheme-theoretic closure and the normalization. For a non-negative rational number r , let $X(r)$ be the admissible open subset of X defined by*

$$X(r)(\bar{K}) = \{x \in X(\bar{K}) \mid \mathrm{Hdg}(\mathfrak{G}_x) < r\}.$$

Put $r_1 = p/(p+1)$ and $r_n = 1/(2p^{n-1})$ for $n \geq 2$.

Suppose $p > 2$. Then there exists an admissible open subgroup \mathcal{C}_n of $G|_{X(r_n)}$ such that, étale locally on $X(r_n)$, the rigid-analytic group \mathcal{C}_n is isomorphic to the constant group $(\mathbb{Z}/p^n\mathbb{Z})^d$ and, for any finite extension L/K and $x \in X(L)$, the fiber $(\mathcal{C}_n)_x$ coincides with the generic fiber of the level n canonical subgroup of \mathfrak{G}_x .

Note that for a smaller range of w , Theorem 1.1 is also proved in [8] and [22]. The key idea of our approach is, firstly, to lift the conjugate Hodge filtration of $\mathcal{G} \times \mathrm{Spec}(\tilde{\mathcal{O}}_K)$ to the Breuil-Kisin module associated to \mathcal{G} . By an induction, it suffices to consider the case of $n = 1$. We may assume that the residue field k is perfect and that the group scheme \mathcal{G} is associated to a φ -module \mathfrak{M} over the formal power series ring $k[[u]]$ via the Breuil-Kisin classification (see Section 2). The Lie algebra $\mathrm{Lie}(\mathcal{G}^\vee)$ is naturally considered as a φ -stable direct summand of the φ -module $\mathfrak{M}/u^e\mathfrak{M}$. Then we show that the reduction modulo $u^{e(1-w)}$ of this direct summand lifts

uniquely to a φ -stable direct summand of \mathfrak{M} . Our canonical subgroup is defined to be the finite flat closed subgroup scheme of \mathcal{G} associated to the quotient of \mathfrak{M} by this direct summand. Its properties stated in Theorem 1.1 follow easily from the construction, basically except the uniqueness and the coincidence with ramification subgroup schemes. The second key idea is to switch to a complete discrete valuation field of equal characteristic: From the φ -module \mathfrak{M} , we can also construct a finite flat group scheme $\mathcal{H}(\mathfrak{M})$ over $k[[u]]$ ([9]). Then, by the main theorem of [12], the ramification subgroups of \mathcal{G} and $\mathcal{H}(\mathfrak{M})$ are naturally isomorphic to each other. Moreover, it is also proved that reductions of \mathcal{G} and $\mathcal{H}(\mathfrak{M})$ are isomorphic as pointed schemes ([12, Corollary 4.6]). These reduce proofs of the remaining properties of our canonical subgroup to an elementary calculation on the side of equal characteristic, which we can easily accomplish.

Acknowledgments. The author would like to thank the anonymous referee for many helpful comments to improve the paper.

2. REVIEW OF THE BREUIL-KISIN CLASSIFICATION OF FINITE FLAT GROUP SCHEMES AND THEIR RAMIFICATION THEORY

Let the notation be as in the previous section and suppose that the residue field k of K is perfect of characteristic $p > 2$. Let $W = W(k)$ be the Witt ring of k and φ be the Frobenius endomorphism of W . Let us fix once and for all a uniformizer π of K and a system $\{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}}$ of its p -power roots in \bar{K} with $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Put $K_n = K(\pi_n)$ and $K_\infty = \cup K_n$. Let $E(u) \in W[u]$ be the Eisenstein polynomial of π over W and put $c_0 = p^{-1}E(0)$. In this section, we briefly recall a classification theory of Breuil ([4], [5]) and Kisin ([15], [16]) of finite flat group schemes over \mathcal{O}_K and their ramification theory, while we concentrate mainly on the p -torsion case.

2.1. Breuil and Kisin modules. Put $\mathfrak{S} = W[[u]]$ and $\mathfrak{S}_1 = k[[u]]$. The φ -semilinear continuous ring endomorphisms of these rings defined by $u \mapsto u^p$ are also denoted by φ . Then a Kisin module over \mathfrak{S} (of E -height ≤ 1) is an \mathfrak{S} -module \mathfrak{M} endowed with a φ -semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ such that the cokernel of the map $1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow \mathfrak{M}$ is killed by $E(u)$. We write $\varphi_{\mathfrak{M}}$ also as φ if there is no risk of confusion. A morphism of Kisin modules is an \mathfrak{S} -linear map which is compatible with φ 's of the source and the target. Then the Kisin modules form a category and this category has an obvious notion of exact sequences. We let $\text{Mod}_{/\mathfrak{S}_1}^{1, \varphi}$ (*resp.* $\text{Mod}_{/\mathfrak{S}}^{1, \varphi}$) denote its full subcategory consisting of the objects whose underlying \mathfrak{S} -modules are free of finite rank over \mathfrak{S}_1 (*resp.* \mathfrak{S}). For a $k[[u]]$ -algebra B , we write φ also for the p -th power Frobenius endomorphism of B . Then we let $\text{Mod}_{/B}^{1, \varphi}$ denote the category of locally free B -modules \mathfrak{M} of finite rank endowed with a φ -semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$ such that the cokernel of the map $1 \otimes \varphi_{\mathfrak{M}} : B \otimes_{\varphi, B} \mathfrak{M} \rightarrow \mathfrak{M}$ is killed by $E(u)$.

We also have categories $\text{Mod}_{/S_1}^{1,\varphi}$ and $\text{Mod}_{/S}^{1,\varphi}$ of Breuil modules defined as follows. Let S be the p -adic completion of the divided power envelope $W[u]^{\text{DP}}$ of $W[u]$ with respect to the ideal $(E(u))$ and the compatibility condition with the canonical divided power structure on pW . The ring S has a natural filtration $\text{Fil}^i S$ defined as the closure in S of the ideal generated by $E(u)^j/j!$ for integers $j \geq i$. The φ -semilinear continuous ring homomorphism $S \rightarrow S$ defined by $u \mapsto u^p$ is also denoted by φ . We have $\varphi(\text{Fil}^1 S) \subseteq pS$ and put $\varphi_1 = p^{-1}\varphi|_{\text{Fil}^1 S}$. This filtration and the map φ_1 induce a similar structure on the ring $S_n = S/p^n S$. Note that we have $\varphi(\text{Fil}^1 S_1) = 0$. Put $c = \varphi_1(E(u)) \in S^\times$. Then we let $\text{Mod}_{/S}^{1,\varphi}$ denote the category of S -modules \mathcal{M} endowed with an S -submodule $\text{Fil}^1 \mathcal{M}$ containing $(\text{Fil}^1 S)\mathcal{M}$ and a φ -semilinear map $\varphi_{1,\mathcal{M}} : \text{Fil}^1 \mathcal{M} \rightarrow \mathcal{M}$ satisfying $\varphi_{1,\mathcal{M}}(s_1 m) = c^{-1}\varphi_1(s_1)\varphi_{1,\mathcal{M}}(E(u)m)$ for any $s_1 \in \text{Fil}^1 S$ and $m \in \mathcal{M}$. A morphism of this category is defined to be a homomorphism of S -modules compatible with Fil^1 's and φ_1 's. Note that this category also has an obvious notion of exact sequences. We drop the subscript \mathcal{M} of $\varphi_{1,\mathcal{M}}$ if no confusion may occur. We let $\text{Mod}_{/S_1}^{1,\varphi}$ (*resp.* $\text{Mod}_{/S}^{1,\varphi}$) denote the full subcategory of $\text{Mod}_{/S}^{1,\varphi}$ consisting of the objects \mathcal{M} such that \mathcal{M} is free of finite rank over S_1 (*resp.* \mathcal{M} is free of finite rank over S , $\mathcal{M}/\text{Fil}^1 \mathcal{M}$ is p -torsion free) and the image $\varphi_{1,\mathcal{M}}(\text{Fil}^1 \mathcal{M})$ generates the S -module \mathcal{M} .

The categories of Breuil and Kisin modules are in fact equivalent. By the natural map $\mathfrak{S} \rightarrow S$, we consider the ring S as an \mathfrak{S} -algebra. We define an exact functor $\mathcal{M}_{\mathfrak{S}}(-) : \text{Mod}_{/S_1}^{1,\varphi} \rightarrow \text{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$ by putting $\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) = S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ with

$$\begin{aligned} \text{Fil}^1 \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) &= \text{Ker}(S \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \xrightarrow{1 \otimes \varphi} (S_1/\text{Fil}^1 S_1) \otimes_{\mathfrak{S}} \mathfrak{M}), \\ \varphi_1 : \text{Fil}^1 \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) &\xrightarrow{1 \otimes \varphi} \text{Fil}^1 S_1 \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\varphi_1 \otimes 1} S_1 \otimes_{\varphi,\mathfrak{S}_1} \mathfrak{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}). \end{aligned}$$

Then the functor $\mathcal{M}_{\mathfrak{S}}(-)$ is an equivalence of categories ([16, Proposition 1.1.11]). We also have a similar equivalence for the categories $\text{Mod}_{/\mathfrak{S}}^{1,\varphi}$ and $\text{Mod}_{/S}^{1,\varphi}$ ([6, Theorem 2.2.1]).

2.2. Classification of finite flat group schemes and ramification. Set $\tilde{\mathcal{O}}_{\bar{K}} = \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ and \mathbb{C} to be the completion of \bar{K} . Consider the ring

$$R = \varprojlim(\tilde{\mathcal{O}}_{\bar{K}} \leftarrow \tilde{\mathcal{O}}_{\bar{K}} \leftarrow \cdots),$$

where every transition map is the p -th power map. For an element $x = (x_0, x_1, \dots) \in R$ with $x_n \in \tilde{\mathcal{O}}_{\bar{K}}$, we put $x^{(0)} = \lim_{n \rightarrow \infty} \hat{x}_n^{p^n} \in \mathcal{O}_{\mathbb{C}}$, where \hat{x}_n is a lift of x_n in $\mathcal{O}_{\bar{K}}$. Then the ring R is a complete valuation ring of characteristic p with valuation $v_R(x) = v_p(x^{(0)})$ whose fraction field $\text{Frac}(R)$ is algebraically closed, and the absolute Galois group $G_{\bar{K}} = \text{Gal}(\bar{K}/K)$ naturally acts on this ring. We put $m_{\bar{R}}^{\geq i} = \{x \in R \mid v_R(x) \geq i\}$. Define an element π of R with $v_R(\pi) = 1/e$ by $\pi = (\pi, \pi_1, \pi_2, \dots)$, where we abusively

write π_n also for its image in $\tilde{\mathcal{O}}_{\bar{K}}$. The ring R has a natural \mathfrak{S} -algebra structure defined by the continuous map $\mathfrak{S} \rightarrow R$ which sends u to the element π . Then we have the following classification theorem due to Breuil and Kisin.

Theorem 2.1 ([4], [5], [15], [16]). *There exists an anti-equivalence $\mathrm{Gr}(-)$ from the category $\mathrm{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$ (resp. $\mathrm{Mod}_{/S}^{1,\varphi}$) to the category of finite flat group schemes over \mathcal{O}_K killed by p (resp. Barsotti-Tate groups over \mathcal{O}_K). Moreover, put $\mathcal{G}(-) = \mathrm{Gr}(\mathcal{M}_{\mathfrak{S}}(-))$. Then for any object \mathfrak{M} of the category $\mathrm{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$, we have a natural isomorphism of G_{K_∞} -modules*

$$\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}})|_{G_{K_\infty}} \rightarrow T_{\mathfrak{S}}^*(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, R).$$

The anti-equivalence $\mathcal{G}(-)$ is compatible with Cartier duality in the following sense. For an object \mathfrak{M} of the category $\mathrm{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$, we can define a natural dual object \mathfrak{M}^\vee ([6], [18]), as follows. Put $\mathfrak{M}^\vee = \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_1)$. Choose a basis e_1, \dots, e_h of the free \mathfrak{S}_1 -module \mathfrak{M} and let $e_1^\vee, \dots, e_h^\vee$ denote its dual basis. Define a matrix $A \in M_h(\mathfrak{S}_1)$ by

$$\varphi_{\mathfrak{M}}(e_1, \dots, e_h) = (e_1, \dots, e_h)A.$$

The φ -semilinear map $\varphi_{\mathfrak{M}^\vee} : \mathfrak{M}^\vee \rightarrow \mathfrak{M}^\vee$ is defined to be

$$\varphi_{\mathfrak{M}^\vee}(e_1^\vee, \dots, e_h^\vee) = (e_1^\vee, \dots, e_h^\vee)(E(u)/c_0)({}^tA)^{-1},$$

which is independent of the choice of the basis e_1, \dots, e_h . Then we have a natural isomorphism of finite flat group schemes over \mathcal{O}_K

$$\mathcal{G}(\mathfrak{M})^\vee \rightarrow \mathcal{G}(\mathfrak{M}^\vee),$$

where \vee on the left-hand side means the Cartier dual (see [12, Proposition 4.4]). This defines an isomorphism of G_{K_∞} -modules

$$\delta_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})^\vee(\mathcal{O}_{\bar{K}}) \rightarrow \mathcal{G}(\mathfrak{M}^\vee)(\mathcal{O}_{\bar{K}}) \xrightarrow{\varepsilon_{\mathfrak{M}^\vee}} T_{\mathfrak{S}}^*(\mathfrak{M}^\vee).$$

Let \mathfrak{S}_1^\vee be the object of the category $\mathrm{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$ whose underlying \mathfrak{S} -module is \mathfrak{S}_1 and Frobenius map is given by $\varphi_{\mathfrak{S}_1^\vee}(1) = c_0^{-1}E(u)$. Then the natural pairing

$$\langle \cdot, \cdot \rangle_{\mathfrak{M}} : \mathfrak{M} \times \mathfrak{M}^\vee \rightarrow \mathfrak{S}_1^\vee$$

induces a perfect pairing of G_{K_∞} -modules

$$T_{\mathfrak{S}}^*(\mathfrak{M}) \times T_{\mathfrak{S}}^*(\mathfrak{M}^\vee) \rightarrow T_{\mathfrak{S}}^*(\mathfrak{S}_1^\vee),$$

which is denoted also by $\langle \cdot, \cdot \rangle_{\mathfrak{M}}$. This pairing fits into a commutative diagram of G_{K_∞} -modules

$$\begin{array}{ccc} \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \times \mathcal{G}(\mathfrak{M}^\vee)(\mathcal{O}_{\bar{K}}) & \longrightarrow & \mathbb{Z}/p\mathbb{Z}(1) \\ \varepsilon_{\mathfrak{M}} \downarrow \wr & & \delta_{\mathfrak{M}} \downarrow \wr & & \downarrow \wr \\ T_{\mathfrak{S}}^*(\mathfrak{M}) \times T_{\mathfrak{S}}^*(\mathfrak{M}^\vee) & \longrightarrow & T_{\mathfrak{S}}^*(\mathfrak{S}_1^\vee), \\ & & \langle \cdot, \cdot \rangle_{\mathfrak{M}} & & \end{array}$$

where the top arrow is the Cartier pairing of $\mathcal{G}(\mathfrak{M})$ ([12, Proposition 4.4]).

On the other hand, for a $k[[u]]$ -algebra B , we also have an anti-equivalence of categories $\mathcal{H}(-)$ from $\text{Mod}_{/B}^{1,\varphi}$ to the category of finite locally free group schemes \mathcal{H} over B whose Verschiebung $V_{\mathcal{H}}$ is zero such that the cokernel of the induced map $V_{\mathcal{H}^\vee} : \text{Lie}(\mathcal{H}^\vee)^{(p)} \rightarrow \text{Lie}(\mathcal{H}^\vee)$ is killed by u^e ([9, Théorème 7.4]). Note that the anti-equivalence $\mathcal{H}(-)$ commutes with any base change. Suppose that the B -module \mathfrak{M} is free of rank h . Let e_1, \dots, e_h be its basis and put $\varphi(e_1, \dots, e_h) = (e_1, \dots, e_h)A$ for some matrix $A = (a_{i,j}) \in M_h(B)$. Then the group scheme $\mathcal{H}(\mathfrak{M})$ is by definition the additive group scheme whose affine algebra is

$$B[X_1, \dots, X_h] / (X_1^p - \sum_{j=1}^h a_{j,1} X_j, \dots, X_h^p - \sum_{j=1}^h a_{j,h} X_j).$$

Moreover, for the case of $B = k[[u]]$, we have a natural isomorphism of G_{K_∞} -modules

$$\mathcal{H}(\mathfrak{M})(R) \rightarrow \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, R),$$

by which we identify both sides. Hence we obtain the isomorphism of G_{K_∞} -modules

$$\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}})|_{G_{K_\infty}} \rightarrow \mathcal{H}(\mathfrak{M})(R).$$

We normalize the indices of the ramification subgroup schemes of $\mathcal{H}(\mathfrak{M})$ by multiplying the usual ones by $1/e$, so that

$$\mathcal{H}(\mathfrak{M})_i(R) = \text{Ker}(\mathcal{H}(\mathfrak{M})(R) \rightarrow \mathcal{H}(\mathfrak{M})(R/m_R^{\geq i})).$$

Then we have the following correspondence of ramification between the finite flat group schemes $\mathcal{G}(\mathfrak{M})$ and $\mathcal{H}(\mathfrak{M})$.

Theorem 2.2. *Let \mathfrak{M} be an object of the category $\text{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$.*

- (1) ([12, Theorem 1.1]) *The natural isomorphism of G_{K_∞} -modules*

$$\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}})|_{G_{K_\infty}} \rightarrow \mathcal{H}(\mathfrak{M})(R)$$

preserves the upper and the lower ramification subgroups of both sides.

- (2) ([12, Corollary 4.6]) *By the k -algebra isomorphism $k[[u^{1/p}]]/(u^e) \rightarrow \mathcal{O}_{K_1}/p\mathcal{O}_{K_1}$ sending $u^{1/p}$ to π_1 , we identify both sides of the isomorphism. Then there exists an isomorphism of schemes*

$$\mathcal{G}(\mathfrak{M}) \times \text{Spec}(\mathcal{O}_{K_1}/p\mathcal{O}_{K_1}) \rightarrow \mathcal{H}(\mathfrak{M}) \times \text{Spec}(k[[u^{1/p}]]/(u^e))$$

which preserves the zero section.

For a finite flat group scheme \mathcal{G} over \mathcal{O}_K which is killed by p , the upper and the lower ramification subgroups are in duality as in the following theorem.

Theorem 2.3 ([21], Theorem 1.6 or [8], Proposition 6). *Let K/\mathbb{Q}_p be an extension of complete discrete valuation fields and \mathcal{G} be a finite flat group scheme over \mathcal{O}_K killed by p . For $j \leq p/(p-1)$, we have the equality*

$$\mathcal{G}^j(\mathcal{O}_{\bar{K}})^\perp = (\mathcal{G}^\vee)_{l(j)+}(\mathcal{O}_{\bar{K}})$$

of subgroups of $\mathcal{G}^\vee(\mathcal{O}_{\bar{K}})$, where \perp means the orthogonal subgroup with respect to the Cartier pairing and $l(j) = 1/(p-1) - j/p$.

We insert here a lemma which gives an upper bound of the lower ramification for finite flat group schemes killed by p . For an extension $\mathcal{K}/k((u))$ of complete discrete valuation fields, let v_u be the u -adic valuation on \mathcal{K} normalized as $v_u(u) = 1$. Let \mathcal{G} be a finite flat generically etale group scheme over $\mathcal{O}_{\mathcal{K}}$. Fix a positive integer m and we normalize the indices of the ramification subgroup schemes of \mathcal{G} by multiplying the usual ones by $1/m$. Write $\omega_{\mathcal{G}} \simeq \bigoplus_i \mathcal{O}_{\mathcal{K}}/(a_i)$ for some $a_i \in \mathcal{O}_{\mathcal{K}}$ and put $\deg(\mathcal{G}) = m^{-1} \sum_i v_u(a_i)$. In particular, for an object \mathfrak{M} of the category $\text{Mod}_{\mathcal{O}_{\mathcal{K}}}^{1,\varphi}$, we define $\deg(\mathcal{H}(\mathfrak{M}))$ by putting $m = e$.

Lemma 2.4. *Let \mathcal{K}/\mathbb{Q}_p (resp. $\mathcal{K}/k((u))$) be an extension of complete discrete valuation fields and \mathcal{G} be a finite flat generically etale group scheme over $\mathcal{O}_{\mathcal{K}}$ killed by p . Then $\mathcal{G}_i = 0$ for any $i > \deg(\mathcal{G})/(p-1)$.*

Proof. Let \mathcal{K}^{sep} be a separable closure of \mathcal{K} . We may assume $\mathcal{G}(\mathcal{O}_{\mathcal{K}^{\text{sep}}}) = \mathcal{G}(\mathcal{O}_{\mathcal{K}})$. For i as in the lemma, put $x \in \mathcal{G}_i(\mathcal{O}_{\mathcal{K}})$ and let \mathcal{H} be the scheme-theoretic closure of the subgroup $\mathbb{F}_p x \subseteq \mathcal{G}(\mathcal{O}_{\mathcal{K}})$ in \mathcal{G} . Then we have $\deg(\mathcal{H}) \leq \deg(\mathcal{G})$. By the Oort-Tate classification ([20]), the affine algebra of \mathcal{H} is isomorphic to the ring $\mathcal{O}_{\mathcal{K}}[X]/(X^p - aX)$ for some $a \in \mathcal{O}_{\mathcal{K}}$ with $\deg(\mathcal{H}) = v_p(a)$ (resp. $\deg(\mathcal{H}) = m^{-1}v_u(a)$). Hence we obtain $\mathcal{H}_i = 0$ and $x = 0$. \square

2.3. Hodge filtration and Breuil-Kisin modules. Finally, due to the lack of references, we explain how to decode the Hodge filtration, the Hodge height and the Hodge-Tate map for a truncated Barsotti-Tate group of level one over \mathcal{O}_K from its corresponding Breuil-Kisin module. Put $\mathcal{S}_1 = \text{Spec}(\tilde{\mathcal{O}}_K)$ and $E_1 = \text{Spec}(S_1)$. Consider the big crystalline site with the fppf topology $(\mathcal{S}_1/E_1)_{\text{CRYS}}$, as in [3]. For an fppf sheaf \mathcal{E} over $\text{Spec}(\mathcal{O}_K)$, we let \mathcal{E}_1 denote its restriction to $\text{Spec}(\tilde{\mathcal{O}}_K)$ only in this subsection, and for an fppf sheaf \mathcal{F} over $\text{Spec}(\tilde{\mathcal{O}}_K)$, let $\underline{\mathcal{F}}$ denote the associated sheaf on the site $(\mathcal{S}_1/E_1)_{\text{CRYS}}$ ([3, (1.1.4.5)]). We also write the crystalline Dieudonné functor as $\mathbb{D}^*(-) = \mathcal{E}xt_{\mathcal{S}_1/E_1}^1(-, \mathcal{O}_{\mathcal{S}_1/E_1})$ ([3, Définition 3.1.5]). Let v_u be the u -adic valuation on the ring $k[[u]]$ with $v_u(u) = 1$, as before. By the k -algebra isomorphism $k[[u]]/(u^e) \rightarrow \tilde{\mathcal{O}}_K$ sending u to π , we identify both sides of the isomorphism.

Lemma 2.5. (1) *Let \mathcal{G} be a truncated Barsotti-Tate group of level one over \mathcal{O}_K and \mathcal{M} be the object of $\text{Mod}_{S_1}^{1,\varphi}$ which corresponds to \mathcal{G} via the anti-equivalence $\text{Gr}(-)$. Then there exist natural isomorphisms of $\tilde{\mathcal{O}}_K$ -modules*

$$\text{Fil}^1 \mathcal{M}/(\text{Fil}^1 S) \mathcal{M} \rightarrow \omega_{\mathcal{G}_1}, \quad \mathcal{M}/\text{Fil}^1 \mathcal{M} \rightarrow \text{Lie}(\mathcal{G}_1^\vee).$$

(2) *Let \mathcal{G} be a finite flat group scheme over \mathcal{O}_K killed by p and \mathcal{M} be its corresponding object of the category $\text{Mod}_{S_1}^{1,\varphi}$. Then we have a natural isomorphism of $\tilde{\mathcal{O}}_K$ -modules $\text{Fil}^1 \mathcal{M}/(\text{Fil}^1 S) \mathcal{M} \rightarrow \omega_{\mathcal{G}_1}$.*

(3) Let \mathfrak{M} be an object of the category $\text{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$. Then we have $\deg(\mathcal{G}(\mathfrak{M})) = \deg(\mathcal{H}(\mathfrak{M})) = e^{-1}v_u(\det(\varphi_{\mathfrak{M}}))$.

Proof. Let us consider the assertion (1). By [13, Théorème 4.4 (e)], we can find a Barsotti-Tate group Γ over \mathcal{O}_K such that its p -torsion subgroup scheme $\Gamma[p]$ is isomorphic to \mathcal{G} . Let \mathcal{N} be the object of the category $\text{Mod}_{/S}^{1,\varphi}$ corresponding to Γ and put $\mathcal{N}_1 = \mathcal{N}/p\mathcal{N}$, which is naturally considered as an object of the category $\text{Mod}_{/S_1}^{1,\varphi}$. By the construction of the anti-equivalence ([15]), we have a natural isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Fil}^1\mathcal{N}_1/(\text{Fil}^1S)\mathcal{N}_1 & \longrightarrow & \mathcal{N}_1/(\text{Fil}^1S)\mathcal{N}_1 & \longrightarrow & \mathcal{N}_1/\text{Fil}^1\mathcal{N}_1 \longrightarrow 0 \\ & & \wr \downarrow & & \wr \downarrow & & \wr \downarrow \\ 0 & \longrightarrow & \omega_{\Gamma_1} & \longrightarrow & \mathbb{D}^*(\Gamma_1)_{\tilde{\mathcal{O}}_K} & \longrightarrow & \text{Lie}(\Gamma_1^\vee) \longrightarrow 0, \end{array}$$

where $\mathbb{D}^*(\Gamma_1)_{\tilde{\mathcal{O}}_K}$ is the set of sections on the divided power thickening $\tilde{\mathcal{O}}_K \rightarrow \tilde{\mathcal{O}}_K$. Thus we also have the natural isomorphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Fil}^1\mathcal{M}/(\text{Fil}^1S)\mathcal{M} & \longrightarrow & \mathcal{M}/(\text{Fil}^1S)\mathcal{M} & \longrightarrow & \mathcal{M}/\text{Fil}^1\mathcal{M} \longrightarrow 0 \\ & & \wr \downarrow & & \wr \downarrow & & \wr \downarrow \\ 0 & \longrightarrow & \omega_{\mathcal{G}_1} & \longrightarrow & \mathbb{D}^*(\mathcal{G}_1)_{\tilde{\mathcal{O}}_K} & \longrightarrow & \text{Lie}(\mathcal{G}_1^\vee) \longrightarrow 0. \end{array}$$

For the assertion (2), take a resolution

$$0 \rightarrow \mathcal{G} \rightarrow \Gamma \rightarrow \Gamma' \rightarrow 0$$

of \mathcal{G} by Barsotti-Tate groups Γ and Γ' over \mathcal{O}_K . Let \mathcal{N} and \mathcal{N}' be the objects of the category $\text{Mod}_{/S}^{1,\varphi}$ corresponding to Γ and Γ' , respectively. Then we have exact sequences

$$0 \rightarrow \omega_{\Gamma'} \rightarrow \omega_{\Gamma} \rightarrow \omega_{\mathcal{G}} \rightarrow 0, \quad 0 \rightarrow \mathcal{N}' \rightarrow \mathcal{N} \rightarrow \mathcal{M} \rightarrow 0.$$

By tensoring $\tilde{\mathcal{O}}_K$ to the first sequence and using the assertion (1), we obtain an exact sequence

$$\text{Fil}^1\mathcal{N}'/(p\text{Fil}^1\mathcal{N}' + (\text{Fil}^1S)\mathcal{N}') \rightarrow \text{Fil}^1\mathcal{N}/(p\text{Fil}^1\mathcal{N} + (\text{Fil}^1S)\mathcal{N}) \rightarrow \omega_{\mathcal{G}_1} \rightarrow 0.$$

This and the second sequence induce the isomorphism $\text{Fil}^1\mathcal{M}/(\text{Fil}^1S)\mathcal{M} \rightarrow \omega_{\mathcal{G}_1}$.

For the assertion (3), choose a basis e_1, \dots, e_h of \mathfrak{M} and put

$$\varphi(e_1, \dots, e_h) = (e_1, \dots, e_h)A$$

for some $A \in M_h(k[[u]])$. Let u^{s_1}, \dots, u^{s_h} be the elementary divisors of A , which satisfy $0 \leq s_i \leq e$ for any i . Note that the matrix $u^e A^{-1}$ is contained in $M_h(k[[u]])$ and its elementary divisors are $u^{e-s_1}, \dots, u^{e-s_h}$. By the definition of the functor $\mathcal{M}_{\mathfrak{S}}(-)$, we have the equality

$$\text{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})/(\text{Fil}^1S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) = \text{Span}_{\mathfrak{S}_1}((1 \otimes e_1, \dots, 1 \otimes e_h)u^e A^{-1}).$$

Thus, by the assertion (2), we obtain an isomorphism

$$\omega_{\mathcal{G}(\mathfrak{M})_1} \simeq \bigoplus_i u^{e-s_i} k[[u]]/u^e k[[u]],$$

which implies the equalities $\deg(\mathcal{G}(\mathfrak{M})) = \sum_i e^{-1} s_i = e^{-1} v_u(\det \varphi)$. From the explicit description of the affine algebra of $\mathcal{H}(\mathfrak{M})$ given before, we see that this is also equal to $\deg(\mathcal{H}(\mathfrak{M}))$. \square

Let \mathcal{G} and \mathcal{M} be as in Lemma 2.5 (1) and set \mathfrak{M} to be the object of $\text{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$ which corresponds to \mathcal{M} via the equivalence $\mathcal{M}_{\mathfrak{S}}(-)$. We let h and d denote the height and the dimension of \mathcal{G} , respectively. Let e_1, \dots, e_h be a basis of \mathfrak{M} and A be the element of $M_h(k[[u]])$ with

$$\varphi(e_1, \dots, e_h) = (e_1, \dots, e_h)A.$$

We also put $\mathfrak{M}_1 = \mathfrak{M}/u^e \mathfrak{M}$ and $\text{Fil}^1 \mathfrak{M}_1 = (1 \otimes \varphi)(\tilde{\mathcal{O}}_K \otimes_{\varphi, \tilde{\mathcal{O}}_K} \mathfrak{M}_1) \subseteq \mathfrak{M}_1$. Then we have an isomorphism of $\tilde{\mathcal{O}}_K$ -modules

$$\mathcal{M}/(\text{Fil}^1 S)\mathcal{M} \rightarrow \tilde{\mathcal{O}}_K \otimes_{\varphi, \tilde{\mathcal{O}}_K} \mathfrak{M}_1.$$

From Lemma 2.5 (1) and the definition of the functor $\mathcal{M}_{\mathfrak{S}}(-)$, we also have natural isomorphisms of $\tilde{\mathcal{O}}_K$ -modules

$$\text{Lie}(\mathcal{G}_1^\vee) \rightarrow \mathcal{M}/\text{Fil}^1 \mathcal{M} \xrightarrow{1 \otimes \varphi} \text{Fil}^1 \mathfrak{M}_1.$$

Hence the $\tilde{\mathcal{O}}_K$ -module $\text{Fil}^1 \mathfrak{M}_1$ is free of rank $h - d$ and each elementary divisor of the matrix A is either 1 or u^e . This implies that the $\tilde{\mathcal{O}}_K$ -module $\mathfrak{M}_1/\text{Fil}^1 \mathfrak{M}_1$ is free of rank d .

Consider the diagram of $\tilde{\mathcal{O}}_K$ -modules

$$\begin{array}{ccccc} \tilde{\mathcal{O}}_K \otimes_{\varphi, \tilde{\mathcal{O}}_K} (\tilde{\mathcal{O}}_K \otimes_{\varphi, \tilde{\mathcal{O}}_K} \mathfrak{M}_1) & \xrightarrow{1 \otimes 1 \otimes \varphi} & \tilde{\mathcal{O}}_K \otimes_{\varphi, \tilde{\mathcal{O}}_K} \text{Fil}^1 \mathfrak{M}_1 & \longrightarrow & \tilde{\mathcal{O}}_K \otimes_{\varphi, \tilde{\mathcal{O}}_K} \mathfrak{M}_1 \\ \downarrow 1 \otimes \varphi \otimes \varphi & & \downarrow & & \downarrow 1 \otimes \varphi \\ \tilde{\mathcal{O}}_K \otimes_{\varphi, \tilde{\mathcal{O}}_K} \mathfrak{M}_1 & \xrightarrow{1 \otimes \varphi} & \text{Fil}^1 \mathfrak{M}_1 & \longrightarrow & \mathfrak{M}_1 \end{array}$$

whose left horizontal arrows are surjections and right horizontal arrows are natural inclusions. The left vertical arrow is the map induced by the Frobenius $F_{\mathcal{G}_1}$ via the natural isomorphism $\mathcal{M} \simeq \mathbb{D}^*(\mathcal{G}_1)(S_1 \rightarrow \tilde{\mathcal{O}}_K)$ and the middle vertical arrow is the map induced by the left vertical arrow. Lemma 2.5 (1) implies that the truncated valuation of the determinant of the latter map is equal to $\text{Hdg}(\mathcal{G})$, since the natural isomorphism $\text{Hom}(\mathcal{G}_1, \mathbb{G}_a) \rightarrow \text{Lie}(\mathcal{G}_1^\vee)$ takes the action of the Frobenius $F_{\mathcal{G}_1}$ on the left-hand side to the action of the Verschiebung $V_{\mathcal{G}_1^\vee}$ on the right-hand side. We see that the outer square is commutative and thus the right square is also commutative. Hence we have an exact sequence

$$0 \rightarrow \text{Fil}^1 \mathfrak{M}_1 \rightarrow \mathfrak{M}_1 \rightarrow \mathfrak{M}_1/\text{Fil}^1 \mathfrak{M}_1 \rightarrow 0$$

of the category $\text{Mod}_{/\tilde{\mathcal{O}}_K}^{1,\varphi}$ with $v_p(\det(\varphi_{\text{Fil}^1 \mathfrak{M}_1})) = \text{Hdg}(\mathcal{G})$.

Let $i \leq 1$ be a non-negative rational number. Then the zeroth projection $\text{pr}_0 : R \rightarrow \tilde{\mathcal{O}}_{\bar{K}}$ induces an isomorphism $R/m_{\bar{R}}^{\geq i} \rightarrow \mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i}$, by which we identify both sides. Moreover, we also have natural isomorphisms

$$\text{Hom}_{\tilde{\mathcal{O}}_{\bar{K}}}(\text{Fil}^1 \mathfrak{M}_1, R/m_{\bar{R}}^{\geq i}) \rightarrow \text{Hom}_{\tilde{\mathcal{O}}_{\bar{K}}}(\text{Lie}(\mathcal{G}_1^\vee), \mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i}) \rightarrow \omega_{\mathcal{G}_1^\vee} \otimes \mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i}.$$

Lemma 2.6. *Let \mathcal{G} be a truncated Barsotti-Tate group of level one over \mathcal{O}_K and \mathfrak{M} be the object of $\text{Mod}_{/\mathfrak{S}_1}^{1, \varphi}$ which corresponds to \mathcal{G} via the anti-equivalence $\mathcal{G}(-)$. Then the composite*

$$\mathcal{G}(\mathcal{O}_{\bar{K}}) \rightarrow \mathcal{H}(\mathfrak{M})(R) \rightarrow \text{Hom}_{\tilde{\mathcal{O}}_{\bar{K}}}(\text{Fil}^1 \mathfrak{M}_1, R/m_{\bar{R}}^{\geq i}) \rightarrow \omega_{\mathcal{G}_1^\vee} \otimes \mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i}$$

coincides with the Hodge-Tate map HT_i for any $i \leq 1$.

Proof. For a sheaf \mathcal{E} on the site $(\mathcal{S}_1/E_1)_{\text{CRYS}}$, we let $\mathcal{E}_{\tilde{\mathcal{O}}_{\bar{K}}}$ denote the set of sections on the natural divided power thickening $\tilde{\mathcal{O}}_{\bar{K}} \rightarrow \tilde{\mathcal{O}}_{\bar{K}}$. For a valued point $x : \text{Spec}(\mathcal{O}_{\bar{K}}) \rightarrow \mathcal{G}$, we let $x^\vee : \mathcal{G}^\vee \times \text{Spec}(\mathcal{O}_{\bar{K}}) \rightarrow \mu_p$ denote its dual map. Since \mathcal{G} is a truncated Barsotti-Tate group of level one, we have a commutative diagram

$$\begin{array}{ccc} \text{Hom}(\mathbb{D}^*(\mathbb{Z}/p\mathbb{Z})_{\tilde{\mathcal{O}}_{\bar{K}}}, \mathbb{D}^*(\mathbb{Z}/p\mathbb{Z})_{\tilde{\mathcal{O}}_{\bar{K}}}) & \xrightarrow{\mathbb{D}^*(x)^\vee} & \text{Hom}(\mathbb{D}^*(\mathcal{G}_1)_{\tilde{\mathcal{O}}_{\bar{K}}}, \mathbb{D}^*(\mathbb{Z}/p\mathbb{Z})_{\tilde{\mathcal{O}}_{\bar{K}}}) \\ \wr \downarrow & & \wr \downarrow \\ \mathbb{D}^*(\mu_p)_{\tilde{\mathcal{O}}_{\bar{K}}} & \xrightarrow{\mathbb{D}^*(x^\vee)} & \mathbb{D}^*(\mathcal{G}_1^\vee)_{\tilde{\mathcal{O}}_{\bar{K}}} \end{array}$$

where the vertical arrows are isomorphisms compatible with Hodge filtrations ([3, Proposition 5.3.6]). Put $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$. In particular, by Lemma 2.5 (1) we also have a commutative diagram whose vertical arrows are isomorphisms

$$\begin{array}{ccc} \text{Hom}_S(\mathcal{M}/\text{Fil}^1 \mathcal{M}, \tilde{\mathcal{O}}_{\bar{K}}) & \hookrightarrow & \text{Hom}_S(\mathcal{M}, \tilde{\mathcal{O}}_{\bar{K}}) \\ \wr \downarrow & & \wr \downarrow \\ \omega_{\mathcal{G}_1^\vee} \otimes \tilde{\mathcal{O}}_{\bar{K}} & \hookrightarrow & \mathbb{D}^*(\mathcal{G}_1^\vee)_{\tilde{\mathcal{O}}_{\bar{K}}}. \end{array}$$

Then $\text{HT}_1(x)$ coincides with the image of the identity map on the upper left corner of the former diagram by the lower composite, which is contained in the submodule $\omega_{\mathcal{G}_1^\vee} \otimes \tilde{\mathcal{O}}_{\bar{K}}$.

Now we consider a diagram

$$\begin{array}{ccccc}
\mathcal{G}(\mathcal{O}_{\bar{K}}) & \xrightarrow{\quad \dashrightarrow \quad} & \text{Hom}_S(\mathcal{M}, R^{\text{DP}}) & \xleftarrow{\quad \dashrightarrow \quad} & \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, R) \\
& \searrow & \downarrow & & \downarrow \\
& & \text{Hom}_S(\mathcal{M}, \tilde{\mathcal{O}}_{\bar{K}}) & \xleftarrow{\quad \dashrightarrow \quad} & \text{Hom}_{\mathfrak{S}, \varphi}(\mathfrak{M}, \tilde{\mathcal{O}}_{\bar{K}}) \\
& & \uparrow & & \downarrow \\
& & \text{Hom}_S(\mathcal{M}/\text{Fil}^1 \mathcal{M}, \tilde{\mathcal{O}}_{\bar{K}}) & \xleftarrow{\sim} & \text{Hom}_{\tilde{\mathcal{O}}_{\bar{K}}}(\text{Fil}^1 \mathfrak{M}_1, \tilde{\mathcal{O}}_{\bar{K}}).
\end{array}$$

Here R^{DP} is the divided power envelope of the ring R with respect to the ideal $m_{\bar{R}}^{\geq 1}$. The right-pointing solid arrows are defined by $x \mapsto \mathbb{D}^*(x)$ on the sets of sections over the divided power thickenings $R^{\text{DP}} \rightarrow \tilde{\mathcal{O}}_{\bar{K}}$ and $\tilde{\mathcal{O}}_{\bar{K}} \rightarrow \tilde{\mathcal{O}}_{\bar{K}}$. The top left-pointed horizontal arrow maps a homomorphism $f : \mathfrak{M} \rightarrow R$ to the homomorphism $\mathcal{M} = S \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \rightarrow R^{\text{DP}}$ defined by $s \otimes m \mapsto s\varphi(f(m))$ and the middle left-pointed horizontal arrow is defined similarly. The bottom one is induced by the isomorphism $1 \otimes \varphi : \mathcal{M}/\text{Fil}^1 \mathcal{M} \rightarrow \text{Fil}^1 \mathfrak{M}_1$. The dotted arrow is the isomorphism $\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathcal{O}_{\bar{K}}) \rightarrow T_{\mathfrak{S}}^*(\mathfrak{M})$. From the definition, we see that the bottom square is commutative and thus the diagram is also commutative. Hence the lemma follows. \square

3. LEVEL ONE CANONICAL SUBGROUP

In this section, we prove the following level one case of Theorem 1.1.

Theorem 3.1. *Let the notation be as in Section 1 and suppose $p > 2$. Let \mathcal{G} be a truncated Barsotti-Tate group of level one, height h and dimension d over \mathcal{O}_K with $0 < d < h$ and Hodge height $w = \text{Hdg}(\mathcal{G})$.*

- (1) *If $w < p/(p+1)$, then there exists a unique finite flat closed subgroup scheme \mathcal{C} of \mathcal{G} of order p^d over \mathcal{O}_K such that $\mathcal{C} \times \text{Spec}(\mathcal{O}_K/m_{\bar{K}}^{\geq 1-w})$ coincides with the kernel of the Frobenius homomorphism of $\mathcal{G} \times \text{Spec}(\mathcal{O}_K/m_{\bar{K}}^{\geq 1-w})$. We refer the subgroup scheme \mathcal{C} as the canonical subgroup of \mathcal{G} . Moreover, the subgroup scheme \mathcal{C} has the following properties:*
 - (a) $\deg(\mathcal{G}/\mathcal{C}) = w$.
 - (b) *Let \mathcal{C}' be the canonical subgroup of \mathcal{G}^\vee . Then we have the equality of subgroup schemes $\mathcal{C}' = (\mathcal{G}/\mathcal{C})^\vee$, or equivalently $\mathcal{C}(\mathcal{O}_{\bar{K}}) = \mathcal{C}'(\mathcal{O}_{\bar{K}})^\perp$, where \perp means the orthogonal subgroup with respect to the Cartier pairing.*
 - (c) $\mathcal{C} = \mathcal{G}_{(1-w)/(p-1)} = \mathcal{G}^{pw/(p-1)+}$.
- (2) *If $w < (p-1)/p$, then the subgroup $\mathcal{C}(\mathcal{O}_{\bar{K}})$ coincides with the kernel of the Hodge-Tate map $\text{HT}_b : \mathcal{G}(\mathcal{O}_{\bar{K}}) \rightarrow \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq b}$ for $w/(p-1) < b \leq 1-w$.*
- (3) *If $w < 1/2$, then \mathcal{C} coincides both with the lower ramification subgroup scheme \mathcal{G}_b for $w/(p-1) < b \leq (1-w)/(p-1)$ and the*

upper ramification subgroup scheme \mathcal{G}^{j+} for $pw/(p-1) \leq j < p(1-w)/(p-1)$.

Proof. Let $W(\bar{k})$ be the Witt ring of an algebraic closure \bar{k} of k . Choose a Cohen ring $C(k)$ with residue field k and inclusions $C(k) \rightarrow W(\bar{k})$ and $C(k) \rightarrow \mathcal{O}_K$. Then the \mathcal{O}_K -algebra $\mathcal{O}_{K'} = \mathcal{O}_K \otimes_{C(k)} W(\bar{k})$ is a complete discrete valuation ring with residue field \bar{k} and relative ramification index one over \mathcal{O}_K . Note that finite flat closed subgroup schemes \mathcal{D} and \mathcal{D}' of \mathcal{G} of the same order over \mathcal{O}_K coincide if and only if the map $\mathcal{D} \rightarrow \mathcal{G}/\mathcal{D}'$ is zero and the latter can be checked after the base change to $\mathcal{O}_{K'}$. Hence if we show the theorem for $\mathcal{G} \times \text{Spec}(\mathcal{O}_{K'})$, then the upper ramification subgroup scheme $\mathcal{C} = \mathcal{G}^{pw/(p-1)+}$ satisfies all the properties in the theorem. Therefore, replacing \mathcal{O}_K with $\mathcal{O}_{K'}$, we may assume that k is perfect.

We adopt the notation of the previous section. Let \mathfrak{M} be the object of the category $\text{Mod}_{\mathfrak{S}_1}^{1,\varphi}$ corresponding to \mathcal{G} via the anti-equivalence $\mathcal{G}(-)$. Put $\mathfrak{M}_1 = \mathfrak{M}/u^e\mathfrak{M}$ and $\text{Fil}^1\mathfrak{M}_1 = (1 \otimes \varphi)(\tilde{\mathcal{O}}_K \otimes_{\varphi, \tilde{\mathcal{O}}_K} \mathfrak{M}_1)$. We identify the k -algebras $k[[u]]/(u^e)$ and $\tilde{\mathcal{O}}_K$ by $u \mapsto \pi$, as before. As we have seen in Subsection 2.3, we have the exact sequence

$$0 \rightarrow \text{Fil}^1\mathfrak{M}_1 \rightarrow \mathfrak{M}_1 \rightarrow \mathfrak{M}_1/\text{Fil}^1\mathfrak{M}_1 \rightarrow 0$$

of the category $\text{Mod}_{\tilde{\mathcal{O}}_K}^{1,\varphi}$, where the $\tilde{\mathcal{O}}_K$ -module $\text{Fil}^1\mathfrak{M}_1$ (*resp.* $\mathfrak{M}_1/\text{Fil}^1\mathfrak{M}_1$) is free of rank $h-d$ (*resp.* rank d) and the equality $v_p(\det(\varphi_{\text{Fil}^1\mathfrak{M}_1})) = w$ holds.

We choose once and for all a basis e_1, \dots, e_h of \mathfrak{M} such that e_1, \dots, e_{h-d} is a lift of a basis of the $\tilde{\mathcal{O}}_K$ -module $\text{Fil}^1\mathfrak{M}_1$ to the $k[[u]]$ -submodule $(1 \otimes \varphi)(\mathfrak{S}_1 \otimes_{\varphi, \mathfrak{S}_1} \mathfrak{M})$ and e_{h-d+1}, \dots, e_h is a lift of a basis of the $\tilde{\mathcal{O}}_K$ -module $\mathfrak{M}_1/\text{Fil}^1\mathfrak{M}_1$. Then the elements $e_1, \dots, e_{h-d}, u^e e_{h-d+1}, \dots, u^e e_h$ form a basis of the $k[[u]]$ -module $(1 \otimes \varphi)(\mathfrak{S}_1 \otimes_{\varphi, \mathfrak{S}_1} \mathfrak{M})$ and we have

$$\varphi(e_1, \dots, e_h) = (e_1, \dots, e_h) \begin{pmatrix} P_1 & P_2 \\ u^e P_3 & u^e P_4 \end{pmatrix}$$

for some matrices P_i with entries in the ring $k[[u]]$, where P_4 is a $d \times d$ -matrix. Since the elements on the right-hand side also form a basis of $(1 \otimes \varphi)(\mathfrak{S}_1 \otimes_{\varphi, \mathfrak{S}_1} \mathfrak{M})$, the matrix

$$\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$$

is contained in $GL_h(k[[u]])$. Moreover, since $w < 1$, we also have the equality $v_u(\det(P_1)) = ew$ and thus there exists a matrix $\hat{P}_1 \in M_{h-d}(k[[u]])$ satisfying $P_1 \hat{P}_1 = u^{ew} I_{h-d}$, where I_{h-d} is the identity matrix. Here we note that the number ew is a non-negative integer.

For the uniqueness assertion in (1), we first show the following lemma.

Lemma 3.2. *Let \mathcal{D} be a finite flat closed subgroup scheme of \mathcal{G} over \mathcal{O}_K and \mathfrak{L} be the subobject of \mathfrak{M} in the category $\text{Mod}_{\mathfrak{S}_1}^{1,\varphi}$ corresponding to the*

quotient \mathcal{G}/\mathcal{D} via the anti-equivalence $\mathcal{G}(-)$. Then, for any $i \in \frac{1}{e}\mathbb{Z}$ with $0 < i \leq 1$, the following conditions are equivalent:

- (1) The subgroup scheme $\mathcal{D} \times \text{Spec}(\mathcal{O}_K/m_K^{\geq i})$ coincides with the kernel of the Frobenius homomorphism of $\mathcal{G} \times \text{Spec}(\mathcal{O}_K/m_K^{\geq i})$.
- (2) $\mathfrak{L}/u^{ei}\mathfrak{L} = \text{Fil}^1\mathfrak{M}_1/u^{ei}\text{Fil}^1\mathfrak{M}_1$.

Proof. Put $\mathfrak{N} = \mathfrak{M}/\mathfrak{L}$, which is the object of the category $\text{Mod}_{\mathfrak{S}_1}^{1,\varphi}$ corresponding to the finite flat group scheme \mathcal{D} . Note that the Frobenius map is compatible with any morphism between schemes of characteristic p , and that the kernel of the Frobenius of the truncated Barsotti-Tate group $\mathcal{G} \times \text{Spec}(\mathcal{O}_K/m_K^{\geq i})$ is a finite flat closed subgroup scheme of order p^d . By Theorem 2.2 (2), we see that the kernel of the Frobenius of the group scheme $\mathcal{H}(\mathfrak{M}) \times \text{Spec}(k[[u]]/(u^{ei}))$ is also finite flat of rank p^d . Now the condition (1) in the lemma is equivalent to saying that $\mathcal{D} \times \text{Spec}(\mathcal{O}_K/m_K^{\geq i})$ is killed by the Frobenius and of order p^d . By Theorem 2.2 (2), this holds if and only if $\mathcal{H}(\mathfrak{N}) \times \text{Spec}(k[[u]]/(u^{ei}))$ is killed by the Frobenius and of order p^d , namely if the latter subgroup scheme coincides with the kernel of the Frobenius of $\mathcal{H}(\mathfrak{N}) \times \text{Spec}(k[[u]]/(u^{ei}))$.

From the definition of the anti-equivalence $\mathcal{H}(-)$, we see that the Frobenius of $\mathcal{H}(\mathfrak{M}/u^{ei}\mathfrak{M})$ corresponds to the natural map

$$1 \otimes \varphi : k[[u]]/(u^{ei}) \otimes_{\varphi, k[[u]]} \mathfrak{M}/u^{ei}\mathfrak{M} \rightarrow \mathfrak{M}/u^{ei}\mathfrak{M}.$$

Since $\text{Coker}(1 \otimes \varphi) = \mathfrak{M}_1/\text{Fil}^1\mathfrak{M}_1 \otimes k[[u]]/(u^{ei})$ is free of finite rank over $k[[u]]/(u^{ei})$, the kernel of the Frobenius coincides with $\mathcal{H}(\text{Coker}(1 \otimes \varphi))$. Thus the subgroup scheme $\mathcal{H}(\mathfrak{N}/u^{ei}\mathfrak{N})$ coincides with the kernel of the Frobenius of $\mathcal{H}(\mathfrak{M}/u^{ei}\mathfrak{M})$ if and only if $\mathfrak{L}/u^{ei}\mathfrak{L} = \text{Im}(1 \otimes \varphi) = \text{Fil}^1\mathfrak{M}_1/u^{ei}\text{Fil}^1\mathfrak{M}_1$. \square

By this lemma, the unique existence of \mathcal{C} as in Theorem 3.1 (1) follows from the lemma below, by putting $\mathfrak{N} = \mathfrak{M}/\mathfrak{L}$ and $\mathcal{C} = \mathcal{G}(\mathfrak{N})$.

Lemma 3.3. *There exists a unique direct summand \mathfrak{L} of the $k[[u]]$ -module \mathfrak{M} which is free of rank $h-d$ such that \mathfrak{L} is stable under $\varphi = \varphi_{\mathfrak{M}}$ and \mathfrak{L} satisfies the condition (2) of Lemma 3.2 for $i = 1-w$. Moreover, the φ -module \mathfrak{L} defines a subobject of \mathfrak{M} in the category $\text{Mod}_{\mathfrak{S}_1}^{1,\varphi}$ with $v_u(\det(\varphi_{\mathfrak{L}})) = ew$.*

Proof. Let \mathfrak{L} be a direct summand of \mathfrak{M} satisfying the condition (2) of Lemma 3.2 for $i = 1-w$. Let $\delta_1, \dots, \delta_{h-d}$ be a basis of \mathfrak{L} . Then we have

$$(\delta_1, \dots, \delta_{h-d}) = (e_1, \dots, e_h) \begin{pmatrix} I_{h-d} + u^{e(1-w)}B' \\ u^{e(1-w)}B \end{pmatrix}$$

with some $B \in M_{d,h-d}(k[[u]])$ and $B' \in M_{h-d}(k[[u]])$. By multiplying the inverse of the invertible matrix $I_{h-d} + u^{e(1-w)}B'$, we may assume $B' = 0$. It is enough to show that there exists B uniquely such that the resulting \mathfrak{L} is stable under $\varphi = \varphi_{\mathfrak{M}}$, and that \mathfrak{L} defines an element of the category $\text{Mod}_{\mathfrak{S}_1}^{1,\varphi}$ satisfying $v_u(\det \varphi_{\mathfrak{L}}) = ew$.

Note that we have

$$\varphi(\delta_1, \dots, \delta_{h-d}) = (e_1, \dots, e_h) \begin{pmatrix} P_1 & P_2 \\ u^e P_3 & u^e P_4 \end{pmatrix} \begin{pmatrix} I_{h-d} \\ u^{ep(1-w)} \varphi(B) \end{pmatrix}.$$

Consider the equation

$$\varphi(\delta_1, \dots, \delta_{h-d}) = (\delta_1, \dots, \delta_{h-d})D$$

for $D \in M_{h-d}(k[[u]])$. This is equivalent to the following equations:

$$\begin{cases} P_1 + u^{ep(1-w)} P_2 \varphi(B) = D, \\ u^e P_3 + u^{e+ep(1-w)} P_4 \varphi(B) = u^{e(1-w)} BD. \end{cases}$$

From this we obtain the equation for B

$$BP_1 = u^{ew} P_3 - u^{ep(1-w)} BP_2 \varphi(B) + u^{e+ep(1-w)} P_4 \varphi(B).$$

By multiplying \hat{P}_1 , we have

$$B = P_3 \hat{P}_1 - u^{ep(1-w)-ew} BP_2 \varphi(B) \hat{P}_1 + u^{ep(1-w)} P_4 \varphi(B) \hat{P}_1.$$

The assumption $w < p/(p+1)$ implies the inequalities $ep(1-w) - ew > 0$ and $ep(1-w) > 0$. Therefore we can solve this equation uniquely by recursion to obtain B and D satisfying the above equations. Moreover, we also have $D = P_1(I_{h-d} + u^{ep(1-w)-ew} \hat{P}_1 P_2 \varphi(B))$ and the matrix $I_{h-d} + u^{ep(1-w)-ew} \hat{P}_1 P_2 \varphi(B)$ is invertible. Hence we see that the module \mathcal{L} defines an object of the category $\text{Mod}_{\mathfrak{S}_1}^{1, \varphi}$ and $v_u(\det(\varphi_{\mathcal{L}})) = ew$. This concludes the proof of the lemma. \square

By Lemma 2.5 (3), we obtain the equalities $\deg(\mathcal{G}/\mathcal{G}(\mathfrak{N})) = \deg(\mathcal{G}(\mathcal{L})) = e^{-1}v_u(\det(\varphi_{\mathcal{L}})) = w$ and the part (a) of Theorem 3.1 follows.

For the part (b), put $\bar{\mathcal{G}} = \mathcal{G} \times \text{Spec}(\mathcal{O}_K/m_K^{\geq 1-w})$. Since $\bar{\mathcal{G}}$ is a truncated Barsotti-Tate group of level one, we have the equalities of finite flat closed subgroup schemes

$$(\text{Im}(F_{\bar{\mathcal{G}}}))^\vee = \text{Im}(V_{\bar{\mathcal{G}}^\vee}) = \text{Ker}(F_{\bar{\mathcal{G}}^\vee}).$$

Thus the subgroup scheme $(\mathcal{G}/\mathcal{C})^\vee \times \text{Spec}(\mathcal{O}_K/m_K^{\geq 1-w})$ coincides with the kernel of the Frobenius of $\bar{\mathcal{G}}^\vee$. By the uniqueness of the canonical subgroup we have just proved, we obtain the equality $\mathcal{C}' = (\mathcal{G}/\mathcal{C})^\vee$.

Let us prove the part (c). By Theorem 2.3 and the part (b), it suffices to show the equality $\mathcal{C} = \mathcal{G}_{(1-w)/(p-1)}$. By Theorem 2.2 (1), we are reduced to show the equality $\mathcal{H}(\mathfrak{N}) = \mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}$. For this, we identify an element of the group $\mathcal{H}(\mathfrak{M})(R) = \text{Hom}_{\mathfrak{S}_1, \varphi}(\mathfrak{M}, R)$ defined by $e_i \mapsto x_i$ for $1 \leq i \leq h-d$ and $e_{h-d+i} \mapsto y_i$ for $1 \leq i \leq d$ with a solution $(\underline{x}, \underline{y}) = (x_1, \dots, x_{h-d}, y_1, \dots, y_d)$ in R^h of the equation

$$(\underline{x}^p, \underline{y}^p) = (\underline{x}, \underline{y}) \begin{pmatrix} P_1 & P_2 \\ u^e P_3 & u^e P_4 \end{pmatrix},$$

where we put $\underline{x}^p = (x_1^p, \dots, x_{h-d}^p)$ and similarly for \underline{y}^p . We use the notation in the proof of Lemma 3.3. Then we can also identify the group $\mathcal{H}(\mathcal{L})(R)$ with the set of solutions $\underline{z} = (z_1, \dots, z_{h-d})$ in R^{h-d} of the equation

$$\underline{z}^p = \underline{z}D$$

and the natural map $\mathcal{H}(\mathfrak{M}) \rightarrow \mathcal{H}(\mathcal{L})$ is identified with the map $(\underline{x}, \underline{y}) \mapsto \underline{z} = \underline{x} + u^{e(1-w)}\underline{y}B$.

Let $(\underline{x}, \underline{y})$ be an element of $\mathcal{H}(\mathfrak{N})(R) = \text{Ker}(\mathcal{H}(\mathfrak{M})(R) \rightarrow \mathcal{H}(\mathcal{L})(R))$. Then we have $\underline{x} + u^{e(1-w)}\underline{y}B = 0$ and thus, by eliminating \underline{x} from the equation for $(\underline{x}, \underline{y})$, we obtain

$$\underline{y}^p = u^{e(1-w)}\underline{y}(u^{ew}P_4 - BP_2).$$

Hence the inequality

$$p \min_{1 \leq i \leq d} v_R(y_i) \geq 1 - w + \min_{1 \leq i \leq d} v_R(y_i)$$

follows, which implies $v_R(y_i) \geq (1-w)/(p-1)$ and thus $v_R(x_i) \geq p(1-w)/(p-1)$ for any i . This shows the inclusion $\mathcal{H}(\mathfrak{N})(R) \subseteq \mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}(R)$.

Conversely, let $(\underline{x}, \underline{y})$ be an element of $\mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}(R)$. Then we can write $(\underline{x}, \underline{y}) = u^{e(1-w)/(p-1)}(\underline{a}, \underline{b})$ with some tuples $\underline{a}, \underline{b}$ in R and a $(p-1)$ -st root $u^{e(1-w)/(p-1)}$ of $u^{e(1-w)}$ in R . These tuples satisfy the equality

$$u^{e(1-w)}(\underline{a}^p, \underline{b}^p) = (\underline{a}, \underline{b}) \begin{pmatrix} P_1 & P_2 \\ u^e P_3 & u^e P_4 \end{pmatrix}.$$

Let $\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$ be the inverse of the matrix $\begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$. By multiplying this, we obtain

$$\underline{a} = u^{e(1-w)}(\underline{a}^p Q_1 + \underline{b}^p Q_3)$$

and thus $v_R(x_i) \geq p(1-w)/(p-1)$ for any i . Let \underline{z} be the image of the element $(\underline{x}, \underline{y})$ by the natural map $\mathcal{H}(\mathfrak{M})(R) \rightarrow \mathcal{H}(\mathcal{L})(R)$. Then we have $v_R(z_i) \geq p(1-w)/(p-1) > w/(p-1)$. By Lemma 2.5 (3), the equality $\text{deg}(\mathcal{H}(\mathcal{L})) = w$ holds and Lemma 2.4 implies $\underline{z} = 0$. Hence the inclusion $\mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}(R) \subseteq \mathcal{H}(\mathfrak{N})(R)$ follows and we conclude the proof of the part (c).

For the assertion (2) of Theorem 3.1, let us assume $w < (p-1)/p$. Let b be a rational number with $w/(p-1) < b \leq 1-w$. By the definition of the submodule \mathcal{L} , we have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}(\mathfrak{M})(R) & \longrightarrow & \mathcal{H}(\mathcal{L})(R) \\ \downarrow & & \downarrow \\ \mathcal{H}(\mathfrak{M}/u^{e(1-w)}\mathfrak{M})(R/m_R^{\geq b}) & \longrightarrow & \mathcal{H}(\mathcal{L}/u^{e(1-w)}\mathcal{L})(R/m_R^{\geq b}) \\ & \searrow & \parallel \\ & & \mathcal{H}(\text{Fil}^1\mathfrak{M}_1/u^{e(1-w)}\text{Fil}^1\mathfrak{M}_1)(R/m_R^{\geq b}) \end{array}$$

whose upper right vertical arrow is an injection by Lemma 2.4. Therefore the assertion (2) follows from Lemma 2.6.

Finally, let us prove the assertion (3). Assume $w < 1/2$. Take $i > w/(p-1)$ and consider the commutative diagram with exact upper row

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}(\mathcal{O}_{\bar{K}}) & \longrightarrow & \mathcal{G}(\mathcal{O}_{\bar{K}}) & \longrightarrow & (\mathcal{G}/\mathcal{C})(\mathcal{O}_{\bar{K}}) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{G}(\mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i}) & \longrightarrow & (\mathcal{G}/\mathcal{C})(\mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq i}). \end{array}$$

By Lemma 2.4, we have $(\mathcal{G}/\mathcal{C})_i = 0$ and the right vertical arrow in the diagram is an injection. This implies the inclusion $\mathcal{G}_{w/(p-1)+} \subseteq \mathcal{C}$. Thus we obtain the inclusions

$$\mathcal{C} = \mathcal{G}_{(1-w)/(p-1)} \subseteq \mathcal{G}_{w/(p-1)+} \subseteq \mathcal{C}$$

and the equality $\mathcal{C} = \mathcal{G}_b$ holds for any b satisfying $w/(p-1) < b \leq (1-w)/(p-1)$. The assertion for upper ramification subgroups follows from the part (b) and Theorem 2.3. \square

4. HIGHER CANONICAL SUBGROUPS

In this section, we prove Theorem 1.1 and Corollary 1.2. Though this can be done basically by repeating arguments in [1], [8] and [22], we give a proof here with necessary modifications for the convenience of the reader. First we recall the following two lemmas in [8] whose proofs depend only on elementary arguments and are independent of the theory of Harder-Narasimhan filtrations or Hodge-Tate maps developed there. Note that for Lemma 4.1, the proof of [8, Théorème 5] remains valid also for our subgroup scheme \mathcal{C} by Theorem 3.1 (1).

Lemma 4.1 ([8], Théorème 5). *Let \mathcal{G} be a truncated Barsotti-Tate group of level two, height h and dimension $d < h$ over \mathcal{O}_K with $\text{Hdg}(\mathcal{G}) < 1/(p+1)$. Let \mathcal{C} be the level one canonical subgroup of $\mathcal{G}[p]$ constructed in Theorem 3.1. Then the group scheme $p^{-1}\mathcal{C}/\mathcal{C}$ is a truncated Barsotti-Tate group of level one, height h and dimension d with $\text{Hdg}(p^{-1}\mathcal{C}/\mathcal{C}) = p\text{Hdg}(\mathcal{G})$.*

Lemma 4.2 ([8], Proposition 12). *Let \mathcal{G} be a truncated Barsotti-Tate group of level n and dimension d over \mathcal{O}_K . Let \mathcal{C} be a finite flat closed subgroup scheme of \mathcal{G} over \mathcal{O}_K of order p^{nd} . Suppose that we have the inequality $\deg(\mathcal{C}) > nd - 1$. Then the group $\mathcal{C}(\mathcal{O}_{\bar{K}})$ is isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^d$.*

Proof of Theorem 1.1. We proceed by induction on n . The case of $n = 1$ is Theorem 3.1. For $n \geq 2$, suppose that the theorem holds for any level less than n . By assumption, we have the level one canonical subgroup \mathcal{C} of $\mathcal{G}[p]$ as in Theorem 3.1. Then Lemma 4.1 implies that, for $n \geq 2$, the group scheme $p^{-(n-1)}\mathcal{C}/\mathcal{C}$ is a truncated Barsotti-Tate group of level $n-1$ over \mathcal{O}_K with $\text{Hdg}(p^{-(n-1)}\mathcal{C}/\mathcal{C}) = pw$. Using the induction hypothesis for $p^{-(n-1)}\mathcal{C}/\mathcal{C}$, we define the level n canonical subgroup \mathcal{C}_n of \mathcal{G} to be the

unique finite flat closed subgroup scheme of \mathcal{G} over \mathcal{O}_K containing \mathcal{C} such that the quotient $\mathcal{C}_n/\mathcal{C}$ is the level $n-1$ canonical subgroup of $p^{-(n-1)}\mathcal{C}/\mathcal{C}$. Then \mathcal{C}_n is of order p^{nd} .

By the assertion on the Frobenius kernel for $p^{-(n-1)}\mathcal{C}/\mathcal{C}$, we have the equality $\mathcal{C}_n = (F^{n-1})^{-1}(\mathcal{C}^{(p^{n-1})})$ of fppf sheaves over $\text{Spec}(\mathcal{O}_K/m_K^{\geq 1-p^{n-1}w})$. Since the subgroup scheme $\mathcal{C} \times \text{Spec}(\mathcal{O}_K/m_K^{\geq 1-p^{n-1}w})$ also coincides with the Frobenius kernel of $\mathcal{G} \times \text{Spec}(\mathcal{O}_K/m_K^{\geq 1-p^{n-1}w})$, we obtain the assertion on the Frobenius kernel for \mathcal{G} .

Since the multiplication by p^{n-1} induces an isomorphism $\mathcal{G}/p^{-(n-1)}\mathcal{C} \rightarrow \mathcal{G}[p]/\mathcal{C}$, we have the equality

$$\deg(\mathcal{G}/\mathcal{C}_n) = \deg(\mathcal{G}[p]/\mathcal{C}) + \deg((p^{-(n-1)}\mathcal{C}/\mathcal{C})/(\mathcal{C}_n/\mathcal{C}))$$

and the part (a) of the theorem follows from the induction hypothesis.

For the part (b), it is enough to show the Cartier pairing $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ kills the subset $\mathcal{C}_n(\mathcal{O}_{\bar{K}}) \times \mathcal{C}'_n(\mathcal{O}_{\bar{K}})$. Let \mathcal{C}' be the canonical subgroup of $\mathcal{G}^{\vee}[p]$. By the construction of \mathcal{C}'_n , we have $\mathcal{C}' \subseteq \mathcal{C}'_n$ and $\mathcal{C}'_n/\mathcal{C}' \subseteq p^{-(n-1)}\mathcal{C}'/\mathcal{C}'$. For $x \in \mathcal{C}(\mathcal{O}_{\bar{K}})$ and $y \in \mathcal{C}'_n(\mathcal{O}_{\bar{K}})$, we have $p^{n-1}y \in \mathcal{C}'(\mathcal{O}_{\bar{K}})$ and $\langle x, y \rangle_{\mathcal{G}} = \langle x, p^{n-1}y \rangle_{\mathcal{G}[p]} = 0$ by Theorem 3.1 (b). Thus the subgroup $\mathcal{C}'_n(\mathcal{O}_{\bar{K}})$ is contained in $\mathcal{C}(\mathcal{O}_{\bar{K}})^{\perp} = (\mathcal{G}/\mathcal{C})^{\vee}(\mathcal{O}_{\bar{K}}) \subseteq \mathcal{G}^{\vee}(\mathcal{O}_{\bar{K}})$ and we have $\langle x, y \rangle_{\mathcal{G}} = \langle \bar{x}, y \rangle_{\mathcal{G}/\mathcal{C}}$ for any $x \in \mathcal{C}_n(\mathcal{O}_{\bar{K}})$ and $y \in \mathcal{C}'_n(\mathcal{O}_{\bar{K}})$, where \bar{x} is the image of x in $(\mathcal{G}/\mathcal{C})(\mathcal{O}_{\bar{K}})$. Since we have an exact sequence

$$0 \rightarrow p^{-(n-1)}\mathcal{C}/\mathcal{C} \rightarrow \mathcal{G}/\mathcal{C} \xrightarrow{p^{n-1}} \mathcal{G}[p]/\mathcal{C} \rightarrow 0$$

and an isomorphism $\mathcal{C}' \simeq (\mathcal{G}[p]/\mathcal{C})^{\vee}$ of subgroup schemes of $\mathcal{G}^{\vee}[p]$, a similar argument on the side of \mathcal{C}'_n implies the equality $\langle \bar{x}, y \rangle_{\mathcal{G}/\mathcal{C}} = \langle \bar{x}, \bar{y} \rangle_{p^{-(n-1)}\mathcal{C}/\mathcal{C}}$, where \bar{y} is the image of y by the surjection $(\mathcal{G}/\mathcal{C})^{\vee}(\mathcal{O}_{\bar{K}}) \rightarrow (p^{-(n-1)}\mathcal{C}/\mathcal{C})^{\vee}(\mathcal{O}_{\bar{K}})$. Moreover, we have an isomorphism $(\mathcal{G}/\mathcal{C})^{\vee} \simeq p^{-(n-1)}\mathcal{C}'$ of subgroup schemes of \mathcal{G}^{\vee} fitting into the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\mathcal{G}[p]/\mathcal{C})^{\vee} & \longrightarrow & (\mathcal{G}/\mathcal{C})^{\vee} & \longrightarrow & (p^{-(n-1)}\mathcal{C}/\mathcal{C})^{\vee} \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ 0 & \longrightarrow & \mathcal{C}' & \longrightarrow & p^{-(n-1)}\mathcal{C}' & \longrightarrow & p^{-(n-1)}\mathcal{C}'/\mathcal{C}' \longrightarrow 0. \end{array}$$

Thus we obtain an isomorphism $(p^{-(n-1)}\mathcal{C}/\mathcal{C})^{\vee} \rightarrow p^{-(n-1)}\mathcal{C}'/\mathcal{C}'$ and the equality $\langle \bar{x}, \bar{y} \rangle_{p^{-(n-1)}\mathcal{C}/\mathcal{C}} = 0$ holds by induction hypothesis.

Moreover, for $w < (p-1)/(p^n-1)$, the inequality $\deg(\mathcal{C}_n) = nd - w(p^n - 1)/(p-1) > nd - 1$ holds by the part (a) and Lemma 4.2 implies the part (d).

Next we show the part (e). By induction hypothesis, we have the subgroup scheme \mathcal{C}_i of $\mathcal{G}[p^i]$ as in the theorem. Note that, by the part (d), a subgroup of $\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ which is isomorphic to $(\mathbb{Z}/p^{n-1}\mathbb{Z})^d$ is equal to $\mathcal{C}_n(\mathcal{O}_{\bar{K}})[p^{n-1}]$. Hence, by induction hypothesis, it is enough to check that \mathcal{C}_{n-1} is contained in \mathcal{C}_n . The case of $n=2$ follows from the definition of \mathcal{C}_n . Suppose $n > 2$.

For any i with $2 \leq i \leq n$, the group scheme $\mathcal{C}_i/\mathcal{C}$ is the level $i-1$ canonical subgroup of the truncated Barsotti-Tate group $p^{-(i-1)}\mathcal{C}/\mathcal{C}$ of Hodge height pw . The inequality $p(p-1)/(p^n-1) < (p-1)/(p^{n-1}-1)$ and the induction hypothesis imply that $\mathcal{C}_{n-1}/\mathcal{C}$ is contained in $\mathcal{C}_n/\mathcal{C}$. Thus the part (e) follows.

Let us show the assertion (3). Assume $w < (p-1)/p^n$ and set \mathcal{K}_n to be the scheme-theoretic closure in \mathcal{G} of the subgroup $\text{Ker}(\text{HT}_{n-w(p^n-1)/(p-1)})$. We have the inequality $w/(p-1) < 1-\epsilon \leq 1-w$ for $\epsilon = w(p^n-1)/(p-1)$ and Theorem 3.1 (2) implies that the kernel of the map $\text{HT}_{1-\epsilon} : \mathcal{G}[p](\mathcal{O}_{\bar{K}}) \rightarrow \omega_{\mathcal{G}[p]^\vee} \otimes \mathcal{O}_{\bar{K}}/m_{\bar{K}}^{\geq 1-\epsilon}$ is of order p^d . Using this, we can show that the order of the group scheme \mathcal{K}_n is no more than p^{nd} , by an elementary argument just as in [8, Proposition 13]. On the other hand, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{C}(\mathcal{O}_{\bar{K}}) & \longrightarrow & \mathcal{G}(\mathcal{O}_{\bar{K}}) & \longrightarrow & (\mathcal{G}/\mathcal{C})(\mathcal{O}_{\bar{K}}) \longrightarrow 0 \\ & & \text{HT} \downarrow & & \text{HT} \downarrow & & \text{HT} \downarrow \\ 0 & \longrightarrow & \omega_{\mathcal{C}^\vee} \otimes \mathcal{O}_{\bar{K}} & \longrightarrow & \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K}} & \longrightarrow & \omega_{(\mathcal{G}/\mathcal{C})^\vee} \otimes \mathcal{O}_{\bar{K}} \longrightarrow 0, \end{array}$$

where we put $\text{HT}(x) = (x^\vee)^*(dt/t)$ as before. Take $x \in \mathcal{C}_n(\mathcal{O}_{\bar{K}})$ and let \bar{x} denote its image in $(\mathcal{G}/\mathcal{C})(\mathcal{O}_{\bar{K}})$. By the construction of \mathcal{C}_n and induction hypothesis, we see that the element $\text{HT}(\bar{x})$ is killed by the ideal $m_{\bar{K}}^{\geq w(p^n-p)/(p-1)}$. Since $\deg(\mathcal{C}^\vee) = w$, we have $m_{\bar{K}}^{\geq w}(\omega_{\mathcal{C}^\vee} \otimes \mathcal{O}_{\bar{K}}) = 0$. Hence the element $\text{HT}(x)$ is killed by the ideal $m_{\bar{K}}^{\geq w(p^n-1)/(p-1)}$ and we obtain the inclusion $\mathcal{C}_n \subseteq \mathcal{K}_n$, which implies the assertion (3) by comparing orders.

Finally, the assertion (4) follows just as in the proof of [22, Theorem 2.5], using induction hypothesis and assertions of the theorem we have already proved. This concludes the proof of Theorem 1.1. \square

We can also prove the following result on anti-canonical isogenies, slightly generalizing [8, Proposition 16].

Proposition 4.3. *Let \mathcal{G} be a truncated Barsotti-Tate group over \mathcal{O}_K of level two, height h , dimension d with $0 < d < h$ and Hodge height $w = \text{Hdg}(\mathcal{G})$. Suppose $w < 1/2$ and let \mathcal{C} be the canonical subgroup of $\mathcal{G}[p]$ as in Theorem 3.1. Let \mathcal{D} be a finite flat closed subgroup scheme of $\mathcal{G}[p]$ over \mathcal{O}_K such that the natural map $\mathcal{C}(\mathcal{O}_{\bar{K}}) \oplus \mathcal{D}(\mathcal{O}_{\bar{K}}) \rightarrow \mathcal{G}[p](\mathcal{O}_{\bar{K}})$ is an isomorphism.*

- (1) *The truncated Barsotti-Tate group $p^{-1}\mathcal{D}/\mathcal{D}$ of level one has Hodge height $\text{Hdg}(p^{-1}\mathcal{D}/\mathcal{D}) = p^{-1}w$.*
- (2) *The subgroup scheme $\mathcal{G}[p]/\mathcal{D}$ is the canonical subgroup of $p^{-1}\mathcal{D}/\mathcal{D}$.*
- (3) *$\deg(\mathcal{D}) = p^{-1}w$.*

Proof. By a base change argument as before, we may assume that the residue field k of K is perfect. Then, by [13, Théorème 4.4 (e)], there exists a Barsotti-Tate group Γ over \mathcal{O}_K satisfying $\mathcal{G} \simeq \Gamma[p^2]$.

Note that the truncated Barsotti-Tate group $p^{-1}\mathcal{D}/\mathcal{D}$ is also of height h and dimension d . The natural homomorphism $\mathcal{C} \rightarrow \mathcal{G}[p]/\mathcal{D}$ induces an

isomorphism between the generic fibers of both sides. Now we claim that the group scheme $(\mathcal{G}[p]/\mathcal{D}) \times \text{Spec}(\mathcal{O}_K/m_K^{\geq 1-w})$ is killed by the Frobenius. Indeed, let \mathfrak{N} and \mathfrak{N}' be the objects of the category $\text{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$ corresponding to the finite flat group schemes \mathcal{C} and $\mathcal{G}[p]/\mathcal{D}$ via the anti-equivalence $\mathcal{G}(-)$, respectively. By [18, Corollary 2.2.2], the generic isomorphism $(\mathcal{G}[p]/\mathcal{D})^\vee \rightarrow \mathcal{C}^\vee$ corresponds to an injection $\mathfrak{N}^\vee \rightarrow (\mathfrak{N}')^\vee$. Then the \mathfrak{S}_1 -modules $\wedge^d \mathfrak{N}^\vee$ and $\wedge^d (\mathfrak{N}')^\vee$ are free of rank one and we also have an injection $\wedge^d \mathfrak{N}^\vee \rightarrow \wedge^d (\mathfrak{N}')^\vee$. Hence we obtain the inequality $v_u(\det \varphi_{(\mathfrak{N}')^\vee}) \leq v_u(\det \varphi_{\mathfrak{N}^\vee})$. By Theorem 3.1 (b), the object \mathfrak{N}^\vee corresponds to the quotient $\mathcal{G}[p]^\vee/\mathcal{C}'$ of the Cartier dual $\mathcal{G}[p]^\vee$ by its canonical subgroup \mathcal{C}' . Thus Lemma 2.5 (3) and Theorem 3.1 (a) imply the equality $v_u(\det \varphi_{\mathfrak{N}^\vee}) = ew$. From this and the definition of the dual object of the category $\text{Mod}_{/\mathfrak{S}_1}^{1,\varphi}$, we see that the map $\varphi_{\mathfrak{N}'}$ is zero modulo $u^{e(1-w)}$ and the claim follows from Theorem 2.2 (2).

Since the group scheme $p^{-1}\mathcal{D}/\mathcal{D}$ is a truncated Barsotti-Tate group of level one and dimension d , the group scheme in the claim coincides with the kernel of the Frobenius of the group scheme $(p^{-1}\mathcal{D}/\mathcal{D}) \times \text{Spec}(\mathcal{O}_K/m_K^{\geq 1-w})$. Note the equality $p^{-1}(\mathcal{G}[p]/\mathcal{D}) = \mathcal{G}/\mathcal{D}$ of closed subgroup schemes of the truncated Barsotti-Tate group $p^{-2}\mathcal{D}/\mathcal{D}$. Therefore the Frobenius induces an isomorphism of group schemes over $\text{Spec}(\mathcal{O}_K/m_K^{\geq 1-w})$

$$\mathcal{G}[p] \simeq (\mathcal{G}/\mathcal{D})/(\mathcal{G}[p]/\mathcal{D}) = p^{-1}(\mathcal{G}[p]/\mathcal{D})/(\mathcal{G}[p]/\mathcal{D}) \xrightarrow{F} (p^{-1}\mathcal{D}/\mathcal{D})^{(p)}.$$

Considering the Hodge heights of both sides, we have the equality

$$\min\{w, 1-w\} = \min\{p\text{Hdg}(p^{-1}\mathcal{D}/\mathcal{D}), 1-w\},$$

from which the assertion (1) follows. The uniqueness of the canonical subgroup in Theorem 3.1 (1) implies the assertion (2). The last assertion follows from the isomorphism $(p^{-1}\mathcal{D}/\mathcal{D})/(\mathcal{G}[p]/\mathcal{D}) \simeq \mathcal{D}$ and Theorem 3.1 (a). \square

We give here remarks on the uniqueness of the canonical subgroup \mathcal{C}_n . Let \mathcal{G} be a truncated Barsotti-Tate group over \mathcal{O}_K of level n , height h , dimension d and Hodge height $w < 1/(p^{n-2}(p+1))$. Let \mathcal{D}_n be a finite flat closed subgroup scheme of \mathcal{G} over \mathcal{O}_K . Then an induction and the uniqueness assertion in Theorem 3.1 (1) show the following uniqueness of \mathcal{C}_n : Suppose that there exists a filtration $0 = \mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots \subseteq \mathcal{D}_{n-1} \subseteq \mathcal{D}_n$ of finite flat closed subgroup schemes over \mathcal{O}_K such that $\mathcal{D}_i/\mathcal{D}_{i-1}$ is killed by p , of order p^d and its modulo $m_K^{\geq 1-p^{i-1}w}$ is killed by the Frobenius for any i . Then we have $\mathcal{C}_n = \mathcal{D}_n$.

On the other hand, the following stronger uniqueness also holds for $w < p(p-1)/(p^{n+1}-1)$. Note the inequalities

$$(p-1)/p^n < p(p-1)/(p^{n+1}-1) < (p-1)/(p^n-1).$$

Proposition 4.4. *Let \mathcal{G} be a truncated Barsotti-Tate group over \mathcal{O}_K of level n , height h , dimension d and Hodge height $w < p(p-1)/(p^{n+1}-1)$. Let \mathcal{D}_n be a finite flat closed subgroup scheme of \mathcal{G} over \mathcal{O}_K such that $\mathcal{D}_n(\mathcal{O}_K) \simeq$*

$(\mathbb{Z}/p^n\mathbb{Z})^d$ and the group scheme $\mathcal{D}_n \times \text{Spec}(\mathcal{O}_K/m_K^{\geq 1-p^{n-1}w})$ is killed by the n -th iterated Frobenius F^n . Then we have $\mathcal{C}_n = \mathcal{D}_n$.

Proof. We proceed by induction on n . The case of $n = 1$ follows from Theorem 3.1 (1). Suppose that $n \geq 2$ and the assertion holds for $n - 1$. Let \mathcal{C}_1 be the scheme-theoretic closure of $\mathcal{C}_n(\mathcal{O}_{\bar{K}})[p]$ in \mathcal{C}_n and define \mathcal{D}_1 similarly. By Theorem 1.1 (e), the subgroup scheme \mathcal{C}_1 is the canonical subgroup of $\mathcal{G}[p]$. First we claim $\mathcal{C}_1 = \mathcal{D}_1$. For this, by a base change argument as before, we may assume that the residue field k of K is perfect and $\mathcal{G}(\mathcal{O}_{\bar{K}}) = \mathcal{G}(\mathcal{O}_K)$. Suppose $\mathcal{C}_1 \neq \mathcal{D}_1$. Then we can find a finite flat closed subgroup scheme \mathcal{E} of $\mathcal{G}[p]$ such that $\mathcal{C}_1(\mathcal{O}_{\bar{K}}) \oplus \mathcal{E}(\mathcal{O}_{\bar{K}}) = \mathcal{G}[p](\mathcal{O}_{\bar{K}})$ and $\mathcal{D}_1(\mathcal{O}_{\bar{K}}) \cap \mathcal{E}(\mathcal{O}_{\bar{K}}) \neq 0$. Let \mathcal{F} be the scheme-theoretic closure of the latter intersection in $\mathcal{G}[p]$, which is a closed subgroup scheme of \mathcal{E} . Let \mathfrak{L} , \mathfrak{E} and \mathfrak{F} be the objects of the category $\text{Mod}_{\mathfrak{S}_1}^{1,\varphi}$ corresponding to $\mathcal{G}[p]/\mathcal{C}_1$, \mathcal{E} and \mathcal{F} , respectively. Since we have the inequality $w < p(p-1)/(p^3-1) < 1/2$, Proposition 4.3 (3) implies $v_u(\det \varphi_{\mathfrak{F}}) \leq v_u(\det \varphi_{\mathfrak{E}}) = p^{-1}ew$. On the other hand, since \mathcal{F} is also a closed subgroup scheme of \mathcal{D}_n , the n -th iterated Frobenius of $\mathcal{F} \times \text{Spec}(\mathcal{O}_K/m_K^{\geq 1-p^{n-1}w})$ is zero. By Theorem 2.2 (2), we see that the entries of a representing matrix of the map

$$\mathfrak{S}_1 \otimes_{\varphi^n, \mathfrak{S}_1} \mathfrak{F} \xrightarrow{1 \otimes \varphi_{\mathfrak{F}}} \mathfrak{S}_1 \otimes_{\varphi^{n-1}, \mathfrak{S}_1} \mathfrak{F} \xrightarrow{1 \otimes \varphi_{\mathfrak{F}}} \cdots \xrightarrow{1 \otimes \varphi_{\mathfrak{F}}} \mathfrak{S}_1 \otimes_{\varphi, \mathfrak{S}_1} \mathfrak{F} \xrightarrow{1 \otimes \varphi_{\mathfrak{F}}} \mathfrak{F}$$

have u -adic valuations no less than $e(1 - p^{n-1}w)$. Taking the valuation of the determinant of this map, we obtain the inequalities

$$1 - p^{n-1}w \leq e^{-1}v_u(\det \varphi_{\mathfrak{F}})(p^n - 1)/(p - 1) \leq w(p^n - 1)/(p^2 - p),$$

which contradict the assumption on w and the equality $\mathcal{C}_1 = \mathcal{D}_1$ follows.

Now the group scheme $p^{-(n-1)}\mathcal{C}_1/\mathcal{C}_1$ is a truncated Barsotti-Tate group of level $n - 1$, height h , dimension d and Hodge height pw with the level $n - 1$ canonical subgroup $\mathcal{C}_n/\mathcal{C}_1$. The finite flat closed subgroup scheme $\mathcal{D}_n/\mathcal{C}_1$ satisfies $(\mathcal{D}_n/\mathcal{C}_1)(\mathcal{O}_{\bar{K}}) \simeq (\mathbb{Z}/p^{n-1}\mathbb{Z})^d$ and its modulo $m_K^{\geq 1-p^{n-2}pw}$ is killed by F^{n-1} . Thus the induction hypothesis implies the equality $\mathcal{C}_n/\mathcal{C}_1 = \mathcal{D}_n/\mathcal{C}_1$. This concludes the proof of the proposition. \square

To show Corollary 1.2, we need a patching lemma for upper ramification subgroups, which is a slight generalization of [1, Proposition 8.2.2]. Let K/\mathbb{Q}_p be an extension of complete discrete valuation fields and j be a positive rational number. Let \mathfrak{X} be an admissible formal scheme over $\text{Spf}(\mathcal{O}_K)$ which is quasi-compact and X be its Raynaud generic fiber. For a finite extension L/K , an \mathcal{O}_L -valued point $x : \text{Spf}(\mathcal{O}_L) \rightarrow \mathfrak{X}$ and an ideal \mathcal{I} of $\mathcal{O}_{\mathfrak{X}}$, we let $\mathcal{I}(x)$ denote the ideal of \mathcal{O}_L generated by the image of \mathcal{I} . The p -adic valuation of a generator of this ideal is denoted by $v_p(\mathcal{I}(x))$. For any $x \in X(L)$, we let x also denote the map $\text{Spf}(\mathcal{O}_L) \rightarrow \mathfrak{X}$ obtained from x by taking the scheme-theoretic closure and the normalization.

Lemma 4.5. *Let the notation be as above. Let \mathfrak{G} be a finite locally free formal group scheme over \mathfrak{X} , G be its Raynaud generic fiber and \mathcal{H} be a*

coherent open ideal of $\mathcal{O}_{\mathfrak{X}}$. Then there exists an admissible open subgroup $G^{j\mathcal{H}^+}$ of G such that for any finite extension L/K and $x \in X(L)$, the fiber $G_x^{j\mathcal{H}^+}$ coincides with the generic fiber of the upper ramification subgroup scheme $\mathfrak{G}_x^{jv_p(\mathcal{H}(x))^+}$ of the finite flat (formal) group scheme $\mathfrak{G}_x = \mathfrak{G} \times_{\mathfrak{X}, x} \mathrm{Spf}(\mathcal{O}_L)$ over \mathcal{O}_L .

Proof. By replacing \mathfrak{X} by the admissible blow-up of \mathfrak{X} along the ideal \mathcal{H} , we may assume that the ideal \mathcal{H} is invertible. It suffices to show the existence of an admissible open subgroup as in the lemma locally. Thus we may assume that $\mathfrak{X} = \mathrm{Spf}(B)$ is affine, the ideal \mathcal{H} is generated by $h \in B$ and there exists a closed immersion of \mathfrak{G} to a projective abelian scheme \mathcal{A} over $\mathrm{Spec}(B)$, by a theorem of Raynaud ([3, Théorème 3.1.1]). Let \mathfrak{P} be the formal completion of \mathcal{A} along the special fiber, P be its Raynaud generic fiber and \mathcal{I} be the defining ideal of the closed immersion $\mathfrak{G} \rightarrow \mathfrak{P}$. For positive rational numbers j and j' , write $j = m/n$ and $j' = m'/n$ with positive integers m, m', n and put $\mathcal{J} = \mathcal{I}^n + p^{m'} h^m \mathcal{O}_{\mathfrak{P}}$. Let \mathfrak{B} be the admissible blow-up of \mathfrak{P} along the ideal \mathcal{J} and $\mathfrak{Z}^{m, m', n, h}$ be the formal open subscheme of \mathfrak{B} where $p^{m'} h^m$ generates the ideal $\mathcal{J} \mathcal{O}_{\mathfrak{B}}$.

The Raynaud generic fiber of $\mathfrak{Z}^{m, m', n, h}$ is the admissible open subset of P whose set of \bar{K} -valued points is given by

$$\{y \in P(\bar{K}) \mid v_p(\mathcal{J}(y)) \geq jv_p(h(y)) + j'\}$$

and it is independent of the choice of m, m', n . We write this Raynaud generic fiber as $Z^{jh+j'}$. We claim that the admissible open subset $Z^{jh+j'}$ is a rigid-analytic subgroup of P . For this, take a finite extension L/K and $x \in X(L)$. It is enough to show that the fiber $(Z^{jh+j'})_x$ is a rigid-analytic subgroup of P_x . This is the admissible open subset of P_x consisting of $y \in P_x(\bar{L})$ such that the inequality $v_L(\mathcal{J}(y)) \geq e(L/\mathbb{Q}_p)(jv_p(h(x)) + j')$ holds, where v_L is the normalized valuation of L . Taking a sufficiently large L , we may assume that the constant $e(L/\mathbb{Q}_p)(jv_p(h(x)) + j')$ is an integer. Then the universality of the dilatation implies the claim, as in the proof of [1, Proposition 8.2.2].

By [1, Proposition A.1.2], there exists an admissible open subgroup $Z^{jh+j', 0}$ of $Z^{jh+j'}$ such that for any $x \in X(\bar{K})$, the fiber $(Z^{jh+j', 0})_x$ coincides with the unit component $(Z_x^{jh+j'})^0$ of the rigid-analytic group $Z_x^{jh+j'}$. Put

$$Z^{jh+} = \bigcup_{j' > 0} Z^{jh+j'}, \quad Z^{jh+, 0} = \bigcup_{j' > 0} Z^{jh+j', 0}$$

and $G^{j\mathcal{H}^+} = G \cap Z^{jh+, 0}$. These are admissible open subgroups of P and G , respectively. To show that $G^{j\mathcal{H}^+}$ satisfies the property as in the lemma, it is enough to show that, for any finite extension L/K and $x \in X(L)$, the fiber $(Z^{jh+j'})_x$ is naturally isomorphic to the tubular neighborhood $X^{jv_p(h(x))+j'}(\mathfrak{G}_x \rightarrow \mathfrak{Z})$ of a closed immersion from \mathfrak{G}_x to a formal affine scheme \mathfrak{Z} which is an object of the category $\mathrm{SFA}_{\mathcal{O}_L}$ ([1, Notation 1.5]). Take $x \in X(L)$. Since \mathcal{A} is projective and \mathfrak{G}_x is finite over \mathcal{O}_L , there exists

an affine open subscheme \mathcal{U}_x of \mathcal{A}_x such that the closed immersion $\mathfrak{G}_x \rightarrow \mathcal{A}_x$ factors through \mathcal{U}_x . Let $\hat{\mathcal{U}}_x$ be its formal completion along the special fiber and U_x be its Raynaud generic fiber. Then $\hat{\mathcal{U}}_x$ is an object of the category $\text{SFA}_{\mathcal{O}_L}$. Moreover, $\hat{\mathcal{U}}_x$ is a formal affine open subscheme of \mathfrak{P}_x through which the closed immersion $\mathfrak{G}_x \rightarrow \mathfrak{P}_x$ factors and the restriction $(Z^{jh+j'})_x|_{U_x}$ coincides with the tubular neighborhood $X^{jv_p(h(x))+j'}(\mathfrak{G}_x \rightarrow \hat{\mathcal{U}}_x)$. Therefore we are reduced to show the equality $(Z^{jh+j'})_x|_{U_x} = (Z^{jh+j'})_x$. For this, take $y \in (Z^{jh+j'})_x(\bar{L})$. Then we have $v_p(\mathcal{I}(y)) \geq jv_p(h(x)) + j' > 0$. Put $\mathfrak{P}_x \setminus \hat{\mathcal{U}}_x = V(\mathcal{I}'_x)$ with some ideal $\mathcal{I}'_x \subseteq \mathcal{O}_{\mathfrak{P}_x}$. Let $\bar{\mathfrak{P}}_x$ and \bar{y} denote the special fibers of \mathfrak{P}_x and y , respectively. Consider the inverse image $\bar{\mathcal{I}}_x$ (resp. $\bar{\mathcal{I}}'_x$) of \mathcal{I}_x (resp. \mathcal{I}'_x) in $\bar{\mathfrak{P}}_x$. Since $V(\bar{\mathcal{I}}_x) \cap V(\bar{\mathcal{I}}'_x) = \emptyset$, we have $\bar{\mathcal{I}}_x + \bar{\mathcal{I}}'_x = (1)$ in any open affine neighborhood of the image of \bar{y} in $\bar{\mathfrak{P}}_x$. Thus the image is not contained in $V(\bar{\mathcal{I}}'_x)$ and we obtain $y \in (Z^{jh+j'})_x|_{U_x}$. \square

Proof of Corollary 1.2. Define a positive rational number j and an open coherent ideal \mathcal{H} of $\mathcal{O}_{\mathfrak{X}}$ by $j = p(p^n - 1)/(p - 1)^2$ and $\mathcal{H} = \text{Ha}_{\mathfrak{J}}(\mathfrak{G}[p])$ ([8, 2.2.2]). We define C_n to be the admissible open subgroup $G^{j\mathcal{H}^+}$ of G as in Lemma 4.5. Then, by Theorem 1.1 (c) and (4), each fiber $(C_n)_x$ coincides with the generic fiber of the level n canonical subgroup of \mathfrak{G}_x . Since G is etale over X , the admissible open subgroup C_n of G is also etale over X . Since its fibers over $X(r_n)$ are isomorphic to the group $(\mathbb{Z}/p^n\mathbb{Z})^d$ by Theorem 1.1 (d), [1, Lemme A.1.1] implies that C_n is finite over $X(r_n)$. Hence the rigid-analytic group C_n over $X(r_n)$ is etale locally constant, and thus etale locally on $X(r_n)$ this is isomorphic to the constant group $(\mathbb{Z}/p^n\mathbb{Z})^d$. \square

REFERENCES

- [1] A. Abbes and A. Mokrane: *Sous-groupes canoniques et cycles évanescents p-adiques pour les variétés abéliennes*, Publ. Math. Inst. Hautes Etudes Sci. **99** (2004), 117–162.
- [2] F. Andreatta and C. Gasbarri: *The canonical subgroup for families of abelian varieties*, Compos. Math. **143** (2007), no. 3, 566–602.
- [3] P. Berthelot, L. Breen and W. Messing: *Théorie de Dieudonné cristalline II*, Lecture Notes in Mathematics, **930**. Springer-Verlag, Berlin, 1982. x+261 pp.
- [4] C. Breuil: *Groupes p-divisibles, groupes finis et modules filtrés*, Ann. of Math. (2) **152** (2000), no. 2, 489–549.
- [5] C. Breuil: *Integral p-adic Hodge theory*, Algebraic geometry 2000, Azumino (Hotaka), 51–80, Adv. Stud. Pure Math., **36**, Math. Soc. Japan, Tokyo, 2002.
- [6] X. Caruso and T. Liu: *Quasi-semi-stable representations*, Bull. Soc. Math. France **137** (2009), no. 2, 185–223.
- [7] B. Conrad: *Higher-level canonical subgroups in abelian varieties*, preprint.
- [8] L. Fargues: *La filtration canonique des points de torsion des groupes p-divisibles (avec la collaboration de Yichao Tian)*, Ann. Sci. Ecole Norm. Sup. (4) **44** (2011), no. 6, 905–961.
- [9] P. Gabriel: *Étude infinitésimale des schémas en groupes et groupes formels*, Schémas en Groupes (Sém. Géométrie Algébrique, Inst. Hautes Etudes Sci., 1963/64), Fasc. 2b, Exposé 7a pp. 1-65+4 Inst. Hautes Etudes Sci., Paris.
- [10] E. Z. Goren and P. L. Kassaei: *The canonical subgroup: a “subgroup-free” approach*, Comment. Math. Helv. **81** (2006), no. 3, 617–641.

- [11] E. Z. Goren and P. L. Kassaei: *Canonical subgroups over Hilbert modular varieties*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 17–18, 985–990.
- [12] S. Hattori: *Ramification correspondence of finite flat group schemes over equal and mixed characteristic local fields*, J. of Number Theory **132** (2012), no. 10, 2084–2102.
- [13] L. Illusie: *Déformations de groupes de Barsotti-Tate (d’après A. Grothendieck)*, Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84), Asterisque No. **127** (1985), 151–198.
- [14] N. M. Katz: *p-adic properties of modular schemes and modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 69–190. Lecture Notes in Mathematics **350**, Springer, Berlin, 1973.
- [15] M. Kisin: *Crystalline representations and F-crystals*, Algebraic geometry and number theory, 459–496, Progr. Math. **253**, Birkhauser Boston, Boston, MA, 2006.
- [16] M. Kisin: *Moduli of finite flat group schemes and modularity*, Ann. of Math. (3) **170** (2009), 1085–1180.
- [17] M. Kisin and K. F. Lai: *Overconvergent Hilbert modular forms*, Amer. J. Math. **127** (2005), no. 4, 735–783.
- [18] T. Liu: *Torsion p-adic Galois representations and a conjecture of Fontaine*, Ann. Sci. Ecole Norm. Sup. (4) **40** (2007), no. 4, 633–674.
- [19] J. Rabinoff: *Higher-level canonical subgroups for p-divisible groups*, J. Inst. Math. Jussieu **11** (2012), no. 2, 363–419.
- [20] J. Tate and F. Oort: *Group schemes of prime order*, Ann. Sci. Ecole Norm. Sup. (4) **3** (1970), 1–21.
- [21] Y. Tian: *Canonical subgroups of Barsotti-Tate groups*, Ann. of Math. (2) **172** (2010), no. 2, 955–988.
- [22] Y. Tian: *An upper bound on the Abbes-Saito filtration for finite flat group schemes and applications*, Algebra & Number Theory **6** (2012), no. 2, 231–242.

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