CANONICAL SUBGROUPS VIA BREUIL-KISIN MODULES FOR $p = 2$

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Abstract. Let p be a rational prime and $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields. Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level n, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$. In this paper, we prove the existence of higher canonical subgroups with standard properties for $\mathcal{G}$ if the Hasse invariant of $\mathcal{G}$ is less than $1/(2p^{n-1})$, including the case of $p = 2$.

1. Introduction

Let p be a rational prime and $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields. Let k be its residue field, π be its uniformizer, e be its absolute ramification index, $\bar{K}$ be its algebraic closure and v_p be its valuation extended to $\bar{K}$ and normalized as $v_p(p) = 1$. We let $\mathbb{C}$ denote the completion of $\bar{K}$. For any valuation field F (of height one) with valuation v_F, we let $\mathcal{O}_F$ denote its valuation ring and put $m_F^{\geq i} = \{x \in F \mid v_F(x) \geq i\}$ and $\mathcal{O}_{F,i} = \mathcal{O}_F/m_F^{\geq i}$ for any positive real number i. We also put $\tilde{\mathcal{O}}_K = \mathcal{O}_{K,1}$ and $\tilde{\mathcal{O}}_{\bar{K}} = \mathcal{O}_{\bar{K},1}$.

One of the key ingredients of the theory of p-adic Siegel modular forms is the existence theorem of canonical subgroups. Let $\mathfrak{X}$ be the p-adic completion of the Siegel modular variety of genus g and level prime to p over the Witt ring $W(k)$, X be its Raynaud generic fiber, X^{ord} be its ordinary locus considered as an admissible open of X and $\mathfrak{A}$ be the universal abelian scheme over $\mathfrak{X}$. Consider the unit component $\mathfrak{A}[p^n]^0$ of the p^n-torsion of $\mathfrak{A}$. The rigid-analytic subgroup $\mathfrak{A}[p^n]^0|_{X^{\mathrm{ord}}}$ is etale locally isomorphic to the constant group $(\mathbb{Z}/p^n\mathbb{Z})^g$ and it is a lift of the kernel of the n-th iterated Frobenius of the special fiber of $\mathfrak{A}$. Then the theorem asserts that this subgroup can be extended to a subgroup C_n with the same properties over a larger rigid-analytic subspace $X(r)$ of X containing X^{ord}. In [9], the author proved the existence of such a subgroup over the locus of the Hasse invariant less than $1/(2p^{n-1})$ for $p \geq 3$. The aim of this paper is to generalize the result to the case of $p = 2$.

To state the main theorem, we introduce some notation. For any finite torsion $\mathcal{O}_K$-module M which is isomorphic to $\mathcal{O}_K/(\pi^{r_1}) \oplus \cdots \oplus \mathcal{O}_K/(\pi^{r_l})$, we define its degree by $\deg(M) = e^{-1}(r_1 + \cdots + r_l)$. For any finite flat

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(commutative) group scheme $\mathcal{G}$ (*resp.* Barsotti-Tate group Γ) over $\mathcal{O}_K$, we let $\omega_{\mathcal{G}}$ (*resp.* ω_Γ) denote its module of invariant differentials and put $\deg(\mathcal{G})=\deg(\omega_{\mathcal{G}})$. For any positive rational number i, the Hodge-Tate map for a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ killed by p^n is defined to be the natural homomorphism

$$\mathrm{HT}_i : \mathcal{G}(\mathcal{O}_{\bar{K}}) \simeq \mathrm{Hom}(\mathcal{G}^\vee \times \mathrm{Spec}(\mathcal{O}_{\bar{K}}), \mu_{p^n} \times \mathrm{Spec}(\mathcal{O}_{\bar{K}})) \to \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K},i}$$

defined by $g \mapsto g^*(dT/T)$, where $\vee$ means the Cartier dual and $\mu_{p^n} = \mathrm{Spec}(\mathcal{O}_K[T]/(T^{p^n}-1))$ is the group scheme of p^n-th roots of unity. We normalize the upper and the lower ramification subgroups of $\mathcal{G}$ to be adapted to the valuation v_p. Namely, writing the affine algebra of $\mathcal{G}$ as

$$\mathcal{O}_K[T_1,\ldots,T_r]/(f_1,\ldots,f_s)$$

and an r-tuple $(x_1,\ldots,x_r)\in\mathcal{O}_{\bar{K}}^r$ as $\underline{x}$, we put

$$\mathcal{G}^j(\mathcal{O}_{\bar{K}}) = \mathcal{G}(\mathcal{O}_{\bar{K}}) \cap \{\underline{x}\in\mathcal{O}_{\bar{K}}^r \mid v_p(f_l(\underline{x})) \geq j \text{ for any } l\}^0,$$
$$\mathcal{G}_i(\mathcal{O}_{\bar{K}}) = \mathrm{Ker}(\mathcal{G}(\mathcal{O}_{\bar{K}}) \to \mathcal{G}(\mathcal{O}_{\bar{K},i})),$$

where $(-)^0$ in the first equality means the geometric connected component as an affinoid variety over K containing the zero section (see [1, Section 2]). We also put $\mathcal{G}^{j+}(\mathcal{O}_{\bar{K}}) = \cup_{j'>j}\mathcal{G}^{j'}(\mathcal{O}_{\bar{K}})$ for any non-negative rational number j. The scheme-theoretic closure of $\mathcal{G}^j(\mathcal{O}_{\bar{K}})$ in $\mathcal{G}$ is denoted by $\mathcal{G}^j$ and define $\mathcal{G}^{j+}$ and $\mathcal{G}_i$ similarly. Finally, for any truncated Barsotti-Tate group $\mathcal{G}$ ([10]) of level n, height h and dimension d over $\mathcal{O}_K$ with $d<h$, we define the Hasse invariant $\mathrm{Ha}(\mathcal{G})$ to be the truncated valuation $v_p(\det(V))\in[0,1]$ of the determinant of the natural action of the Verschiebung V of the group scheme $\mathcal{G}[p]^\vee \times \mathrm{Spec}(\tilde{\mathcal{O}}_K)$ on the free $\tilde{\mathcal{O}}_K$-module of finite rank $\mathrm{Lie}(\mathcal{G}[p]^\vee\times\mathrm{Spec}(\tilde{\mathcal{O}}_K))$. Then our main theorem is the following, which is proved in [9] except the case of $p=2$.

Theorem 1.1. *Let p be a rational prime and $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields. Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level n, height h and dimension d over $\mathcal{O}_K$ with $0<d<h$ and Hasse invariant $w=\mathrm{Ha}(\mathcal{G})$. If $w<1/(2p^{n-1})$, then the upper ramification subgroup scheme $\mathcal{C}_n=\mathcal{G}^{j+}$ for*

$$pw(p^n-1)/(p-1)^2 \leq j < p(1-w)/(p-1)$$

satisfies $\mathcal{C}_n(\mathcal{O}_{\bar{K}})\simeq(\mathbb{Z}/p^n\mathbb{Z})^d$. Moreover, the group scheme $\mathcal{C}_n$ has the following properties:

(a) $\deg(\mathcal{G}/\mathcal{C}_n)=w(p^n-1)/(p-1)$.

(b) $\mathcal{C}_n\times\mathrm{Spec}(\mathcal{O}_{K,1-p^{n-1}w})$ coincides with the kernel of the n-th iterated Frobenius of $\mathcal{G}\times\mathrm{Spec}(\mathcal{O}_{K,1-p^{n-1}w})$.

(c) The scheme-theoretic closure of $\mathcal{C}_n(\mathcal{O}_{\bar{K}})[p^i]$ in $\mathcal{C}_n$ coincides with the subgroup scheme $\mathcal{C}_i$ of $\mathcal{G}[p^i]$ for $1\leq i\leq n-1$.

(d) The subgroup $\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ coincides with the kernel of the Hodge-Tate map $\mathrm{HT}_{n-w(p^n-1)/(p-1)}:\mathcal{G}(\mathcal{O}_{\bar{K}})\to\omega_{\mathcal{G}^\vee}\otimes\mathcal{O}_{\bar{K},n-w(p^n-1)/(p-1)}$.

(e) Put $\mathcal{C}'_n = (\mathcal{G}^\vee)^{j+}$ for j as above. Then we have the equality $\mathcal{C}_n(\mathcal{O}_{\bar{K}}) = \mathcal{C}'_n(\mathcal{O}_{\bar{K}})^\perp$, where $\perp$ means the orthogonal subgroup with respect to the Cartier pairing

$$\langle\ ,\ \rangle_{\mathcal{G}} : \mathcal{G}(\mathcal{O}_{\bar{K}}) \times \mathcal{G}^\vee(\mathcal{O}_{\bar{K}}) \to \mu_{p^n}(\mathcal{O}_{\bar{K}}).$$

From Theorem 1.1, we can show the following corollary just as in the proof of [9, Corollary 1.2].

Corollary 1.2. *Let $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields. Let $\mathfrak{X}$ be an admissible formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$ *which is quasi-compact and $\mathfrak{G}$ be a truncated Barsotti-Tate group of level n over $\mathfrak{X}$ of constant height h and dimension d with $0 < d < h$. We let X and G denote the Raynaud generic fibers of the formal schemes $\mathfrak{X}$ and $\mathfrak{G}$, respectively. Let G^{j+} be the family of j-th upper ramification subgroups as in* [9, Lemma 4.3]. *For any finite extension L/K and $x \in X(L)$, we put $\mathfrak{G}_x = \mathfrak{G} \times_{\mathfrak{X},x} \mathrm{Spf}(\mathcal{O}_L)$, where we let x also denote the map $\mathrm{Spf}(\mathcal{O}_L) \to \mathfrak{X}$ obtained from x by taking the scheme-theoretic closure and the normalization. For any non-negative rational number r, let $X(r)$ be the rigid-analytic subspace of X defined by*

$$X(r)(\bar{K}) = \{x \in X(\bar{K}) \mid \mathrm{Ha}(\mathfrak{G}_x) < r\}.$$

Then the rigid-analytic group $G^{j+}|_{X(r)}$ over $X(r)$ is etale locally isomorphic to the constant group $(\mathbb{Z}/p^n\mathbb{Z})^d$ for $r = 1/(2p^{n-1})$ and $j = (2p^{n-1} - 1)/(2p^{n-2}(p-1))$.

The basic strategy of the proof of the main theorem is the same as in [9]: we reduce to the case where k is perfect and then switch to a study of ramification of finite flat group schemes over a complete discrete valuation ring of equal characteristic p with residue field k. In [9], we carried out this switching by the ramification correspondence theorem of the author ([8]). The latter theorem is based on a classification of finite flat group schemes over $\mathcal{O}_K$ killed by p for $p \geq 3$ in terms of ϕ-modules over the formal power series ring $k[[u]]$, which is due to Breuil ([3], [4]) and Kisin ([11], [12]). Instead, here we use a similar classification of unipotent finite flat group schemes also due to Kisin which holds for any p ([13]) to prove a correspondence result of lower ramification subgroups between mixed and equal characteristics. We seek the canonical subgroup $\mathcal{C}_n$ of the truncated Barsotti-Tate group $\mathcal{G}$ in its upper ramification subgroup schemes and they depend only on its unit component $\mathcal{G}^0$. Moreover, we prove properties of $\mathcal{C}_n$ by studying lower ramification subgroups of the unipotent group scheme $(\mathcal{G}^0)^\vee$ via Cartier duality. Thus the classification theorem of unipotent group schemes suffices to prove our existence theorem of canonical subgroups.

2. Classification of unipotent finite flat group schemes

In this and the next section, we assume that the residue field k of K is perfect. For $p \geq 3$, we have a classification theory of Barsotti-Tate groups and finite flat group schemes over $\mathcal{O}_K$ due to Breuil ([3], [4]) and Kisin ([11],

[12]) in terms of so-called Breuil-Kisin modules. Kisin ([13]) also extended this classification to the case of $p = 2$ and where groups are connected, using Zink's classification of formal Barsotti-Tate groups ([16], [17]). In this section, we briefly recall this result of Kisin. Since we adopt a contravariant notation contrary to his, what we describe here is a classification of unipotent Barsotti-Tate groups and unipotent finite flat group schemes.

Let $W = W(k)$ be the Witt ring of k and ϕ be its natural Frobenius endomorphism which lifts the p-th power map of k. Natural ϕ-semilinear Frobenius endomorphisms of various W-algebras are denoted also by ϕ. Let $E(u) \in W[u]$ be the Eisenstein polynomial of π over W. Let us fix once and for all a system $\{\pi_n\}_{n\geq 0}$ of p-power roots of π in $\bar{K}$ with $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Put $K_\infty = \cup_{n\geq 0} K(\pi_n)$, $\mathfrak{S} = W[[u]]$ and $\mathfrak{S}_1 = k[[u]]$. We write the ϕ-semilinear continuous ring endomorphisms of the latter two rings defined by $u \mapsto u^p$ also as ϕ. Then a Kisin module is an $\mathfrak{S}$-module $\mathfrak{M}$ such that a ϕ-semilinear map $\phi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ is given. We write $\phi_{\mathfrak{M}}$ also as ϕ if no confusion may occur. We follow the notation of [9, Subsection 2.1]. In particular, we have categories $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}}$, $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$, $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_\infty}$ of Kisin modules of E-height ≤ 1 and a category $\mathrm{Mod}^{1,\phi}_{/B}$ for any $k[[u]]$-algebra B.

Let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_\infty}$ and put $\phi^*\mathfrak{M} = \mathfrak{S} \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$. Then the map $1 \otimes \phi : \phi^*\mathfrak{M} \to \mathfrak{M}$ is injective and we have a unique map $\psi_{\mathfrak{M}} : \mathfrak{M} \to \phi^*\mathfrak{M}$ satisfying $(1 \otimes \phi) \circ \psi_{\mathfrak{M}} = E(u)$. We say $\mathfrak{M}$ is V-nilpotent if the composite

$$\phi^{n-1*}(\psi_{\mathfrak{M}}) \circ \phi^{n-2*}(\psi_{\mathfrak{M}}) \circ \cdots \circ \psi_{\mathfrak{M}} : \mathfrak{M} \to \phi^{n*}\mathfrak{M}$$

factors through the submodule $(p, u)\phi^{n*}\mathfrak{M}$ for any sufficiently large n. Similarly, we say an object $\mathfrak{M}$ of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}}$ is topologically V-nilpotent if the same condition holds. The full subcategories of V-nilpotent (*resp.* topologically V-nilpotent) objects are denoted by $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_1}$ and $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_\infty}$ (*resp.* $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}}$). Note that these notions are called connected and formal in [13], respectively.

Let S be the p-adic completion of the divided power envelope of $W[u]$ with respect to the ideal $(E(u))$. The ring S has a natural filtration $\mathrm{Fil}^1 S$ induced by the divided power structure, a ϕ-semilinear Frobenius endomorphism denoted also by ϕ and a ϕ-semilinear map $\phi_1 : \mathrm{Fil}^1 S \to S$. Then a Breuil module is an S-module $\mathcal{M}$ such that an S-submodule $\mathrm{Fil}^1\mathcal{M}$ containing $(\mathrm{Fil}^1 S)\mathcal{M}$ and a ϕ-semilinear map $\phi_{1,\mathcal{M}} : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ are given and these satisfy some compatibility conditions. The map $\phi_{1,\mathcal{M}}$ is also denoted by ϕ_1 if there is no risk of confusion. We also have categories of Breuil modules $\mathrm{Mod}^{1,\phi}_{/S}$, $\mathrm{Mod}^{1,\phi}_{/S_1}$ and $\mathrm{Mod}^{1,\phi}_{/S_\infty}$ (for the definitions, see [8, Subsection 2.1]). For any object $\mathcal{M}$ of these categories, we define a ϕ-semilinear map $\phi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ by $\phi_{\mathcal{M}}(x) = \phi_1(E(u))^{-1}\phi_1(E(u)x)$, which we abusively write as ϕ.

Let $\mathcal{M}$ be an object of the category $\mathrm{Mod}^{1,\phi}_{/S}$ and put $\phi^*\mathcal{M} = S \otimes_{\phi,S} \mathcal{M}$. Then the map $1 \otimes \phi : \phi^*\mathcal{M} \to \mathcal{M}$ is injective and we have a unique map $\psi_{\mathcal{M}} : \mathcal{M} \to \phi^*\mathcal{M}$ satisfying $(1\otimes\phi)\circ\psi_{\mathcal{M}} = p$. Then we say $\mathcal{M}$ is topologically V-nilpotent if the composite

$$\phi^{n-1*}(\psi_{\mathcal{M}}) \circ \phi^{n-2*}(\psi_{\mathcal{M}}) \circ \cdots \circ \psi_{\mathcal{M}} : \mathcal{M} \to \phi^{n*}\mathcal{M}$$

factors through the submodule $(p, \mathrm{Fil}^1 S)\phi^{n*}\mathcal{M}$ for any sufficiently large n. This notion is called S-window over $\mathcal{O}_K$ in [13] and [16]. The full subcategory of topologically V-nilpotent objects is denoted by $\mathrm{Mod}^{1,\phi,V}_{/S}$. For any Kisin module $\mathfrak{M}$, define a Breuil module $\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) = S \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$ by putting

$$\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Ker}(S \otimes_{\phi,\mathfrak{S}} \mathfrak{M} \overset{1\otimes\phi}{\to} S/\mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M}),$$

$$\phi_1 : \mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \overset{1\otimes\phi}{\to} \mathrm{Fil}^1 S \otimes_{\mathfrak{S}} \mathfrak{M} \overset{\phi_1\otimes 1}{\to} \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}).$$

This gives exact functors $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^{1,\phi}_{/S_\infty}$ and $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}} \to \mathrm{Mod}^{1,\phi}_{/S}$, which are both denoted by $\mathcal{M}_{\mathfrak{S}}(-)$, and the latter induces a functor $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}} \to \mathrm{Mod}^{1,\phi,V}_{/S}$ ([13, Proposition 1.2.5]).

We can associate Galois representations to Kisin and Breuil modules. Consider the ring $R = \varprojlim(\tilde{\mathcal{O}}_{\bar{K}} \leftarrow \tilde{\mathcal{O}}_{\bar{K}} \leftarrow \cdots)$, where the transition maps are p-th power maps. An element $r \in R$ is written as $r = (r_n)_{n\geq 0}$ with $r_n \in \tilde{\mathcal{O}}_{\bar{K}}$, and define $r^{(0)} \in \mathcal{O}_{\mathbb{C}}$ by $r^{(0)} = \lim_{n\to\infty} \hat{r}_n^{p^n}$, where $\hat{r}_n$ is a lift of r_n in $\mathcal{O}_{\bar{K}}$. Then the ring R is a valuation ring of characteristic p with its valuation defined by $v_R(r) = v_p(r^{(0)})$ whose fraction field is algebraically closed, and we put $m_R^{\geq i} = \{r \in R \mid v_R(r) \geq i\}$ and $R_i = R/m_R^{\geq i}$. We have a natural ring surjection $W(R) \to \mathcal{O}_{\mathbb{C}}$ which lifts the zeroth projection $\mathrm{pr}_0 : R \to \tilde{\mathcal{O}}_{\bar{K}}$. The ring A_{crys} is the p-adic completion of the divided power envelope of $W(R)$ with respect to the kernel of this surjection. Thus we have the induced surjection $A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}}$. Put $\underline{\pi} = (\pi, \pi_1, \pi_2, \ldots) \in R$ and consider the rings $W(R)$ and A_{crys} as $\mathfrak{S}$-algebras by the map $u \mapsto [\underline{\pi}]$. In particular, we consider the ring $k[[u]]$ as a subring of R by the map $u \mapsto \underline{\pi}$ and let v_R also denote the induced valuation on the former ring, which satisfies $v_R(u) = 1/e$.

For any objects $\mathfrak{M}$ of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}}$ and $\mathcal{M}$ of $\mathrm{Mod}^{1,\phi}_{/S}$, we associate to them G_{K_∞}-modules

$$T^*_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M}, W(R)),$$
$$T^*_{\mathrm{crys}}(\mathcal{M}) = \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi}(\mathcal{M}, A_{\mathrm{crys}})$$

([6, Proposition B1.8.3] and [13, Subsection 1.2.6]). If the $\mathfrak{S}$-module $\mathfrak{M}$ is free of rank h, then the $\mathbb{Z}_p$-module $T^*_{\mathfrak{S}}(\mathfrak{M})$ is also free of rank h ([11, Corollary 2.1.4]). We also have a natural injection of G_{K_∞}-modules $T^*_{\mathfrak{S}}(\mathfrak{M}) \to T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}))$ defined by $f \mapsto 1 \otimes (\phi \circ f)$ and this is a bijection if $\mathfrak{M}$ is topologically V-nilpotent ([13, Proposition 1.2.7]). Similarly, for any object

$\mathfrak{M}$ of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_\infty}$, we have the associated G_{K_∞}-module

$$T^*_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M}, W(R) \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

Put $\mathscr{S}_n = \mathrm{Spec}(\mathcal{O}_{K,n})$, $S_n = S/p^nS$ and $E_n = \mathrm{Spec}(S_n)$. Let us consider the big crystalline site $\mathrm{CRYS}(\mathscr{S}_n/E_n)$ with the fppf topology and its topos $(\mathscr{S}_n/E_n)_{\mathrm{CRYS}}$. For any Barsotti-Tate group Γ over $\mathcal{O}_K$, we have the contravariant Dieudonné crystal $\mathbb{D}^*(\Gamma \times \mathscr{S}_n) = \mathscr{E}xt^1_{\mathscr{S}_n/E_n}(\underline{\Gamma \times \mathscr{S}_n}, \mathcal{O}_{\mathscr{S}_n/E_n})$ (for the notation, see [2]). We put

$$\mathbb{D}^*(\Gamma)(S \to \mathcal{O}_K) = \varprojlim_n \mathbb{D}^*(\Gamma \times \mathscr{S}_n)(S_n \to \mathcal{O}_{K,n}).$$

This module is considered as an object $\mathrm{Mod}(\Gamma)$ of the category $\mathrm{Mod}^{1,\phi}_{/S}$ with the natural ϕ-semilinear Frobenius map induced by the Frobenius of $\Gamma \times \mathscr{S}_1$ and the filtration defined as the inverse image of the natural inclusion

$$\omega_\Gamma \subseteq \varprojlim_n \mathbb{D}^*(\Gamma \times \mathscr{S}_n)(\mathcal{O}_{K,n} \to \mathcal{O}_{K,n}).$$

The A_{crys}-module

$$\mathbb{D}^*(\Gamma)(A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}}) = \varprojlim_n \mathbb{D}^*(\Gamma \times \mathscr{S}_n)(A_{\mathrm{crys}}/p^n A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C},n})$$

also has a ϕ-semilinear Frobenius map and a filtration defined in the same way. Similarly, for any finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, the S-module

$$\mathbb{D}^*(\mathcal{G})(S \to \mathcal{O}_K) = \varprojlim_n \mathbb{D}^*(\mathcal{G} \times \mathscr{S}_n)(S_n \to \mathcal{O}_{K,n})$$

is endowed with a natural ϕ-semilinear Frobenius map which is induced by the Frobenius of the group scheme $\mathcal{G} \times \mathscr{S}_1$ and is also denoted by ϕ, and a filtration defined by the submodule

$$\mathrm{Fil}^1\mathbb{D}^*(\mathcal{G})(S \to \mathcal{O}_K) = \varprojlim_n \mathscr{E}xt^1_{\mathscr{S}_n/E_n}(\underline{\mathcal{G} \times \mathscr{S}_n}, \mathcal{J}_{\mathscr{S}_n/E_n})(S_n \to \mathcal{O}_{K,n}),$$

where $\mathcal{J}_{\mathscr{S}_n/E_n}$ is the canonical divided power ideal sheaf of the structure sheaf $\mathcal{O}_{\mathscr{S}_n/E_n}$.

We say a Barsotti-Tate group or a finite locally free group scheme is unipotent if its Cartier dual is connected. We let $(\mathrm{BT}/\mathcal{O}_K)^{\mathrm{u}}$ (*resp.* $(p\text{-}\mathrm{Gr}/\mathcal{O}_K)^{\mathrm{u}}$) denote the category of unipotent Barsotti-Tate groups (*resp.* the category of unipotent finite flat group schemes killed by some p-power) over $\mathcal{O}_K$. If a Barsotti-Tate group Γ over $\mathcal{O}_K$ is unipotent, then the object $\mathrm{Mod}(\Gamma)$ is topologically V-nilpotent ([13, Lemma 1.1.3]). Moreover, we have the following classification theorem of unipotent Barsotti-Tate groups and unipotent finite flat group schemes, whose second assertion follows from the first assertion by an argument of taking a resolution ([13, Subsection 1.3]).

Theorem 2.1. *(1) ([13], Theorem 1.2.8) There exists an anti-equivalence of categories*

$$\mathcal{G}(-) : \mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}} \to (\mathrm{BT}/\mathcal{O}_K)^{\mathrm{u}}$$

with a natural isomorphism $\mathrm{Mod}(\mathcal{G}(\mathfrak{M})) \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$*. Moreover, we also have a natural isomorphism of* G_{K_∞}*-modules* $\varepsilon_{\mathfrak{M}} : T_p(\mathcal{G}(\mathfrak{M})) \to T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}))$.

(2) *(*[13]*, Theorem 1.3.9) There exists an anti-equivalence of categories*

$$\mathcal{G}(-) : \mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_\infty} \to (p\text{-}\mathrm{Gr}/\mathcal{O}_K)^{\mathrm{u}}$$

with a natural isomorphism of G_{K_∞}*-modules* $\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \to T^*_{\mathfrak{S}}(\mathfrak{M})$.

□

Let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}}$. If we identify an element $g \in T_p(\mathcal{G}(\mathfrak{M}))$ with a homomorphism of Barsotti-Tate groups from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\mathcal{G}(\mathfrak{M})$ over $\mathcal{O}_{\mathbb{C}}$, then by the natural isomorphism $\mathrm{Mod}(\mathcal{G}(\mathfrak{M})) \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ the element $\varepsilon_{\mathfrak{M}}(g)$ is identified with the induced map

$$\mathbb{D}^*(g) : \mathbb{D}^*(\mathcal{G}(\mathfrak{M}))(A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}}) \to \mathbb{D}^*(\mathbb{Q}_p/\mathbb{Z}_p)(A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}}) = A_{\mathrm{crys}}.$$

A similar argument as in the proof of [12, Proposition 1.1.11] shows that for any exact sequence

$$0 \to \mathfrak{N} \to \mathfrak{N}' \to \mathfrak{M} \to 0$$

of Kisin modules such that $\mathfrak{N}$ and $\mathfrak{N}'$ are objects of the category of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}}$ and $\mathfrak{M}$ is of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_\infty}$, the functor $\mathcal{M}_{\mathfrak{S}}(-)$ induces an exact sequence of Breuil modules

$$0 \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{N}) \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{N}') \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to 0.$$

This and [2, Lemme 4.2.5 (ii)] imply that, for any object $\mathfrak{M}$ of the category $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_\infty}$, there exists a natural isomorphism of S-modules

$$\mathbb{D}^*(\mathcal{G}(\mathfrak{M}))(S \to \mathcal{O}_K) \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$$

which is compatible with Fil^1 and ϕ. Then we can show the following lemma just as in the proof of [9, Lemma 2.4].

Lemma 2.2. *(1) Let* $\mathcal{G}$ *be a unipotent truncated Barsotti-Tate group of level one over* $\mathcal{O}_K$ *and* $\mathfrak{M}$ *be the corresponding object of* $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_1}$ *via the anti-equivalence* $\mathcal{G}(-)$*. Then there exist natural isomorphisms of* $\tilde{\mathcal{O}}_K$*-modules*

$$\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})/(\mathrm{Fil}^1 S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to \omega_{\mathcal{G}}, \quad \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})/\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to \mathrm{Lie}(\mathcal{G}^\vee).$$

(2) Let $\mathcal{G}$ *be a unipotent finite flat group scheme over* $\mathcal{O}_K$ *killed by* p *and* $\mathfrak{M}$ *be the corresponding object of the category* $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_1}$ *via the anti-equivalence* $\mathcal{G}(-)$*. Then there exists a natural isomorphism of* $\tilde{\mathcal{O}}_K$*-modules* $\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})/(\mathrm{Fil}^1 S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to \omega_{\mathcal{G}}$.

□

3. Ramification correspondence

Let K be as in the previous section. Recall that, for any $k[[u]]$-algebra B, we have an anti-equivalence $\mathcal{H}(-)$ from the category $\mathrm{Mod}^{1,\phi}_{/B}$ to a category of finite locally free group schemes over B whose Verschiebung is the zero map ([7, Théorème 7.4]. See also [8, Subsection 3.2]). Moreover, for $B = k[[u]]$ and $\mathfrak{M} \in \mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_1}$, the anti-equivalences $\mathcal{G}(-)$ and $\mathcal{H}(-)$ are related by the natural isomorphism $\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \to T^*_{\mathfrak{S}}(\mathfrak{M}) = \mathcal{H}(\mathfrak{M})(R)$ of Theorem 2.1. Consider the lower ramification subgroups of the group scheme $\mathcal{H}(\mathfrak{M})$ adapted to the valuation v_R. Namely, we define

$$\mathcal{H}(\mathfrak{M})_i(R) = \mathrm{Ker}(\mathcal{H}(\mathfrak{M})(R) \to \mathcal{H}(\mathfrak{M})(R_i))$$

for any positive rational number i. In this section, we prove that the isomorphism $\varepsilon_{\mathfrak{M}}$ preserves the lower ramification subgroups of both sides. In [8] where we treated the case of $p \geq 3$, we proved this assertion for any finite flat group scheme killed by p by using an explicit description of the affine algebra of $\mathcal{G}(\mathfrak{M})$ due to Breuil ([3]). For the case where group schemes are unipotent and $p \geq 2$, a crystalline method as in [2] is enough to show the correspondence of lower ramification subgroups, though it seems insufficient for obtaining a similar description of the affine algebra.

Let $i \leq 1$ be a positive rational number and put $\bar{\mathscr{S}}_i = \mathrm{Spec}(\mathcal{O}_{\bar{K},i})$. We let R_i^{DP} abusively denote the divided power envelope of the ring R with respect to the kernel of the natural surjection $\mathrm{pr}_0 : R \to \mathcal{O}_{\bar{K},i}$. The induced surjection $R_i^{\mathrm{DP}} \to \mathcal{O}_{\bar{K},i}$ is considered as an element of the crystalline site $\mathrm{CRYS}(\mathscr{S}_1/E_1)$ and let this thickening be denoted by A_i. Note that, by fixing a generator x of the principal ideal $m_R^{\geq i}$, we have an isomorphism of R-algebras

$$R_{pi}[Y_1, Y_2, \ldots]/(Y_1^p, Y_2^p, \ldots) \to R_i^{\mathrm{DP}}$$

which sends Y_l to the p^l-th divided power $\gamma_{p^l}(x)$. The ring R_i^{DP} is endowed with the natural filtration defined by the divided power structure and we have a natural isomorphism $R_{pi} \to R_i^{\mathrm{DP}}/\mathrm{Fil}^p R_i^{\mathrm{DP}}$. The ring R_1^{DP} is written also as R^{DP} and is naturally identified with the ring $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$. We also identify the module of sections $\mathbb{D}^*(\mathbb{Z}/p\mathbb{Z})(A_i)$ with the ring R_i^{DP}.

Lemma 3.1. *Let $i \leq 1$ be a positive rational number and $\mathcal{G}$ be a unipotent finite flat group scheme over $\mathcal{O}_{K,i}$ killed by p. Then the natural homomorphism of abelian groups*

$$\mathcal{G}(\mathcal{O}_{\bar{K},i}) = \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, \mathcal{G} \times \bar{\mathscr{S}}_i) \to \mathrm{Hom}_{R_i^{\mathrm{DP}},\phi}(\mathbb{D}^*(\mathcal{G})(A_i), R_i^{\mathrm{DP}}/\mathrm{Fil}^p R_i^{\mathrm{DP}})$$

defined by $g \mapsto \mathbb{D}^*(g)$ and the identification $\mathbb{D}^*(\mathbb{Z}/p\mathbb{Z})(A_i) \simeq R_i^{\mathrm{DP}}$ above is an injection.

Proof. Note that, for any exact sequence of finite locally free group schemes over $\mathcal{O}_{\bar{K},i}$ killed by p

$$0 \to \mathcal{H}' \to \mathcal{H} \to \mathcal{H}'' \to 0,$$

we have an exact sequence of abelian groups

$$0 \to \mathcal{H}'(\mathcal{O}_{\bar{K},i}) \to \mathcal{H}(\mathcal{O}_{\bar{K},i}) \to \mathcal{H}''(\mathcal{O}_{\bar{K},i})$$

and an exact sequence of ϕ-modules over R_i^{DP}

$$0 \to \mathbb{D}^*(\mathcal{H}'')(A_i) \to \mathbb{D}^*(\mathcal{H})(A_i) \to \mathbb{D}^*(\mathcal{H}')(A_i) \to 0$$

by [2, Proposition 4.3.1]. Thus, by a dévissage argument, we may assume that the Verschiebung of the group scheme $\mathcal{G}$ is zero. In this case, by [2, Proposition 4.3.6] there exists a natural isomorphism

$$\mathbb{D}^*(\mathcal{G})(A_i) \to \mathrm{Lie}(\mathcal{G}^\vee \times \bar{\mathscr{S}}_i) \otimes_{\mathcal{O}_{\bar{K},i},\phi} R_i^{\mathrm{DP}},$$

where the map $\phi : \mathcal{O}_{\bar{K},i} \to R_i^{\mathrm{DP}}$ is induced by $\phi : R_i^{\mathrm{DP}} \to R_i^{\mathrm{DP}}$. By the natural isomorphism $R_{pi} \to R_i^{\mathrm{DP}}/\mathrm{Fil}^p R_i^{\mathrm{DP}}$, the group on the right-hand side of the theorem is identified with

$$\mathrm{Hom}_{R,\phi}(\mathrm{Lie}(\mathcal{G}^\vee \times \bar{\mathscr{S}}_i) \otimes_{\mathcal{O}_{\bar{K},i},\phi} R_{pi}, \mathrm{Lie}((\mathbb{Z}/p\mathbb{Z})^\vee) \otimes_{\mathcal{O}_{\bar{K},i},\phi} R_{pi}).$$

Since the map $\mathrm{pr}_0 : R \to \mathcal{O}_{\bar{K},i}$ induces an isomorphism $R_i \to \mathcal{O}_{\bar{K},i}$ and the map $\phi : \mathcal{O}_{\bar{K},i} \simeq R_i \to R_{pi}$ is also an isomorphism, the claim follows from [7, Théorème 7.4]. $\square$

Theorem 3.2. *For any object $\mathfrak{M}$ of the category $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_1}$, the natural isomorphism $\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \to \mathcal{H}(\mathfrak{M})(R)$ induces an isomorphism of lower ramification subgroups*

$$\mathcal{G}(\mathfrak{M})_i(\mathcal{O}_{\bar{K}}) \to \mathcal{H}(\mathfrak{M})_i(R)$$

for any positive rational number i.

Proof. Put $\mathcal{G} = \mathcal{G}(\mathfrak{M})$. Since the i-th lower ramification subgroups of $\mathcal{G}$ and $\mathcal{H}(\mathfrak{M})$ vanish for $i > 1/(p-1)$ ([8, Corollary 3.5 and Remark 3.6]), we may assume $i \leq 1/(p-1) \leq 1$. Consider a resolution of $\mathcal{G}$ by unipotent Barsotti-Tate groups

$$0 \to \mathcal{G} \to \Gamma_1 \to \Gamma_2 \to 0$$

and the associated exact sequence of Kisin modules

$$0 \to \mathfrak{N}_2 \to \mathfrak{N}_1 \to \mathfrak{M} \to 0.$$

Put $\mathcal{M}=\mathbb{D}^*(\mathcal{G})(S\to\mathcal{O}_K)$ and $\mathcal{N}_i=\mathbb{D}^*(\Gamma_i)(S\to\mathcal{O}_K)$. Then we have a diagram

$$\begin{tikzcd}
T_p(\Gamma_1) \arrow[r, "\sim"] \arrow[d] & T^*_{\mathrm{crys}}(\mathcal{N}_1) \arrow[d] & T^*_{\mathfrak{S}}(\mathfrak{N}_1) \arrow[l, "\sim"'] \arrow[d] \\
T_p(\Gamma_2) \arrow[r, "\sim"] \arrow[d, "\pi_{\mathcal{G}}"'] & T^*_{\mathrm{crys}}(\mathcal{N}_2) \arrow[d, "\pi_{\mathcal{M}}"'] & T^*_{\mathfrak{S}}(\mathfrak{N}_2) \arrow[l, "\sim"'] \arrow[d, "\pi_{\mathfrak{M}}"'] \\
\mathcal{G}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] \arrow[rr, dotted, bend left, "\varepsilon_{\mathfrak{M}}"] & \mathrm{Hom}_{S,\phi}(\mathcal{M},R^{\mathrm{DP}}) \arrow[d] & T^*_{\mathfrak{S}}(\mathfrak{M}) \arrow[l] \arrow[d] \\
\mathcal{G}(\mathcal{O}_{\bar{K},i}) \arrow[r, hook] & \mathrm{Hom}_{S,\phi}(\mathcal{M},R_i^{\mathrm{DP}}/\mathrm{Fil}^p R_i^{\mathrm{DP}}) & \mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M},R_i), \arrow[l, "\sim"']
\end{tikzcd}$$

where the left horizontal arrows are induced by $g\mapsto\mathbb{D}^*(g)$ and the right horizontal arrows are the maps sending f to $1\otimes(\phi\circ f)$. The middle left vertical arrow $\pi_{\mathcal{G}}:T_p(\Gamma_2)\to\mathcal{G}(\mathcal{O}_{\bar{K}})$ is defined as follows: For $g\in T_p(\Gamma_2)$, the element pg is contained in the image of $T_p(\Gamma_1)=\varprojlim_n\Gamma_1[p^n](\mathcal{O}_{\bar{K}})$ and put $pg=h=(h_n)_{n>0}$. Then the element $h_1\in\Gamma_1[p](\mathcal{O}_{\bar{K}})$ is contained in the subgroup $\mathcal{G}(\mathcal{O}_{\bar{K}})$ and the map $\pi_{\mathcal{G}}$ is defined by $g\mapsto h_1$. We define the map $\pi_{\mathcal{M}}:T^*_{\mathrm{crys}}(\mathcal{N}_2)\to\mathrm{Hom}_{S,\phi}(\mathcal{M},R^{\mathrm{DP}})$ similarly: For any map $f:\mathcal{N}_2\to A_{\mathrm{crys}}$, the map pf induces a map $\mathcal{N}_1\to A_{\mathrm{crys}}$. Its composite with the natural map $A_{\mathrm{crys}}\to R^{\mathrm{DP}}$ factors through $\mathcal{M}$ and defines the map $\pi_{\mathcal{M}}(f):\mathcal{M}\to R^{\mathrm{DP}}$. The map $\pi_{\mathfrak{M}}$ is defined in the same way. From these definitions, we see that the diagram is commutative. Note that the bottom right horizontal arrow is an isomorphism since $\phi:R_i\to R_{pi}$ is an isomorphism and the bottom left horizontal arrow is an injection by Lemma 3.1. The upper squares induce the isomorphism $\varepsilon_{\mathfrak{M}}:\mathcal{G}(\mathcal{O}_{\bar{K}})\to T^*_{\mathfrak{S}}(\mathfrak{M})=\mathcal{H}(\mathfrak{M})(R)$ and we also have the equality $\mathcal{H}(\mathfrak{M})(R_i)=\mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M},R_i)$. Thus we obtain a commutative diagram of abelian groups

$$\begin{tikzcd}
\mathcal{G}(\mathcal{O}_{\bar{K}}) \arrow[r, "\sim", "\varepsilon_{\mathfrak{M}}"'] \arrow[d] & \mathcal{H}(\mathfrak{M})(R) \arrow[d] \\
\mathcal{G}(\mathcal{O}_{\bar{K},i}) \arrow[r, hook] & \mathcal{H}(\mathfrak{M})(R_i),
\end{tikzcd}$$

where the upper horizontal arrow is a bijection and the lower horizontal arrow is an injection. Hence the theorem follows. □

4. Canonical subgroups

In this section, we prove Theorem 1.1. The proof is a modification of the argument in [9], where we had to exclude the case of $p=2$. A key step is the following theorem treating the case of level one.

Theorem 4.1. *Let $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields. Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level one, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$ and Hasse invariant $w = \mathrm{Ha}(\mathcal{G})$.*

(1) *Suppose $w < (p^2 - p - 1)/(p^2 - 1)$ for $p \geq 3$ and $w < 1/2$ for $p = 2$. Then the upper ramification subgroup scheme $\mathcal{C} = \mathcal{G}^{pw/(p-1)+}$ is of order p^d. Moreover, the group scheme $\mathcal{C}$ has the following properties:*
   (a) $\deg(\mathcal{G}/\mathcal{C}) = w$.
   (b) *$\mathcal{C} \times \mathrm{Spec}(\mathcal{O}_{K,1-w})$ coincides with the kernel of the Frobenius of $\mathcal{G} \times \mathrm{Spec}(\mathcal{O}_{K,1-w})$.*
(2) *If $w < 1/2$, then $\mathcal{C}$ coincides with the upper ramification subgroup scheme $\mathcal{G}^{j+}$ for $pw/(p-1) \leq j < p(1-w)/(p-1)$. The group scheme $\mathcal{C}$ also has the following properties:*
   (c) *The subgroup $\mathcal{C}(\mathcal{O}_{\bar{K}})$ coincides with the kernel of the Hodge-Tate map $\mathrm{HT}_b : \mathcal{G}(\mathcal{O}_{\bar{K}}) \to \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K},b}$ for $w/(p-1) < b \leq (1-w)/(p-1)$.*
   (d) *Put $\mathcal{C}' = (\mathcal{G}^\vee)^{j+}$ for j as above. Then we have the equality $\mathcal{C}(\mathcal{O}_{\bar{K}}) = \mathcal{C}'(\mathcal{O}_{\bar{K}})^\perp$, where $\perp$ means the orthogonal subgroup with respect to the Cartier pairing $\langle\ ,\ \rangle_{\mathcal{G}}$.*

Proof. By a base change argument as in the proof of [9, Theorem 3.2], we may assume that the residue field k of K is perfect. Let $\mathcal{G}^0$ (*resp.* $\mathcal{G}^{\mathrm{et}}$) be the unit component (*resp.* the maximal etale quotient) of the group scheme $\mathcal{G}$, and consider their Cartier duals $(\mathcal{G}^0)^\vee$ and $(\mathcal{G}^{\mathrm{et}})^\vee$. These four group schemes are all truncated Barsotti-Tate groups of level one over $\mathcal{O}_K$ and let h_0 be the height of $\mathcal{G}^0$. Then $(\mathcal{G}^0)^\vee$ is a unipotent truncated Barsotti-Tate group of level one, height h_0 and dimension $h_0 - d$. If $h_0 = d$, then the group scheme $\mathcal{G}$ is ordinary (namely, $(\mathcal{G}^0)^\vee$ is etale) and the assertions are clear. Thus we may assume $h_0 > d$.

Let $\mathfrak{M}$ be the object of the category $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_1}$ corresponding to the unipotent group scheme $(\mathcal{G}^0)^\vee$ via the anti-equivalence $\mathcal{G}(-)$. Put $\mathfrak{M}_1 = \mathfrak{M}/u^e\mathfrak{M}$ and $A\mathfrak{M}_1 = (1 \otimes \phi)(\phi^*\mathfrak{M}_1)$. By the k-algebra isomorphism $k[[u]]/(u^e) \to \tilde{\mathcal{O}}_K$ defined by $u \mapsto \pi$, we identify both sides of the isomorphism. Then the modules $\mathfrak{M}_1$ and $A\mathfrak{M}_1$ are naturally considered as objects of the category $\mathrm{Mod}^{1,\phi}_{/\tilde{\mathcal{O}}_K}$ and by Lemma 2.2 we can show that there exists a natural isomorphism of $\tilde{\mathcal{O}}_K$-modules $\mathrm{Lie}(\mathcal{G}^0) \to A\mathfrak{M}_1$ as in [9, Subsection 2.3]. Since $\mathrm{Ha}(\mathcal{G}) = \mathrm{Ha}(\mathcal{G}^\vee)$ and $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G}^0)$, we obtain an exact sequence of ϕ-modules over $\tilde{\mathcal{O}}_K$

$$0 \to A\mathfrak{M}_1 \to \mathfrak{M}_1 \to \mathfrak{M}_1/A\mathfrak{M}_1 \to 0$$

which splits as a sequence of $\tilde{\mathcal{O}}_K$-modules and the equality of truncated valuation $v_p(\det(\phi_{A\mathfrak{M}_1})) = w$. We choose a basis $e_1, \ldots, e_{h_0}$ of $\mathfrak{M}$ such that the images of $e_1, \ldots, e_d$ form a basis of $A\mathfrak{M}_1$ and the images of $e_{d+1}, \ldots, e_{h_0}$ form a basis of $\mathfrak{M}_1/A\mathfrak{M}_1$. Define matrices P_1, P_2, P_3, P_4 with coefficients in the ring $k[[u]]$ by $P_1 \in M_d(k[[u]])$ and

$$\phi_{\mathfrak{M}}(e_1, \dots, e_{h_0}) = (e_1, \dots, e_{h_0}) \begin{pmatrix} P_1 & P_2 \\ u^e P_3 & u^e P_4 \end{pmatrix}.$$

We have the equality $v_R(\det(P_1)) = w$ and thus there exists an element $\hat{P}_1$ of $M_d(k[[u]])$ satisfying $P_1\hat{P}_1 = \hat{P}_1 P_1 = u^{ew} I_d$, where I_d is the identity matrix. Put $\mathfrak{L} = k[[u]]e_1 \oplus \dots \oplus k[[u]]e_d$, which we consider as an object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ by putting

$$\phi_{\mathfrak{L}}(e_1, \dots, e_d) = (e_1, \dots, e_d) P_1.$$

We can lift the modulo $u^{e(1-w)}$ of the above sequence to an exact sequence of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$

$$0 \to \mathfrak{L} \to \mathfrak{M} \to \mathfrak{N} \to 0$$

as in the proof of [9, Lemma 3.4] and the objects $\mathfrak{L}$ and $\mathfrak{N}$ are also contained in the full subcategory $\mathrm{Mod}^{1,\phi,V}_{/\mathfrak{S}_1}$.

Lemma 4.2. *The subgroup $\mathcal{H}(\mathfrak{N})(R)$ coincides with the lower ramification subgroup $\mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}(R)$.*

Proof. Let x be an element of the subgroup $\mathcal{H}(\mathfrak{N})(R) \subseteq \mathcal{H}(\mathfrak{M})(R)$ and identify this with a map $\mathfrak{M} \to R$ defined by $e_i \mapsto x_i$ for $1 \leq i \leq d$ and $e_{d+i} \mapsto z_i$ for $1 \leq i \leq h_0 - d$. The h_0-tuple $(x_1, \dots, x_d, z_1, \dots, z_{h_0-d})$ is a solution in R with $v_R(x_i) \geq 1 - w$ of the equation

$$\begin{aligned} \underline{\pi}^{ew}(x_1, \dots, x_d) &= (x_1^p, \dots, x_d^p)\hat{P}_1 - \underline{\pi}^e(z_1, \dots, z_{h_0-d}) P_3 \hat{P}_1, \\ (z_1^p, \dots, z_{h_0-d}^p) &= (x_1, \dots, x_d) P_2 + \underline{\pi}^e(z_1, \dots, z_{h_0-d}) P_4. \end{aligned}$$

Then the proof of [9, Lemma 3.5] works verbatim and the lemma follows for $p \geq 3$.

We need a special care for the case of $p = 2$, as follows. Put $\varepsilon = 1/2 - w$. As in *loc. cit.*, we first show the inequality $v_R(x_i) \geq 1$ for any i. Consider the sequences $\{\xi_n\}_n$ and $\{\zeta_n\}_n$ defined by $\xi_0 = 1/2 + \varepsilon$, $\zeta_0 = 1/4 + \varepsilon/2$ and

$$\begin{aligned} \xi_{n+1} &= \min(2\xi_n, 1 + \zeta_n) - w, \\ \zeta_{n+1} &= 2^{-1} \min(\xi_n, 1 + \zeta_n), \end{aligned}$$

which satisfy $\xi_n \geq 0$ and $\zeta_n \geq 0$. It is enough to show $\xi_n \geq 1$ for some n. Suppose on the contrary $\xi_n < 1$ for any n. Then we have $\zeta_{n+1} = \xi_n/2$ and

$$\xi_{n+1} = \min(2\xi_n - 1/2 + \varepsilon, \xi_{n-1}/2 + 1/2 + \varepsilon).$$

Each ξ_n is an increasing function of ε and we may assume $\varepsilon < 1/6$. Then we have $\xi_1 = 1/2 + 3\varepsilon$. Suppose that the minimum on the right-hand side of the above equality is obtained by the first term $2\xi_n - 1/2 + \varepsilon$ for any n. Then we have $\xi_{n+1} = 1/2 + (2^{n+2} - 1)\varepsilon$, which contradicts to the assumption $\xi_n < 1$. Let N be the smallest integer such that the minimum is obtained by the second term $\xi_{N-1}/2 + 1/2 + \varepsilon$. Then we have $\xi_{N+1} = 3/4 + (2^{N-1} + 1/2)\varepsilon$, $\xi_{N+2} = 3/4 + (2^N + 1/2)\varepsilon$ and

$$\xi_{N+2l+1} = (2^{l+2} - 1)/2^{l+2} + (2^{N-l-1} + 2 - 3/2^{l+1})\varepsilon,$$
$$\xi_{N+2l+2} = (2^{l+2} - 1)/2^{l+2} + (2^{N-l} + 2 - 3/2^{l+1})\varepsilon$$

for any non-negative integer l. However, this sequence satisfies $\xi_{N+2l+1} \geq 1$ for sufficiently large l, which is a contradiction and we obtain the inequality $v_R(x_i) \geq 1$ for any i. Then the rest of the argument of *loc. cit.* works verbatim and the lemma follows also for $p = 2$. $\square$

Hence we see that if w is as in the theorem, the equalities

$$\mathcal{H}(\mathfrak{N})(R) = \mathrm{Ker}(\mathcal{H}(\mathfrak{M})(R) \to \mathcal{H}(A\mathfrak{M}_1)(R_i)) = \mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}(R)$$

hold for any rational number i satisfying $w/(p-1) < i \leq 1 - w$ and these subgroups are of order $p^{h_0 - d}$. In this case, we put $b = (1-w)/(p-1)$. Moreover, we also see that if $w < 1/2$, then these subgroups coincide with the subgroup $\mathcal{H}(\mathfrak{M})_b(R)$ for $w/(p-1) < b \leq (1-w)/(p-1)$. By Theorem 3.2, the subgroup scheme $\mathcal{G}(\mathfrak{N})$ of $(\mathcal{G}^0)^\vee$ is equal to the lower ramification subgroup scheme $((\mathcal{G}^0)^\vee)_b$ for both cases.

Now we define the subgroup scheme $\mathcal{C}$ of $\mathcal{G}^0$ to be the scheme-theoretic closure of the orthogonal subgroup $((\mathcal{G}^0)^\vee)_b(\mathcal{O}_{\bar{K}})^\perp$ with respect to the Cartier pairing $\langle\ ,\ \rangle_{\mathcal{G}^0}$. The group scheme $\mathcal{C}$ is of order p^d. We insert here the following lemma due to the lack of references.

Lemma 4.3. *Let $\mathcal{H}$ be a finite flat group scheme over $\mathcal{O}_K$ and $\mathcal{H}^0$ be its unit component. Then we have $\mathcal{H}^j = (\mathcal{H}^0)^j$ for any positive rational number j.*

Proof. Replacing K by a finite extension, we may assume that the maximal etale quotient $\mathcal{H}^{\mathrm{et}}$ is a constant group scheme $\underline{M}$ for some abelian group M. Then we have an isomorphism of schemes over $\mathcal{O}_K$

$$\mathcal{H}^0 \times \underline{M} \to \mathcal{H}$$

which induces the natural isomorphism of group schemes $\mathcal{H}^0 \times \{0\} \to \mathcal{H}^0$. Let $\mathcal{F}^j$ be the functor of the set of geometric connected components of the j-th tubular neighborhood as in [1, Section 2]. By [1, Lemme 2.1.1], the functor $\mathcal{F}^j$ is compatible with products and we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}^0(\mathcal{O}_{\bar{K}}) \times \underline{M}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & \mathcal{H}(\mathcal{O}_{\bar{K}}) \arrow[d] \\
\mathcal{F}^j(\mathcal{H}^0) \times \mathcal{F}^j(\underline{M}) \arrow[r] & \mathcal{F}^j(\mathcal{H}),
\end{tikzcd}$$

where the vertical arrows are homomorphisms and the horizontal arrows are bijections preserving zero elements. For $j > 0$, the natural map $\underline{M}(\mathcal{O}_{\bar{K}}) \to \mathcal{F}^j(\underline{M})$ is an isomorphism and the kernel of the left vertical arrow is the subgroup $(\mathcal{H}^0)^j(\mathcal{O}_{\bar{K}}) \times \{0\}$. Thus the kernel of the right vertical arrow is the subgroup $(\mathcal{H}^0)^j(\mathcal{O}_{\bar{K}})$ and the lemma follows. $\square$

By this lemma and a theorem of Tian and Fargues ([15, Theorem 1.6] or [5, Proposition 6]), we have the equalities $\mathcal{C} = (\mathcal{G}^0)^{j+} = \mathcal{G}^{j+}$ for $j = p/(p-1) - pb$ and the first assertions of (1) and (2) hold.

Let us show the assertion (a). From the definition, we see that the group scheme $\mathcal{C}$ is isomorphic to $((\mathcal{G}^0)^\vee/\mathcal{G}(\mathfrak{N}))^\vee$ and thus we have the equalities

$$\begin{aligned}\deg(\mathcal{G}/\mathcal{C}) &= \deg((\mathcal{G}^0)^\vee/\mathcal{G}(\mathfrak{N})) = \deg(\mathcal{G}(\mathfrak{L}))\\ &= \deg(\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{L})/(\mathrm{Fil}^1 S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{L})),\end{aligned}$$

where the last equality follows from Lemma 2.2. The $\tilde{\mathcal{O}}_K$-module of the last term is equal to

$$\mathrm{Span}_{\tilde{\mathcal{O}}_K}((1\otimes e_1,\ldots,1\otimes e_d)\hat{P}_1 u^{e(1-w)})$$

and we see that its degree is equal to w.

For the assertion (b), we need the following lemma.

Lemma 4.4. *Let $i \leq 1$ be a positive rational number and $\mathcal{H}$ be a finite locally free group scheme over $\mathcal{O}_{\bar{K},i}$ killed by p and of order p^h. Then the Verschiebung $V_{\mathcal{H}}$ of the group scheme $\mathcal{H}$ is zero if and only if the $\mathcal{O}_{\bar{K},i}$-module $\nu_{\mathcal{H}^\vee} = \mathscr{E}xt^1_{\bar{\mathscr{S}}_i}(\mathcal{H},\mathbb{G}_{\mathrm{a}})_{\bar{\mathscr{S}}_i}$ is locally free of rank h.*

Proof. By [2, Proposition 4.3.1], we have an exact sequence of locally free R_i^{DP}-modules

$$\mathbb{D}^*(\mathcal{H})(A_i) \xrightarrow{V} \mathbb{D}^*(\mathcal{H}^{(p)})(A_i) \to \mathbb{D}^*(\mathrm{Ker}(V_{\mathcal{H}}))(A_i) \to 0,$$

where V denotes the induced map $\mathbb{D}^*(V_{\mathcal{H}})$. On the other hand, by [2, Proposition 4.3.9], we also have a natural isomorphism of R_i^{DP}-modules

$$\mathbb{D}^*(\mathcal{H}^{(p)})(A_i)/V\mathbb{D}^*(\mathcal{H})(A_i) \to \nu_{\mathcal{H}^\vee}\otimes_{\mathcal{O}_{\bar{K},i},\phi} R_i^{\mathrm{DP}}.$$

Thus the map $V_{\mathcal{H}}$ is zero if and only if the natural surjection

$$\mathbb{D}^*(\mathcal{H}^{(p)})(A_i) \to \mathbb{D}^*(\mathcal{H}^{(p)})(A_i)/V\mathbb{D}^*(\mathcal{H})(A_i)$$

is an isomorphism, and it is equivalent to saying that the R_i^{DP}-module $\nu_{\mathcal{H}^\vee}\otimes_{\mathcal{O}_{\bar{K},i},\phi} R_i^{\mathrm{DP}}$ is locally free of rank h. Since we have isomorphisms $\mathcal{O}_{\bar{K},i} \xrightarrow{\phi} R_{pi} \to R_i^{\mathrm{DP}}/\mathrm{Fil}^p R_i^{\mathrm{DP}}$, this holds if and only if the $\mathcal{O}_{\bar{K},i}$-module $\nu_{\mathcal{H}^\vee}$ is locally free of rank h. □

Now we put $\mathcal{H} = \mathcal{G}(\mathfrak{L})\times \bar{\mathscr{S}}_i$. We have an exact sequence of $\mathcal{O}_{\bar{K},i}$-modules

$$\omega_{\mathcal{H}} \to \mathbb{D}^*(\mathcal{H})(\mathcal{O}_{\bar{K},i}\to\mathcal{O}_{\bar{K},i}) \to \nu_{\mathcal{H}^\vee} \to 0$$

for any positive rational number $i \leq 1$. Note that the $\mathcal{O}_{\bar{K},i}$-module $\nu_{\mathcal{H}^\vee}$ is the first homology group of the Lie complex of the Cartier dual $\mathcal{H}^\vee$ ([2, Subsection 3.2.1]) and thus its formation is compatible with any base change. Therefore we obtain a natural isomorphism of $\mathcal{O}_{\bar{K},i}$-modules

$$(\mathcal{M}_{\mathfrak{S}}(\mathfrak{L})/\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{L}))\otimes_{\tilde{\mathcal{O}}_K}\mathcal{O}_{\bar{K},i} \to \nu_{\mathcal{H}^\vee}.$$

Let $u^{s_1}, \dots, u^{s_d}$ be the elementary divisors of the matrix P_1 and put $\mathfrak{L}_1 = \mathfrak{L}/u^e\mathfrak{L}$. Then we have isomorphisms

$$\mathcal{M}_{\mathfrak{S}}(\mathfrak{L})/\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{L}) \overset{1\otimes\phi}{\to} (1 \otimes \phi)(\phi^*\mathfrak{L}_1) \to \oplus_{l=1}^{d} k[[u]]/(u^{e-s_l}).$$

The equality $s_1 + \cdots + s_d = ew$ implies the inequality $e(1-w) \leq e - s_l$ for any l and we see that the $\mathcal{O}_{\bar{K},i}$-module $\nu_{\mathcal{H}^\vee}$ is free of rank d for $i = 1-w$. By Lemma 4.4, the Verschiebung of the group scheme $\mathcal{G}(\mathfrak{L}) \times \bar{\mathscr{S}}_{1-w}$ is zero and its Cartier dual $\mathcal{C} \times \bar{\mathscr{S}}_{1-w}$ is killed by the Frobenius. Since the group scheme $\mathcal{C}$ is of order p^d, we obtain the assertion (b).

Next we prove the assertion (d). We first show the following lemmas.

Lemma 4.5. *Let $\mathcal{H}$ be a finite flat group scheme over $\mathcal{O}_K$ killed by p^n. Then the Cartier pairing*

$$\langle\ ,\ \rangle_{\mathcal{H}} : \mathcal{H}(\mathcal{O}_{\bar{K}}) \times \mathcal{H}^\vee(\mathcal{O}_{\bar{K}}) \to \mu_{p^n}(\mathcal{O}_{\bar{K}})$$

sends the subset $\mathcal{H}_i(\mathcal{O}_{\bar{K}}) \times (\mathcal{H}^\vee)_{i'}(\mathcal{O}_{\bar{K}})$ into $(\mu_{p^n})_{i+i'}(\mathcal{O}_{\bar{K}})$ for any positive rational numbers i and i'.

Proof. Let B and J (*resp.* $B^\vee$ and $J^\vee$) be the affine algebra and the augmentation ideal of $\mathcal{H}$ (*resp.* of $\mathcal{H}^\vee$). Let $(h, h^\vee)$ be an element of the subset in the lemma. Note that we have a natural decomposition $B = \mathcal{O}_K \oplus J$ as an $\mathcal{O}_K$-module. We identify the element h with an $\mathcal{O}_K$-algebra homomorphism $h^* : B \to \mathcal{O}_{\bar{K}}$ which sends the ideal J into $m_{\bar{K}}^{\geq i}$. On the other hand, the element $h^\vee$ can be identified with an $\mathcal{O}_{\bar{K}}$-algebra homomorphism $(h^\vee)^* : \mathcal{O}_{\bar{K}}[T]/(T^{p^n} - 1) \to B \otimes \mathcal{O}_{\bar{K}}$ which sends the element $T - 1$ into $J \otimes m_{\bar{K}}^{\geq i'}$. The element $\langle h, h^\vee\rangle_{\mathcal{H}}$ of the group $\mu_{p^n}(\mathcal{O}_{\bar{K}})$ is defined by the composite $h^* \circ (h^\vee)^* : \mathcal{O}_{\bar{K}}[T]/(T^{p^n} - 1) \to \mathcal{O}_{\bar{K}}$. Hence the lemma follows. □

Lemma 4.6. *Let $\mathcal{H}$ be a finite flat group scheme over $\mathcal{O}_K$. Then the i-th lower ramification subgroup scheme $\mathcal{H}_i$ is connected for any positive rational number i.*

Proof. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}^0(\mathcal{O}_{\bar{K}}) \arrow[r] & \mathcal{H}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & \mathcal{H}^{\mathrm{et}}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d, "\wr"] & 0 \\ & & \mathcal{H}(\mathcal{O}_{\bar{K},i}) \arrow[r] & \mathcal{H}^{\mathrm{et}}(\mathcal{O}_{\bar{K},i}), & \end{tikzcd}$$

where the upper row is exact and the right vertical arrow is an isomorphism. This implies that the subgroup $\mathcal{H}_i(\mathcal{O}_{\bar{K}})$ is contained in $\mathcal{H}^0(\mathcal{O}_{\bar{K}})$ and the lemma follows. □

Put $\mathcal{D} = \mathcal{G}(\mathfrak{N}) = ((\mathcal{G}^0)^\vee)_b$ and set $\tilde{D}$ to be the inverse image of $\mathcal{D}$ by the natural epimorphism $f^\vee : \mathcal{G}^\vee \to (\mathcal{G}^0)^\vee$. Then, for any $x \in \mathcal{C}(\mathcal{O}_{\bar{K}})$ and $x^\vee \in \mathcal{G}^\vee(\mathcal{O}_{\bar{K}})$, we have the equality of Cartier pairings

$$\langle x, x^\vee\rangle_{\mathcal{G}} = \langle x, f^\vee(x^\vee)\rangle_{\mathcal{G}^0}$$

and thus the orthogonal subgroup of $\mathcal{C}(\mathcal{O}_{\bar{K}})$ with respect to the Cartier pairing $\langle\ ,\ \rangle_{\mathcal{G}}$ is $\tilde{\mathcal{D}}(\mathcal{O}_{\bar{K}})$. Moreover, since the group scheme $\mathcal{D}$ is connected by Lemma 4.6, the group scheme $\tilde{\mathcal{D}}$, which is an extension of $\mathcal{D}$ by the connected group scheme $\mathcal{T} = (\mathcal{G}^{\mathrm{et}})^\vee$, is also connected and thus it is a closed subgroup scheme of $(\mathcal{G}^\vee)^0$.

Put $\mathcal{D}' = (((\mathcal{G}^\vee)^0)^\vee)_b$. By definition, the subgroup $\mathcal{C}'(\mathcal{O}_{\bar{K}})$ is the orthogonal subgroup of $\mathcal{D}'(\mathcal{O}_{\bar{K}})$ with respect to the Cartier pairing $\langle\ ,\ \rangle_{(\mathcal{G}^\vee)^0}$. To prove the assertion (d), it is enough to show that the Cartier pairing $\langle\ ,\ \rangle_{(\mathcal{G}^\vee)^0}$ kills the subset $\tilde{\mathcal{D}}(\mathcal{O}_{\bar{K}}) \times \mathcal{D}'(\mathcal{O}_{\bar{K}})$, since both of the group schemes $\mathcal{C}'$ and $\tilde{\mathcal{D}}$ are of order p^{h-d}. For this, first note that the group scheme $\mathcal{T}$ is of multiplicative type and thus it is a closed subgroup scheme of the lower ramification subgroup scheme $((\mathcal{G}^\vee)^0)_{1/(p-1)}$. Since $b > 0$, Lemma 4.5 implies that the Cartier pairing $\langle\ ,\ \rangle_{(\mathcal{G}^\vee)^0}$ kills the subset $\mathcal{T}(\mathcal{O}_{\bar{K}}) \times \mathcal{D}'(\mathcal{O}_{\bar{K}})$. Thus we reduce ourselves to showing that the Cartier pairing $\langle\ ,\ \rangle_{(\mathcal{G}^\vee)^0/\mathcal{T}}$ kills the subset $\mathcal{D}(\mathcal{O}_{\bar{K}}) \times \mathcal{D}'(\mathcal{O}_{\bar{K}})$. Since the lower ramification subgroup is compatible with subgroups, we have the equalities $\mathcal{D} = ((\mathcal{G}^\vee)^0/\mathcal{T})_b$ and $\mathcal{D}' = (((\mathcal{G}^\vee)^0/\mathcal{T})^\vee)_b$. Putting $b = (1-w)/(p-1)$ and observing the inequality $2(1-w)/(p-1) > 1/(p-1)$, the assertion (d) follows from Lemma 4.5.

Finally we show the assertion (c). For simplicity, after replacing $\mathcal{G}$ by $\mathcal{G}^\vee$ we show the assertion for $\mathcal{C}'$. Note that we have shown the equality $\mathcal{C}' = \tilde{\mathcal{D}}$. Now we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d, equal] & \tilde{\mathcal{D}}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & \mathcal{D}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{T}(\mathcal{O}_{\bar{K}}) \arrow[r] & \mathcal{G}^\vee(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & (\mathcal{G}^0)^\vee(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & 0 \\
& & \omega_{\mathcal{G}} \otimes \mathcal{O}_{\bar{K},b} \arrow[r, "\sim"] & \omega_{\mathcal{G}^0} \otimes \mathcal{O}_{\bar{K},b}, &
\end{tikzcd}$$

where the lowest vertical arrows are the Hodge-Tate maps HT_b and the lowest horizontal arrow is an isomorphism. The proof of [9, Lemma 2.5] is valid also for $p = 2$ and the subgroup $\mathcal{D}(\mathcal{O}_{\bar{K}})$ coincides with the kernel of the lowest right vertical arrow. This implies that the subgroup $\tilde{\mathcal{D}}(\mathcal{O}_{\bar{K}})$ is the kernel of the lowest left vertical arrow and the assertion (c) follows. $\square$

Since the arguments in [9, Section 4] work verbatim also for $p = 2$, Theorem 1.1 follows from Theorem 4.1.

We can also prove the following result on anti-canonical isogenies for any p, generalizing [5, Proposition 16]. For a Barsotti-Tate group Γ over $\mathcal{O}_K$, we define its Hasse invariant $\mathrm{Ha}(\Gamma)$ by $\mathrm{Ha}(\Gamma) = \mathrm{Ha}(\Gamma[p])$.

Proposition 4.7. *Let Γ be a Barsotti-Tate group over $\mathcal{O}_K$ of height h, dimension d with $0 < d < h$ and Hasse invariant $w = \mathrm{Ha}(\Gamma)$. Suppose*

$w < 1/2$ and let $\mathcal{C}$ be the canonical subgroup of $\Gamma[p]$ as in Theorem 4.1. Let $\mathcal{E}$ be a finite flat closed subgroup scheme of $\Gamma[p]$ such that the natural map $\mathcal{C}(\mathcal{O}_{\bar{K}}) \oplus \mathcal{E}(\mathcal{O}_{\bar{K}}) \to \Gamma[p](\mathcal{O}_{\bar{K}})$ is an isomorphism. Then we have the equality $\mathrm{Ha}(\Gamma/\mathcal{E}) = p^{-1}\mathrm{Ha}(\Gamma)$ and the subgroup scheme $\Gamma[p]/\mathcal{E}$ is the canonical subgroup of $(\Gamma/\mathcal{E})[p]$.

Proof. Note that the Barsotti-Tate group $\Gamma/\mathcal{E}$ is also of height h and dimension d. The natural homomorphism $\mathcal{C} \to \Gamma[p]/\mathcal{E}$ induces an isomorphism between the generic fibers of both sides. Since the group scheme $\mathcal{C}$ is connected, the connected-etale sequence implies that the group scheme $\Gamma[p]/\mathcal{E}$ is also connected. Now we claim that the group scheme $(\Gamma[p]/\mathcal{E}) \times \mathrm{Spec}(\mathcal{O}_{K,1-w})$ is killed by the Frobenius. For this, by replacing K as in the proof of [9, Theorem 3.2], we may assume that the residue field k is perfect. Let $\mathfrak{L}$ and $\mathfrak{L}'$ be the objects of the category $\mathrm{Mod}^{1,\phi,V}_{/S}$ corresponding to the finite flat unipotent group schemes $\mathcal{C}^\vee$ and $(\Gamma[p]/\mathcal{E})^\vee$ via the anti-equivalence $\mathcal{G}(-)$, respectively. By [14, Corollary 2.2.2], the generic isomorphism $(\Gamma[p]/\mathcal{E})^\vee \to \mathcal{C}^\vee$ corresponds to an injection $\mathfrak{L} \to \mathfrak{L}'$. Then the $\mathfrak{S}_1$-modules $\wedge^d\mathfrak{L}$ and $\wedge^d\mathfrak{L}'$ are free of rank one and this injection induces an injection $\wedge^d\mathfrak{L} \to \wedge^d\mathfrak{L}'$. Hence we obtain the inequality $v_u(\det\phi_{\mathfrak{L}'}) \leq v_u(\det\phi_{\mathfrak{L}}) = ew$. As in the proof of Theorem 4.1 (b), this implies the claim and the group scheme in the claim coincides with the kernel of the Frobenius of the group scheme $(p^{-1}\mathcal{E}/\mathcal{E}) \times \mathrm{Spec}(\mathcal{O}_{K,1-w})$. Considering the Hasse invariant of the Barsotti-Tate group $(\Gamma/\mathcal{E})/(\Gamma[p]/\mathcal{E}) \simeq \Gamma$, we have the equality

$$\min\{w, 1-w\} = \min\{p\mathrm{Ha}(\Gamma/\mathcal{E}), 1-w\},$$

from which the equality of the proposition follows.

On the other hand, for any rational number j with $pw/(p-1) < j < p(1-w)/(p-1)$, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{C}(\mathcal{O}_{\bar{K}}) \arrow[d] \arrow[dr] & & \\
\Gamma[p](\mathcal{O}_{\bar{K}}) \arrow[r] & (\Gamma[p]/\mathcal{E})(\mathcal{O}_{\bar{K}}) \arrow[r] & (p^{-1}\mathcal{E}/\mathcal{E})(\mathcal{O}_{\bar{K}}),
\end{tikzcd}$$

where the oblique arrow is an isomorphism. Since $\mathcal{C} = \Gamma[p]^j$, the functoriality of j-th upper ramification subgroups implies that the closed subgroup scheme $\Gamma[p]/\mathcal{E}$ of $p^{-1}\mathcal{E}/\mathcal{E}$ is contained in $(p^{-1}\mathcal{E}/\mathcal{E})^j$. By the inequality $w/(p-1) < j < p(1-w/p)/(p-1)$, the latter group scheme is the canonical subgroup of $p^{-1}\mathcal{E}/\mathcal{E}$ and its height is equal to d. Thus the group scheme coincides with the subgroup scheme $\Gamma[p]/\mathcal{E}$ and we conclude the proof of the proposition. $\square$

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