CANONICAL SUBGROUPS VIA BREUIL-KISIN MODULES FOR $p = 2$

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Abstract. Let p be a rational prime and $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields. Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level n, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$. In this paper, we prove the existence of higher canonical subgroups for $\mathcal{G}$ with standard properties if the Hodge height of $\mathcal{G}$ is less than $1/(p^{n-2}(p+1))$, including the case of $p = 2$.

1. Introduction

Let p be a rational prime and $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields. Let k be its residue field, π be its uniformizer, e be its absolute ramification index, $\bar{K}$ be its algebraic closure and v_p be its valuation extended to $\bar{K}$ and normalized as $v_p(p) = 1$. We let $\mathbb{C}$ denote the completion of $\bar{K}$. For any valuation field F (of height one) with valuation v_F and valuation ring $\mathcal{O}_F$, put $m_F^{\geqslant i} = \{x \in F \mid v_F(x) \geq i\}$ and $\mathcal{O}_{F,i} = \mathcal{O}_F/m_F^{\geqslant i}$ for any positive real number i. We also put $\tilde{\mathcal{O}}_K = \mathcal{O}_{K,1}$, $\tilde{\mathcal{O}}_{\bar{K}} = \mathcal{O}_{\bar{K},1}$ and $\mathscr{S}_i = \mathrm{Spec}(\mathcal{O}_{K,i})$.

One of the key ingredients of the theory of p-adic Siegel modular forms is the existence theorem of canonical subgroups. Let $\mathfrak{X}$ be the formal completion of the Siegel modular variety of genus g and level prime to p over the Witt ring $W(k)$ along the special fiber, X be its Raynaud generic fiber, X^{ord} be its ordinary locus considered as an admissible open subset of X and $\mathfrak{A}$ be the universal abelian scheme over $\mathfrak{X}$. Consider the unit component $\mathfrak{A}[p^n]^0$ of the p^n-torsion of $\mathfrak{A}$ and its Raynaud generic fiber $(\mathfrak{A}[p^n]^0)^{\mathrm{rig}}$. The restriction $(\mathfrak{A}[p^n]^0)^{\mathrm{rig}}|_{X^{\mathrm{ord}}}$ is etale locally isomorphic to the constant group $(\mathbb{Z}/p^n\mathbb{Z})^g$ and it is a lift of the kernel of the n-th iterated Frobenius of the special fiber of $\mathfrak{A}$. Then the theorem asserts that this subgroup can be extended to a subgroup C_n with the same properties over a larger admissible open subset of X containing X^{ord}. In [9], the author proved the existence of such a subgroup over the locus of the Hodge height less than $p/(p+1)$ if $n = 1$ and $1/(2p^{n-1})$ if $n \geq 2$, for $p \geq 3$. The aim of this paper is to generalize the result to the case of $p = 2$.

To state the main theorem, we introduce some notation. For any finite flat (commutative) group scheme $\mathcal{G}$ (*resp.* Barsotti-Tate group Γ) over $\mathcal{O}_K$, we

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let $\omega_{\mathcal{G}}$ (*resp.* ω_Γ) denote its module of invariant differentials. Put $\deg(\mathcal{G}) = \sum_i v_p(a_i)$ by writing $\omega_{\mathcal{G}} \simeq \oplus_i \mathcal{O}_K/(a_i)$. For any positive rational number i, the Hodge-Tate map for a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ killed by p^n is defined to be the natural homomorphism

$$\mathrm{HT}_i : \mathcal{G}(\mathcal{O}_{\bar{K}}) \simeq \mathrm{Hom}(\mathcal{G}^\vee \times \mathrm{Spec}(\mathcal{O}_{\bar{K}}), \mu_{p^n} \times \mathrm{Spec}(\mathcal{O}_{\bar{K}})) \to \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K},i}$$

defined by $g \mapsto g^*(dT/T)$, where $\vee$ means the Cartier dual and $\mu_{p^n} = \mathrm{Spec}(\mathcal{O}_K[T]/(T^{p^n} - 1))$ is the group scheme of p^n-th roots of unity. We normalize the upper and the lower ramification subgroups of $\mathcal{G}$ to be adapted to the valuation v_p. Namely, writing the affine algebra of $\mathcal{G}$ as

$$\mathcal{O}_K[T_1, \ldots, T_r]/(f_1, \ldots, f_s)$$

and an r-tuple $(x_1, \ldots, x_r) \in \mathcal{O}_{\bar{K}}^r$ as $\underline{x}$, we put

$$\mathcal{G}^j(\mathcal{O}_{\bar{K}}) = \mathcal{G}(\mathcal{O}_{\bar{K}}) \cap \{\underline{x} \in \mathcal{O}_{\bar{K}}^r \mid v_p(f_l(\underline{x})) \geq j \text{ for any } l\}^0,$$
$$\mathcal{G}_i(\mathcal{O}_{\bar{K}}) = \mathrm{Ker}(\mathcal{G}(\mathcal{O}_{\bar{K}}) \to \mathcal{G}(\mathcal{O}_{\bar{K},i})),$$

where $(-)^0$ in the first equality means the geometric connected component as an affinoid variety over K containing the zero section (see [1, Section 2]). We also put $\mathcal{G}^{j+}(\mathcal{O}_{\bar{K}}) = \cup_{j'>j} \mathcal{G}^{j'}(\mathcal{O}_{\bar{K}})$ for any non-negative rational number j. The scheme-theoretic closure of $\mathcal{G}^j(\mathcal{O}_{\bar{K}})$ in $\mathcal{G}$ is denoted by $\mathcal{G}^j$ and define $\mathcal{G}^{j+}$ and $\mathcal{G}_i$ similarly. Finally, for any truncated Barsotti-Tate group $\mathcal{G}$ ([11]) of level n, height h and dimension d over $\mathcal{O}_K$ with $d < h$, we define the Hodge height $\mathrm{Hdg}(\mathcal{G})$ to be the truncated valuation $v_p(\det(V)) \in [0,1]$ of the determinant of the natural action of the Verschiebung V of the group scheme $\mathcal{G}[p]^\vee \times \mathrm{Spec}(\tilde{\mathcal{O}}_K)$ on the free $\tilde{\mathcal{O}}_K$-module of finite rank $\mathrm{Lie}(\mathcal{G}[p]^\vee \times \mathrm{Spec}(\tilde{\mathcal{O}}_K))$. Then our main theorem is the following, which is proved in [9] except the case of $p = 2$ (note that for $p \geq 3$, it is also proved by Fargues ([5]) under a slightly stronger assumption on w).

Theorem 1.1. *Let p be a rational prime and $K/\mathbb{Q}_p$ be an extension of complete discrete valuation fields. Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level n, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$ and Hodge height $w = \mathrm{Hdg}(\mathcal{G})$.*

(1) If $w < 1/(p^{n-2}(p+1))$, then there exists a finite flat closed subgroup scheme $\mathcal{C}_n$ of $\mathcal{G}$ of order p^{nd} over $\mathcal{O}_K$, which we call the level n canonical subgroup of $\mathcal{G}$, such that $\mathcal{C}_n \times \mathscr{S}_{1-p^{n-1}w}$ coincides with the kernel of the n-th iterated Frobenius homomorphism F^n of $\mathcal{G} \times \mathscr{S}_{1-p^{n-1}w}$. Moreover, the group scheme $\mathcal{C}_n$ has the following properties:
- *(a)* $\deg(\mathcal{G}/\mathcal{C}_n) = w(p^n - 1)/(p-1)$.
- *(b) Put $\mathcal{C}'_n$ to be the level n canonical subgroup of $\mathcal{G}^\vee$. Then we have the equality of subgroup schemes $\mathcal{C}'_n = (\mathcal{G}/\mathcal{C}_n)^\vee$, or equivalently $\mathcal{C}_n(\mathcal{O}_{\bar{K}}) = \mathcal{C}'_n(\mathcal{O}_{\bar{K}})^\perp$, where $\perp$ means the orthogonal subgroup with respect to the Cartier pairing.*
- *(c) If $n = 1$, then $\mathcal{C}_1 = \mathcal{G}_{(1-w)/(p-1)} = \mathcal{G}^{pw/(p-1)+}$.*

(2) *If* $w < (p-1)/(p^n-1)$, *then the subgroup scheme* $\mathcal{C}_n$ *also satisfies the following:*
   (d) *the group* $\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ *is isomorphic to* $(\mathbb{Z}/p^n\mathbb{Z})^d$.
   (e) *The scheme-theoretic closure of* $\mathcal{C}_n(\mathcal{O}_{\bar{K}})[p^i]$ *in* $\mathcal{C}_n$ *coincides with the subgroup scheme* $\mathcal{C}_i$ *of* $\mathcal{G}[p^i]$ *for* $1 \leq i \leq n-1$.

(3) *If* $w < (p-1)/p^n$, *then the subgroup* $\mathcal{C}_n(\mathcal{O}_{\bar{K}})$ *coincides with the kernel of the Hodge-Tate map* $\mathrm{HT}_{n-w(p^n-1)/(p-1)}$.

(4) *If* $w < 1/(2p^{n-1})$, *then the subgroup scheme* $\mathcal{C}_n$ *coincides with the upper ramification subgroup scheme* $\mathcal{G}^{j+}$ *for any* j *satisfying*

$$pw(p^n-1)/(p-1)^2 \leq j < p(1-w)/(p-1).$$

We also show the uniqueness of the canonical subgroup $\mathcal{C}_n$ for $w < p(p-1)/(p^{n+1}-1)$ (Proposition 3.8). From Theorem 1.1, we can show the following corollary just as in the proof of [9, Corollary 1.2].

Corollary 1.2. *Let* $K/\mathbb{Q}_p$ *be an extension of complete discrete valuation fields. Let* $\mathfrak{X}$ *be an admissible formal scheme over* $\mathrm{Spf}(\mathcal{O}_K)$ *which is quasi-compact and* $\mathfrak{G}$ *be a truncated Barsotti-Tate group of level* n *over* $\mathfrak{X}$ *of constant height* h *and dimension* d *with* $0 < d < h$. *We let* X *and* G *denote the Raynaud generic fibers of the formal schemes* $\mathfrak{X}$ *and* $\mathfrak{G}$, *respectively. For a finite extension* L/K *and* $x \in X(L)$, *we put* $\mathfrak{G}_x = \mathfrak{G} \times_{\mathfrak{X},x} \mathrm{Spf}(\mathcal{O}_L)$, *where we let* x *also denote the map* $\mathrm{Spf}(\mathcal{O}_L) \to \mathfrak{X}$ *obtained from* x *by taking the scheme-theoretic closure and the normalization. For a non-negative rational number* r, *let* $X(r)$ *be the admissible open subset of* X *defined by*

$$X(r)(\bar{K}) = \{x \in X(\bar{K}) \mid \mathrm{Hdg}(\mathfrak{G}_x) < r\}.$$

Put $r_1 = p/(p+1)$ *and* $r_n = 1/(2p^{n-1})$ *for* $n \geq 2$.

Then there exists an admissible open subgroup C_n *of* $G|_{X(r_n)}$ *such that, etale locally on* $X(r_n)$, *the rigid-analytic group* C_n *is isomorphic to the constant group* $(\mathbb{Z}/p^n\mathbb{Z})^d$ *and, for any finite extension* L/K *and* $x \in X(L)$, *the fiber* $(C_n)_x$ *coincides with the generic fiber of the level* n *canonical subgroup of* $\mathfrak{G}_x$.

The basic strategy of the proof of the main theorem is the same as in [9]: we construct the level one canonical subgroup by lifting the conjugate Hodge filtration of a reduction of $\mathcal{G}$ to the subobject of the associated Breuil-Kisin module of $\mathcal{G}$. Since the canonical subgroup is required to be a lift of the Frobenius kernel of a reduction of $\mathcal{G}$, it should be connected. Thus we may use a classification of connected finite flat group schemes allowing the case of $p = 2$ which is due to Kisin ([15]), though all the arguments are valid without the connectedness assumption if we use the theorem of Kim ([12]) instead.

The main obstacle to generalize the results of [9] to the case of $p = 2$ is the use in [9] of the following two results, which are proved only for $p \geq 3$ in [8]: One is the ramification correspondence theorem ([8, Theorem 1.1]), which was used to show that the canonical subgroup coincides with ramification subgroups and that its construction can be reduced to the case of perfect residue field. The other is a congruence of the defining equations of finite flat group schemes of equal and mixed characteristics associated to the same Kisin module ([8, Corollary 4.6]), which was used to show that the canonical subgroup coincides with a lift of a Frobenius kernel and that it has uniqueness properties. While the author recently proved the ramification correspondence theorem also for $p = 2$ ([10]), the latter is still available only for $p \geq 3$ and we need to bypass this.

The idea we adopt here for this purpose is to begin the construction of the canonical subgroup of a truncated Barsotti-Tate group $\mathcal{G}$ of level one lifting the conjugate Hodge filtration, with a dual situation. The resulting subgroup is called the conjugate Hodge subgroup, and we define the canonical subgroup of $\mathcal{G}$ as the orthogonal subgroup with respect to Cartier duality of the conjugate Hodge subgroup of the Cartier dual $(\mathcal{G}^0)^\vee$ of the unit component $\mathcal{G}^0$. By this passage to the dual, we can show the coincidence with a lift of a Frobenius kernel and the uniqueness of the canonical subgroup using [5, Proposition 1] which is valid for any p. By this uniqueness, we can show the compatibility of the canonical subgroup with any finite base extension, which in turn enables us to show the coincidence with ramification subgroups easily, without any use of ramification correspondence theorems.

2. Classification of unipotent finite flat group schemes

In this section, we assume that the residue field k of K is perfect. For $p \geq 3$, we have a classification theory of Barsotti-Tate groups and finite flat group schemes over $\mathcal{O}_K$ due to Breuil ([3], [4]) and Kisin ([13], [14]) in terms of so-called Breuil-Kisin modules. Kisin ([15]) also extended this classification to the case of $p = 2$ and where groups are connected, using Zink's classification of formal Barsotti-Tate groups ([21], [22]). In this section, we briefly recall this result of Kisin. Since we adopt a contravariant notation contrary to his, what we describe here is a classification of unipotent Barsotti-Tate groups and unipotent finite flat group schemes.

Let $W = W(k)$ be the Witt ring of k and φ be its natural Frobenius endomorphism which lifts the p-th power map of k. Natural φ-semilinear Frobenius endomorphisms of various W-algebras are denoted also by φ. Let $E(u) \in W[u]$ be the Eisenstein polynomial of π over W. Let us fix once and for all a system $\{\pi_n\}_{n \geq 0}$ of p-power roots of π in $\bar{K}$ with $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Put $K_\infty = \cup_{n \geq 0} K(\pi_n)$, $\mathfrak{S} = W[[u]]$ and $\mathfrak{S}_1 = k[[u]]$. We write the φ-semilinear continuous ring endomorphisms of the latter two rings defined by $u \mapsto u^p$ also as φ. Then a Kisin module is an $\mathfrak{S}$-module $\mathfrak{M}$ endowed with a φ-semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$. We write $\varphi_{\mathfrak{M}}$ also as φ if no confusion may occur. We follow the notation of [8, Subsection 2.1] and [9, Subsection 2.1]. In particular, we have categories $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}}$, $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_1}$, $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$ of Kisin modules of E-height ≤ 1 and a category $\mathrm{Mod}^{1,\varphi}_{/B}$ for any $k[[u]]$-algebra B.

Let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$ and put $\varphi^*\mathfrak{M}=\mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$. Then the map $1\otimes\varphi:\varphi^*\mathfrak{M}\to\mathfrak{M}$ is injective and we have a unique map $\psi_{\mathfrak{M}}:\mathfrak{M}\to\varphi^*\mathfrak{M}$ satisfying $(1\otimes\varphi)\circ\psi_{\mathfrak{M}}=E(u)$. We say $\mathfrak{M}$ is V-nilpotent if the composite

$$\varphi^{n-1*}(\psi_{\mathfrak{M}})\circ\varphi^{n-2*}(\psi_{\mathfrak{M}})\circ\cdots\circ\psi_{\mathfrak{M}}:\mathfrak{M}\to\varphi^{n*}\mathfrak{M}$$

factors through the submodule $(p,u)\varphi^{n*}\mathfrak{M}$ for any sufficiently large n. Similarly, we say an object $\mathfrak{M}$ of the category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}}$ is topologically V-nilpotent if the same condition holds. The full subcategories of V-nilpotent (*resp.* topologically V-nilpotent) objects are denoted by $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$ and $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_\infty}$ (*resp.* $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}}$). Note that these notions are called connected and formal in [15], respectively.

Let S be the p-adic completion of the divided power envelope of $W[u]$ with respect to the ideal $(E(u))$. The ring S has a natural filtration $\mathrm{Fil}^1 S$ induced by the divided power structure, a φ-semilinear Frobenius endomorphism denoted also by φ and a φ-semilinear map $\varphi_1:\mathrm{Fil}^1 S\to S$, since the inclusion $\varphi(\mathrm{Fil}^1 S)\subseteq pS$ holds for any p. Then a Breuil module is an S-module $\mathcal{M}$ endowed with an S-submodule $\mathrm{Fil}^1\mathcal{M}$ containing $(\mathrm{Fil}^1 S)\mathcal{M}$ and a φ-semilinear map $\varphi_{1,\mathcal{M}}:\mathrm{Fil}^1\mathcal{M}\to\mathcal{M}$ satisfying some compatibility conditions. The map $\varphi_{1,\mathcal{M}}$ is also denoted by φ_1 if there is no risk of confusion. We also have categories of Breuil modules $\mathrm{Mod}^{1,\varphi}_{/S}$, $\mathrm{Mod}^{1,\varphi}_{/S_1}$ and $\mathrm{Mod}^{1,\varphi}_{/S_\infty}$ (for the definitions, see [8, Subsection 2.1]. Though there these are defined only for $p\geq 3$, the definitions are valid also for the case of $p=2$). For any object $\mathcal{M}$ of these categories, we define a φ-semilinear map $\varphi_{\mathcal{M}}:\mathcal{M}\to\mathcal{M}$ by $\varphi_{\mathcal{M}}(x)=\varphi_1(E(u))^{-1}\varphi_1(E(u)x)$, which we abusively write as φ.

Let $\mathcal{M}$ be an object of the category $\mathrm{Mod}^{1,\varphi}_{/S}$ and put $\varphi^*\mathcal{M}=S\otimes_{\varphi,S}\mathcal{M}$. Then the map $1\otimes\varphi:\varphi^*\mathcal{M}\to\mathcal{M}$ is injective and we have a unique map $\psi_{\mathcal{M}}:\mathcal{M}\to\varphi^*\mathcal{M}$ satisfying $(1\otimes\varphi)\circ\psi_{\mathcal{M}}=p$. Then we say $\mathcal{M}$ is topologically V-nilpotent if the composite

$$\varphi^{n-1*}(\psi_{\mathcal{M}})\circ\varphi^{n-2*}(\psi_{\mathcal{M}})\circ\cdots\circ\psi_{\mathcal{M}}:\mathcal{M}\to\varphi^{n*}\mathcal{M}$$

factors through the submodule $(p,\mathrm{Fil}^1 S)\varphi^{n*}\mathcal{M}$ for any sufficiently large n. This notion is called S-window over $\mathcal{O}_K$ in [15] and [21]. The full subcategory of topologically V-nilpotent objects is denoted by $\mathrm{Mod}^{1,\varphi,V}_{/S}$. For any Kisin module $\mathfrak{M}$, define a Breuil module $\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})=S\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$ by putting

$$\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})=\mathrm{Ker}(S\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}\overset{1\otimes\varphi}{\to}S/\mathrm{Fil}^1 S\otimes_{\mathfrak{S}}\mathfrak{M}),$$

$$\varphi_1:\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})\overset{1\otimes\varphi}{\to}\mathrm{Fil}^1 S\otimes_{\mathfrak{S}}\mathfrak{M}\overset{\varphi_1\otimes 1}{\to}\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}).$$

This gives exact functors $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}\to\mathrm{Mod}^{1,\varphi}_{/S_\infty}$ and $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}}\to\mathrm{Mod}^{1,\varphi}_{/S}$, which are both denoted by $\mathcal{M}_{\mathfrak{S}}(-)$, and the latter induces a functor $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}}\to\mathrm{Mod}^{1,\varphi,V}_{/S}$ ([15, Proposition 1.2.5]).

We can associate Galois representations to Kisin and Breuil modules. Consider the ring $R = \varprojlim(\tilde{\mathcal{O}}_{\bar{K}} \leftarrow \tilde{\mathcal{O}}_{\bar{K}} \leftarrow \cdots)$, where the transition maps are p-th power maps. An element $r \in R$ is written as $r = (r_n)_{n\geq 0}$ with $r_n \in \tilde{\mathcal{O}}_{\bar{K}}$, and define $r^{(0)} \in \mathcal{O}_{\mathbb{C}}$ by $r^{(0)} = \lim_{n\to\infty} \hat{r}_n^{p^n}$, where $\hat{r}_n$ is a lift of r_n in $\mathcal{O}_{\bar{K}}$. Then the ring R is a complete valuation ring of characteristic p with its valuation defined by $v_R(r) = v_p(r^{(0)})$ whose fraction field is algebraically closed, and we put $m_R^{\geqslant i} = \{r \in R \mid v_R(r) \geq i\}$ and $R_i = R/m_R^{\geqslant i}$. We have a natural ring surjection $W(R) \to \mathcal{O}_{\mathbb{C}}$ which lifts the zeroth projection $\mathrm{pr}_0 : R \to \tilde{\mathcal{O}}_{\bar{K}}$. The ring A_{crys} is the p-adic completion of the divided power envelope of $W(R)$ with respect to the kernel of this surjection. Thus we have the induced surjection $A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}}$. Put $\underline{\pi} = (\pi, \pi_1, \pi_2, \ldots) \in R$ and consider the rings $W(R)$ and A_{crys} as $\mathfrak{S}$-algebras by the map $u \mapsto [\underline{\pi}]$. In particular, we consider the ring $k[[u]]$ as a subring of R by the map $u \mapsto \underline{\pi}$ and let v_R also denote the induced valuation on the former ring, which satisfies $v_R(u) = 1/e$.

For any objects $\mathfrak{M}$ of the category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}}$ and $\mathcal{M}$ of $\mathrm{Mod}^{1,\varphi}_{/S}$, we associate to them G_{K_∞}-modules

$$T^*_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W(R)),$$
$$T^*_{\mathrm{crys}}(\mathcal{M}) = \mathrm{Hom}_{S,\mathrm{Fil}^1,\varphi}(\mathcal{M}, A_{\mathrm{crys}})$$

([6, Proposition B1.8.3] and [15, Subsection 1.2.6]). If the $\mathfrak{S}$-module $\mathfrak{M}$ is free of rank h, then the $\mathbb{Z}_p$-module $T^*_{\mathfrak{S}}(\mathfrak{M})$ is also free of rank h ([13, Corollary 2.1.4]). We also have a natural injection of G_{K_∞}-modules $T^*_{\mathfrak{S}}(\mathfrak{M}) \to T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}))$ defined by $f \mapsto 1 \otimes (\varphi \circ f)$ and this is a bijection if $\mathfrak{M}$ is topologically V-nilpotent ([15, Proposition 1.2.7]). Similarly, for any object $\mathfrak{M}$ of the category $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$, we have the associated G_{K_∞}-module

$$T^*_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W(R) \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$

Put $\mathscr{S}_n = \mathrm{Spec}(\mathcal{O}_{K,n})$, $S_n = S/p^n S$ and $E_n = \mathrm{Spec}(S_n)$. Let us consider the big crystalline site $\mathrm{CRYS}(\mathscr{S}_n/E_n)$ with the fppf topology and its topos $(\mathscr{S}_n/E_n)_{\mathrm{CRYS}}$. For any Barsotti-Tate group Γ over $\mathcal{O}_K$, we have the contravariant Dieudonné crystal $\mathbb{D}^*(\Gamma \times \mathscr{S}_n) = \mathscr{E}xt^1_{\mathscr{S}_n/E_n}(\underline{\Gamma \times \mathscr{S}_n}, \mathcal{O}_{\mathscr{S}_n/E_n})$ (for the notation, see [2]). We put

$$\mathbb{D}^*(\Gamma)(S \to \mathcal{O}_K) = \varprojlim_n \mathbb{D}^*(\Gamma \times \mathscr{S}_n)(S_n \to \mathcal{O}_{K,n}).$$

This module is considered as an object $\mathrm{Mod}(\Gamma)$ of the category $\mathrm{Mod}^{1,\varphi}_{/S}$ with the natural φ-semilinear Frobenius map induced by the Frobenius of $\Gamma \times \mathscr{S}_1$ and the filtration defined as the inverse image of the natural inclusion

$$\omega_\Gamma \subseteq \varprojlim_n \mathbb{D}^*(\Gamma \times \mathscr{S}_n)(\mathcal{O}_{K,n} \to \mathcal{O}_{K,n}).$$

The A_{crys}-module

$$\mathbb{D}^*(\Gamma)(A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}}) = \varprojlim_n \mathbb{D}^*(\Gamma \times \mathscr{S}_n)(A_{\mathrm{crys}}/p^n A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C},n})$$

also has a φ-semilinear Frobenius map and a filtration defined in the same way. Similarly, for any finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, the S-module

$$\mathbb{D}^*(\mathcal{G})(S \to \mathcal{O}_K) = \varprojlim_n \mathbb{D}^*(\mathcal{G} \times \mathscr{S}_n)(S_n \to \mathcal{O}_{K,n})$$

is endowed with a natural φ-semilinear Frobenius map which is induced by the Frobenius of the group scheme $\mathcal{G} \times \mathscr{S}_1$ and is also denoted by φ, and a filtration defined by the submodule

$$\mathrm{Fil}^1\mathbb{D}^*(\mathcal{G})(S \to \mathcal{O}_K) = \varprojlim_n \mathscr{E}xt^1_{\mathscr{S}_n/E_n}(\underline{\mathcal{G} \times \mathscr{S}_n}, \mathcal{J}_{\mathscr{S}_n/E_n})(S_n \to \mathcal{O}_{K,n}),$$

where $\mathcal{J}_{\mathscr{S}_n/E_n}$ is the canonical divided power ideal sheaf of the structure sheaf $\mathcal{O}_{\mathscr{S}_n/E_n}$.

We say a Barsotti-Tate group or a finite locally free group scheme is unipotent if its Cartier dual is connected. We let $(\mathrm{BT}/\mathcal{O}_K)^{\mathrm{u}}$ (*resp.* $(p\text{-Gr}/\mathcal{O}_K)^{\mathrm{u}}$) denote the category of unipotent Barsotti-Tate groups (*resp.* the category of unipotent finite flat group schemes killed by some p-power) over $\mathcal{O}_K$. If a Barsotti-Tate group Γ over $\mathcal{O}_K$ is unipotent, then the object $\mathrm{Mod}(\Gamma)$ is topologically V-nilpotent ([15, Lemma 1.1.3]). Moreover, we have the following classification theorem of unipotent Barsotti-Tate groups and unipotent finite flat group schemes, whose second assertion follows from the first assertion by an argument of taking a resolution ([15, Subsection 1.3]).

Theorem 2.1. *(1) ([15], Theorem 1.2.8) There exists an anti-equivalence of exact categories*

$$\mathcal{G}(-) : \mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}} \to (\mathrm{BT}/\mathcal{O}_K)^{\mathrm{u}}$$

with a natural isomorphism $\mathrm{Mod}(\mathcal{G}(\mathfrak{M})) \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$. *Moreover, we also have a natural isomorphism of* G_{K_∞}*-modules*

$$\varepsilon_{\mathfrak{M}} : T_p(\mathcal{G}(\mathfrak{M})) \to T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})).$$

(2) ([15], Theorem 1.3.9) There exists an anti-equivalence of exact categories

$$\mathcal{G}(-) : \mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_\infty} \to (p\text{-Gr}/\mathcal{O}_K)^{\mathrm{u}}$$

with a natural isomorphism of G_{K_∞}*-modules*

$$\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \to T^*_{\mathfrak{S}}(\mathfrak{M}).$$

On the other hand, for any $k[[u]]$-algebra B, we have an exact anti-equivalence $\mathcal{H}(-)$ from the category $\mathrm{Mod}^{1,\varphi}_{/B}$ to a category of finite locally free group schemes over B whose Verschiebung is the zero map ([7, Théorème 7.4]. See also [8, Subsection 3.2]). Moreover, for $B = k[[u]]$ and $\mathfrak{M} \in \mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$, the anti-equivalences $\mathcal{G}(-)$ and $\mathcal{H}(-)$ are related by the natural isomorphism

$$\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \to T^*_{\mathfrak{S}}(\mathfrak{M}) = \mathcal{H}(\mathfrak{M})(R)$$

of Theorem 2.1 (2). We define the lower ramification subgroups of the group scheme $\mathcal{H}(\mathfrak{M})$ to be adapted to the valuation v_R. Namely, we define

$$\mathcal{H}(\mathfrak{M})_i(R) = \mathrm{Ker}(\mathcal{H}(\mathfrak{M})(R) \to \mathcal{H}(\mathfrak{M})(R_i))$$

for any positive rational number i. We also define $\deg(\mathcal{H}(\mathfrak{M}))$ by writing $\omega_{\mathcal{H}(\mathfrak{M})} \simeq \oplus_i k[[u]]/(a_i)$ and putting $\deg(\mathcal{H}(\mathfrak{M})) = \sum_i v_R(a_i)$.

Let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}}$. If we identify an element $g \in T_p(\mathcal{G}(\mathfrak{M}))$ with a homomorphism of Barsotti-Tate groups from $\mathbb{Q}_p/\mathbb{Z}_p$ to $\mathcal{G}(\mathfrak{M})$ over $\mathcal{O}_{\mathbb{C}}$, then by the natural isomorphism $\mathrm{Mod}(\mathcal{G}(\mathfrak{M})) \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$ the element $\varepsilon_{\mathfrak{M}}(g)$ is identified with the induced map

$$\mathbb{D}^*(g) : \mathbb{D}^*(\mathcal{G}(\mathfrak{M}))(A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}}) \to \mathbb{D}^*(\mathbb{Q}_p/\mathbb{Z}_p)(A_{\mathrm{crys}} \to \mathcal{O}_{\mathbb{C}}) = A_{\mathrm{crys}}.$$

A similar argument as in the proof of [14, Proposition 1.1.11] shows that for any exact sequence

$$0 \to \mathfrak{N} \to \mathfrak{N}' \to \mathfrak{M} \to 0$$

of Kisin modules such that $\mathfrak{N}$ and $\mathfrak{N}'$ are objects of the category of $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}}$ and $\mathfrak{M}$ is of $\mathrm{Mod}^{1,\varphi}_{/\mathfrak{S}_\infty}$, the functor $\mathcal{M}_{\mathfrak{S}}(-)$ induces an exact sequence of Breuil modules

$$0 \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{N}) \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{N}') \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to 0.$$

This and [2, Lemme 4.2.5 (ii)] imply that, for any object $\mathfrak{M}$ of the category $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_\infty}$, there exists a natural isomorphism of S-modules

$$\mathbb{D}^*(\mathcal{G}(\mathfrak{M}))(S \to \mathcal{O}_K) \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$$

which is compatible with Fil^1 and φ.

Let us consider the k-algebra isomorphism $k[[u]]/(u^e) \to \tilde{\mathcal{O}}_K$ defined by $u \mapsto \pi$. Using this isomorphism, we identify the k-algebras of both sides. Then we can show the following lemma just as in the proof of [9, Lemma 2.4].

Lemma 2.2. *(1) Let $\mathcal{G}$ be a unipotent truncated Barsotti-Tate group of level one over $\mathcal{O}_K$ and $\mathfrak{M}$ be the corresponding object of* $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$ *via the anti-equivalence $\mathcal{G}(-)$. Then there exist natural isomorphisms of $\tilde{\mathcal{O}}_K$-modules*

$$\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})/(\mathrm{Fil}^1 S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to \omega_{\mathcal{G}}, \quad \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})/\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to \mathrm{Lie}(\mathcal{G}^\vee).$$

(2) Let $\mathcal{G}$ be a unipotent finite flat group scheme over $\mathcal{O}_K$ killed by p and $\mathfrak{M}$ be the corresponding object of the category $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$ *via the anti-equivalence $\mathcal{G}(-)$. Then there exists a natural isomorphism of $\tilde{\mathcal{O}}_K$-modules* $\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})/(\mathrm{Fil}^1 S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to \omega_{\mathcal{G}}$.

(3) Let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$. *Then we have the equalities*

$$\deg(\mathcal{G}(\mathfrak{M})) = \deg(\mathcal{H}(\mathfrak{M})) = v_R(\det(\varphi_{\mathfrak{M}})).$$

Let $\mathcal{G}$ be a unipotent truncated Barsotti-Tate group of level one, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$ and Hodge height $\mathrm{Hdg}(\mathcal{G}) = w$. Let $\mathfrak{M}$ be the object of the category $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$ corresponding to $\mathcal{G}$ via the anti-equivalence $\mathcal{G}(-)$. Put $\mathfrak{M}_1 = \mathfrak{M}/u^e\mathfrak{M}$ and

$$\mathrm{Fil}^1\mathfrak{M}_1 = \mathrm{Im}(1 \otimes \varphi : \tilde{\mathcal{O}}_K \otimes_{\varphi,\tilde{\mathcal{O}}_K} \mathfrak{M}_1 \to \mathfrak{M}_1).$$

Then the modules $\mathfrak{M}_1$ and $\mathrm{Fil}^1\mathfrak{M}_1$ are naturally considered as objects of the category $\mathrm{Mod}^{1,\varphi}_{/\tilde{\mathcal{O}}_K}$. By Lemma 2.2 (1), we can show that there exists a natural isomorphism of $\tilde{\mathcal{O}}_K$-modules $\mathrm{Lie}(\mathcal{G}^\vee) \to \mathrm{Fil}^1\mathfrak{M}_1$ as in [9, Subsection 2.3]. Thus the $\tilde{\mathcal{O}}_K$-module $\mathrm{Fil}^1\mathfrak{M}_1$ is free of rank $h-d$. Moreover, we obtain an exact sequence of φ-modules over $\tilde{\mathcal{O}}_K$

$$0 \to \mathrm{Fil}^1\mathfrak{M}_1 \to \mathfrak{M}_1 \to \mathfrak{M}_1/\mathrm{Fil}^1\mathfrak{M}_1 \to 0$$

which splits as a sequence of $\tilde{\mathcal{O}}_K$-modules and the equality of truncated valuation $v_p(\det(\varphi_{\mathrm{Fil}^1\mathfrak{M}_1})) = w$. We can also prove the following lemma as in the proof of [9, Lemma 2.5].

Lemma 2.3. *Let $\mathcal{G}$ be a unipotent truncated Barsotti-Tate group of level one over $\mathcal{O}_K$ and $\mathfrak{M}$ be the object of $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$ which corresponds to $\mathcal{G}$ via the anti-equivalence $\mathcal{G}(-)$. Then the composite*

$$\mathcal{G}(\mathcal{O}_{\bar{K}}) \overset{\varepsilon_{\mathfrak{M}}}{\to} \mathcal{H}(\mathfrak{M})(R) \to \mathrm{Hom}_{\tilde{\mathcal{O}}_K}(\mathrm{Fil}^1\mathfrak{M}_1, R/m_R^{\geqslant i}) \to \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K},i}$$

coincides with the Hodge-Tate map HT_i for any $i \leq 1$.

Remark 2.4. A similar classification for finite flat group schemes over $\mathcal{O}_K$ via Breuil-Kisin modules allowing the case of $p = 2$ and with non-trivial etale part is obtained independently by Kim ([12]), Lau ([16]) and Liu ([18]). By using [12, Corollary 4.3], we can generalize Lemma 2.2 and Lemma 2.3 to this case.

3. Canonical subgroups

In this section, we prove Theorem 1.1. The proof is a modification of the argument in [9], where we had to exclude the case of $p = 2$. We begin with a consideration on a dual situation, as below. By this passage to the dual, we can replace the use of the congruence of the defining equations of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{H}(\mathfrak{M})$ ([8, Corollary 4.6]) in [9] to show the coincidence with a lift of a Frobenius kernel and the uniqueness of the level one canonical subgroup, by that of [5, Proposition 1] which is valid for any p. This uniqueness in turn replaces the use of the ramification correspondence theorem ([8, Theorem 1.1]) to show that the canonical subgroup coincides with ramification subgroups.

Suppose that the residue field k of K is perfect. Let $\mathcal{G}$ be a unipotent truncated Barsotti-Tate group of level one, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$ and Hodge height $\mathrm{Hdg}(\mathcal{G}) = w$. Let $\mathfrak{M}$ be the object of the category $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$ corresponding to $\mathcal{G}$ via the anti-equivalence $\mathcal{G}(-)$.

Consider the objects $\mathfrak{M}_1$ and $\mathrm{Fil}^1\mathfrak{M}_1$ of the category $\mathrm{Mod}^{1,\varphi}_{/\tilde{\mathcal{O}}_K}$ as in the previous section. Then, a verbatim argument as in the proof of [9, Lemma 3.3] shows the following proposition.

Proposition 3.1. *Let the notation be as above. For any finite flat closed subgroup scheme $\mathcal{D}$ of $\mathcal{G}$ over $\mathcal{O}_K$, let $\mathfrak{L}$ be the subobject of $\mathfrak{M}$ in the category $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$ corresponding to the quotient $\mathcal{G}/\mathcal{D}$. Suppose $w < p/(p+1)$. Then there exists a unique $\mathcal{D}$ satisfying $\mathfrak{L}/u^{e(1-w)}\mathfrak{L} = \mathrm{Fil}^1\mathfrak{M}_1/u^{e(1-w)}\mathrm{Fil}^1\mathfrak{M}_1$. Moreover, for this unique $\mathcal{D}$, we have the equality $v_R(\det(\varphi_{\mathfrak{L}})) = w$.*

We temporarily refer to the unique $\mathcal{D}$ in Proposition 3.1 as the conjugate Hodge subgroup of $\mathcal{G}$, which will be shown to be equal to the canonical subgroup of $\mathcal{G}$.

Lemma 3.2. *Let $\mathcal{G}$ be as above and $\mathcal{D}$ be the conjugate Hodge subgroup of $\mathcal{G}$. Then $\mathcal{D}$ is the unique finite flat closed subgroup scheme of $\mathcal{G}$ over $\mathcal{O}_K$ such that $(\mathcal{G}/\mathcal{D})^\vee \times \mathscr{S}_{1-w}$ coincides with the Frobenius kernel of $\mathcal{G}^\vee \times \mathscr{S}_{1-w}$. In particular, the construction of the conjugate Hodge subgroup is compatible with any finite extension of K.*

Proof. Put $i = 1-w$. First let us show that the group scheme $(\mathcal{G}/\mathcal{D})^\vee \times \mathscr{S}_i$ coincides with the Frobenius kernel of $\mathcal{G}^\vee \times \mathscr{S}_i$. By comparing orders, it is enough to show that the group scheme $(\mathcal{G}/\mathcal{D})^\vee \times \mathscr{S}_i$ is contained in the Frobenius kernel of $\mathcal{G}^\vee \times \mathscr{S}_i$. By [5, Proposition 1], it is equivalent to saying that the natural map

$$\omega_{\mathcal{G}/\mathcal{D}} \otimes \mathcal{O}_{K,i} \to \omega_{\mathcal{G}} \otimes \mathcal{O}_{K,i}$$

is zero. By Lemma 2.2 (2), this map can be identified with the top horizontal arrow of the commutative diagram

$$\begin{tikzcd}
(\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{L})/(\mathrm{Fil}^1 S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{L})) \otimes \mathcal{O}_{K,i} \arrow[r] \arrow[d] & (\mathrm{Fil}^1\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})/(\mathrm{Fil}^1 S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{L})) \otimes \mathcal{O}_{K,i} \arrow[d, hook] \\
(\mathcal{M}_{\mathfrak{S}}(\mathfrak{L})/(\mathrm{Fil}^1 S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{L})) \otimes \mathcal{O}_{K,i} \arrow[r, hook] & (\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})/(\mathrm{Fil}^1 S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})) \otimes \mathcal{O}_{K,i},
\end{tikzcd}$$

where the bottom horizontal arrow and the right vertical arrow are injective. Let $\delta_1, \ldots, \delta_{h-d}$ be a basis of the $\mathfrak{S}$-module $\mathfrak{L}$ and D be the element of $M_{h-d}(k[[u]])$ satisfying

$$\varphi(\delta_1, \ldots, \delta_{h-d}) = (\delta_1, \ldots, \delta_{h-d})D.$$

From the definition of the functor $\mathcal{M}_{\mathfrak{S}}(-)$, we see that the module on the top left corner of the diagram is equal to the $\mathcal{O}_{K,i}$-module

$$\mathrm{Span}_{\tilde{\mathcal{O}}_K}((1 \otimes \delta_1, \ldots, 1 \otimes \delta_{h-d})u^e D^{-1}) \otimes \mathcal{O}_{K,i}.$$

The equality $v_R(\det(D)) = w$ implies that the entries of the matrix $u^e D^{-1}$ are divisible by u^{ei} and the left vertical arrow of the diagram is zero. Hence the assertion follows.

On the other hand, let $\mathcal{D}'$ be a finite flat closed subgroup scheme of $\mathcal{G}$ over $\mathcal{O}_K$ such that $(\mathcal{G}/\mathcal{D}')^\vee \times \mathscr{S}_i$ coincides with the Frobenius kernel of $\mathcal{G}^\vee \times \mathscr{S}_i$. Let $\mathfrak{L}'$ be the subobject of $\mathfrak{M}$ corresponding to the quotient $\mathcal{G}/\mathcal{D}'$. Since $\mathcal{G}$ is a truncated Barsotti-Tate group of level one, the group scheme $\mathcal{D}' \times \mathscr{S}_i$ also coincides with the Frobenius kernel of $\mathcal{G} \times \mathscr{S}_i$. Note that the principal ideal $m_R^{\geqslant i}$ of the ring R_{pi} has a unique divided power structure satisfying $\gamma_n(x) = 0$ for any $x \in m_R^{\geqslant i}$ and $n \geq p$. We can consider the surjection $\mathrm{pr}_0 : R_{pi} \to \mathcal{O}_{\bar{K},i}$ as a divided power thickening over the surjection $S \to \mathcal{O}_K$. Evaluating the exact sequence

$$0 \to \mathbb{D}^*(\mathrm{Im}(F)) \to \mathbb{D}^*(\mathcal{G} \times \mathscr{S}_i) \to \mathbb{D}^*(\mathrm{Ker}(F)) \to 0$$

on this divided power thickening, we obtain the equality

$$R_{pi} \otimes_S \mathcal{M}_{\mathfrak{S}}(\mathfrak{L}') = \mathrm{Im}(R_{pi} \otimes_{\varphi,S} \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \overset{1\otimes\varphi}{\to} R_{pi} \otimes_S \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})).$$

Since the $\tilde{\mathcal{O}}_K$-module $\mathrm{Fil}^1\mathfrak{M}_1$ is a direct summand of $\mathfrak{M}_1$, the R_{pi}-module $R_{pi} \otimes_{\varphi,\tilde{\mathcal{O}}_K} \mathrm{Fil}^1\mathfrak{M}_1$ is a submodule of $R_{pi} \otimes_{\varphi,\tilde{\mathcal{O}}_K} \mathfrak{M}_1$ and is equal to the image on the right-hand side of the above equality. Thus we obtain the equality

$$R_{pi} \otimes_{\varphi,\mathcal{O}_{K,i}} (\mathfrak{L}'/u^{ei}\mathfrak{L}') = R_{pi} \otimes_{\varphi,\mathcal{O}_{K,i}} (\mathrm{Fil}^1\mathfrak{M}_1/u^{ei}\mathrm{Fil}^1\mathfrak{M}_1)$$

and the natural map

$$\mathfrak{L}'/u^{ei}\mathfrak{L}' \to (\mathfrak{M}_1/u^{ei}\mathfrak{M}_1)/(\mathrm{Fil}^1\mathfrak{M}_1/u^{ei}\mathrm{Fil}^1\mathfrak{M}_1)$$

is zero after tensoring the injection $\varphi : \mathcal{O}_{K,i} \to R_{pi}$. Since the $\mathcal{O}_{K,i}$-modules of the both sides of this natural map is free, we obtain the inclusion $\mathfrak{L}'/u^{ei}\mathfrak{L}' \subseteq \mathrm{Fil}^1\mathfrak{M}_1/u^{ei}\mathrm{Fil}^1\mathfrak{M}_1$. Since the $\mathcal{O}_{K,i}$-module $\mathfrak{L}'/u^{ei}\mathfrak{L}'$ is also a direct summand of $\mathfrak{M}_1/u^{ei}\mathfrak{M}_1$, the reverse inclusion follows similarly and the equality

$$\mathfrak{L}'/u^{ei}\mathfrak{L}' = \mathrm{Fil}^1\mathfrak{M}_1/u^{ei}\mathrm{Fil}^1\mathfrak{M}_1$$

holds. Then the uniqueness assertion in Proposition 3.1 implies the equality $\mathcal{D} = \mathcal{D}'$.

For the last assertion, let L/K be a finite extension. Put $\mathcal{G}_{\mathcal{O}_L} = \mathcal{G} \times \mathrm{Spec}(\mathcal{O}_L)$ and similarly for $\mathcal{D}_{\mathcal{O}_L}$. The subgroup scheme $(\mathcal{G}_{\mathcal{O}_L}/\mathcal{D}_{\mathcal{O}_L})^\vee \times \mathscr{S}_{1-w}$ also coincides with the Frobenius kernel of $(\mathcal{G}_{\mathcal{O}_L})^\vee \times \mathscr{S}_{1-w}$. By the uniqueness we have just proved, the subgroup scheme $\mathcal{D}_{\mathcal{O}_L}$ coincides with the conjugate Hodge subgroup of $\mathcal{G}_{\mathcal{O}_L}$. This concludes the proof of the lemma. □

Lemma 3.3. *Let $\mathcal{G}$ be as above and $\mathcal{D}$ be the conjugate Hodge subgroup of $\mathcal{G}$.*

1. *(1) $\mathcal{D} = \mathcal{G}_{(1-w)/(p-1)}$.*
2. *(2) If $w < (p-1)/p$, then the subgroup $\mathcal{D}(\mathcal{O}_{\bar{K}})$ coincides with the kernel of the Hodge-Tate map*

$$\mathrm{HT}_b : \mathcal{G}(\mathcal{O}_{\bar{K}}) \to \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K},b}$$

for any b satisfying $w/(p-1) < b \leq 1 - w$.

(3) If $w < 1/2$, then we have $\mathcal{D} = \mathcal{G}_b$ for any b satisfying $w/(p-1) < b \leq (1-w)/(p-1)$.

Proof. First we consider the assertion (1). It is enough to show the equality $\mathcal{D}_{\mathcal{O}_L} = (\mathcal{G}_{\mathcal{O}_L})_{(1-w)/(p-1)}$ for a finite extension L/K. By Lemma 3.2, the subgroup scheme $\mathcal{D}_{\mathcal{O}_L}$ is the conjugate Hodge subgroup of $\mathcal{G}_{\mathcal{O}_L}$. Thus we may assume $\mathcal{G}(\mathcal{O}_{\bar{K}}) = \mathcal{G}(\mathcal{O}_K)$. Let $\mathfrak{M}$ and $\mathfrak{N}$ be the objects of the category $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$ corresponding to $\mathcal{G}$ and $\mathcal{D}$, respectively. Then we can show the equality

$$\mathcal{H}(\mathfrak{N}) = \mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}$$

as in the proof of [9, Theroem 3.1 (c)]. Take $x \in \mathcal{G}(\mathfrak{M})$. Let $\mathcal{G}'$ be the scheme-theoretic closure in $\mathcal{G}$ of the subgroup $\mathbb{F}_p x \subseteq \mathcal{G}(\mathcal{O}_{\bar{K}})$ and $\mathfrak{M}'$ be the quotient of $\mathfrak{M}$ corresponding to $\mathcal{G}'$. By the Oort-Tate classification ([19]) and Lemma 2.2 (3), we have the following equivalences:

$$\begin{aligned}\mathcal{G}(\mathfrak{M}')_{(1-w)/(p-1)} = \mathcal{G}(\mathfrak{M}') &\Leftrightarrow v_R(\det(\varphi_{\mathfrak{M}'})) \geq 1-w \\ &\Leftrightarrow \mathcal{H}(\mathfrak{M}')_{(1-w)/(p-1)} = \mathcal{H}(\mathfrak{M}').\end{aligned}$$

Note the commutative diagram

$$\begin{tikzcd} \mathcal{G}(\mathfrak{M}')(\mathcal{O}_{\bar{K}}) \arrow[r, "\varepsilon_{\mathfrak{M}'}", "\sim"'] \arrow[d, hook] & \mathcal{H}(\mathfrak{M}')(R) \arrow[d, hook] \\ \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \arrow[r, "\varepsilon_{\mathfrak{M}}", "\sim"'] & \mathcal{H}(\mathfrak{M})(R) \\ \mathcal{G}(\mathfrak{N})(\mathcal{O}_{\bar{K}}) \arrow[r, "\varepsilon_{\mathfrak{N}}", "\sim"'] \arrow[u, hook] & \mathcal{H}(\mathfrak{N})(R), \arrow[u, hook] \end{tikzcd}$$

where the horizontal arrows are isomorphisms and the vertical arrows are injections. Then we have

$$\begin{aligned}x \in \mathcal{D}(\mathcal{O}_{\bar{K}}) = \mathcal{G}(\mathfrak{N})(\mathcal{O}_{\bar{K}}) &\Leftrightarrow \varepsilon_{\mathfrak{M}}(x) \in \mathcal{H}(\mathfrak{N})(R) = \mathcal{H}(\mathfrak{M})_{(1-w)/(p-1)}(R) \\ &\Leftrightarrow \mathcal{H}(\mathfrak{M}')_{(1-w)/(p-1)} = \mathcal{H}(\mathfrak{M}') \\ &\Leftrightarrow \mathcal{G}(\mathfrak{M}')_{(1-w)/(p-1)} = \mathcal{G}(\mathfrak{M}') \\ &\Leftrightarrow x \in \mathcal{G}(\mathfrak{M})_{(1-w)/(p-1)}(\mathcal{O}_{\bar{K}})\end{aligned}$$

and the assertion (1) follows. The assertions (2) and (3) can be shown by a verbatim argument as in the proof of [9, Theorem 3.1 (2), (3)]. □

Remark 3.4. By using results of [12] and Remark 2.4, we can drop the assumption that $\mathcal{G}$ is unipotent from Proposition 3.1, Lemma 3.2 and Lemma 3.3, though we do no use this fact in what follows.

Now we proceed to prove the following theorem, which is the level one case of Theorem 1.1.

Theorem 3.5. *Let the notation be as in Section 1. Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level one, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$ and Hodge height $w = \mathrm{Hdg}(\mathcal{G})$.*

(1) *If $w < p/(p+1)$, then there exists a unique finite flat closed subgroup scheme $\mathcal{C}$ of $\mathcal{G}$ of order p^d over $\mathcal{O}_K$ such that $\mathcal{C} \times \mathscr{S}_{1-w}$ coincides with the kernel of the Frobenius homomorphism of $\mathcal{G} \times \mathscr{S}_{1-w}$. We refer to the subgroup scheme $\mathcal{C}$ as the (level one) canonical subgroup of $\mathcal{G}$. Moreover, the subgroup scheme $\mathcal{C}$ has the following properties:*
   (a) $\deg(\mathcal{G}/\mathcal{C}) = w$.
   (b) *Let $\mathcal{C}'$ be the canonical subgroup of $\mathcal{G}^\vee$. Then we have the equality of subgroup schemes $\mathcal{C}' = (\mathcal{G}/\mathcal{C})^\vee$, or equivalently $\mathcal{C}(\mathcal{O}_{\bar{K}}) = \mathcal{C}'(\mathcal{O}_{\bar{K}})^\perp$, where $\perp$ means the orthogonal subgroup with respect to the Cartier pairing.*
   (c) $\mathcal{C} = \mathcal{G}_{(1-w)/(p-1)} = \mathcal{G}^{pw/(p-1)+}$.

(2) *If $w < (p-1)/p$, then the subgroup $\mathcal{C}(\mathcal{O}_{\bar{K}})$ coincides with the kernel of the Hodge-Tate map $\mathrm{HT}_b : \mathcal{G}(\mathcal{O}_{\bar{K}}) \to \omega_{\mathcal{G}^\vee} \otimes \mathcal{O}_{\bar{K},b}$ for any b satisfying $w/(p-1) < b \leq 1 - w$.*

(3) *If $w < 1/2$, then $\mathcal{C}$ coincides both with the lower ramification subgroup scheme $\mathcal{G}_b$ for any b satisfying $w/(p-1) < b \leq (1-w)/(p-1)$ and the upper ramification subgroup scheme $\mathcal{G}^{j+}$ for any j satisfying $pw/(p-1) \leq j < p(1-w)/(p-1)$.*

Proof. By a base change argument as in the proof of [9, Theorem 3.1], we may assume that the residue field k of K is perfect. Let $\mathcal{G}^0$ (*resp.* $\mathcal{G}^{\mathrm{et}}$) be the unit component (*resp.* the maximal etale quotient) of the group scheme $\mathcal{G}$, and consider their Cartier duals $(\mathcal{G}^0)^\vee$ and $(\mathcal{G}^{\mathrm{et}})^\vee$. These four group schemes are all truncated Barsotti-Tate groups of level one over $\mathcal{O}_K$ and let h_0 be the height of $\mathcal{G}^0$. Then $(\mathcal{G}^0)^\vee$ is a unipotent truncated Barsotti-Tate group of level one, height h_0 and dimension $h_0 - d$. If $h_0 = d$, then the group scheme $\mathcal{G}$ is ordinary (namely, $(\mathcal{G}^0)^\vee$ is etale) and the assertions are clear. Thus we may assume $h_0 > d$.

Since $\mathrm{Hdg}(\mathcal{G}) = \mathrm{Hdg}(\mathcal{G}^\vee)$ and $\mathrm{Lie}(\mathcal{G}) = \mathrm{Lie}(\mathcal{G}^0)$, the truncated Barsotti-Tate group $(\mathcal{G}^0)^\vee$ satisfies the assumption on the Hodge height in Proposition 3.1. Let $\mathcal{D}$ be the conjugate Hodge subgroup of $(\mathcal{G}^0)^\vee$, which is of order p^{h_0-d}. We define the canonical subgroup $\mathcal{C}$ by $\mathcal{C} = ((\mathcal{G}^0)^\vee/\mathcal{D})^\vee$, which is a finite flat closed subgroup scheme of $\mathcal{G}^0$ of order p^d over $\mathcal{O}_K$. By Lemma 3.2, the group scheme $\mathcal{C} \times \mathscr{S}_{1-w}$ coincides with the Frobenius kernel of $\mathcal{G}^0 \times \mathscr{S}_{1-w}$, which is equal to the Frobenius kernel of $\mathcal{G} \times \mathscr{S}_{1-w}$. If a finite flat closed subgroup scheme $\mathcal{E}$ of $\mathcal{G}$ has the reduction $\mathcal{E} \times \mathscr{S}_{1-w}$ equal to this Frobenius kernel, then $\mathcal{E}$ is connected and Lemma 3.2 implies $\mathcal{C} = \mathcal{E}$. Thus the uniqueness assertion of Theorem 3.5 (1) follows.

By Lemma 2.2 (3) and Proposition 3.1, we have $\deg((\mathcal{G}^0)^\vee/\mathcal{D}) = w$ and

$$\begin{aligned}\deg(\mathcal{G}/\mathcal{C}) &= \deg(\mathcal{G}^0/\mathcal{C}) = \deg(\mathcal{D}^\vee) = h_0 - d - \deg(\mathcal{D}) \\ &= h_0 - d - \deg((\mathcal{G}^0)^\vee) + \deg((\mathcal{G}^0)^\vee/\mathcal{D}) = w.\end{aligned}$$

Thus the part (a) of the theorem follows. Cartier duality and the uniqueness of the canonical subgroup we have just proved imply the part (b).

For the part (c), we insert here the following lemma due to the lack of references.

Lemma 3.6. *Let $\mathcal{H}$ be a finite flat group scheme over $\mathcal{O}_K$ and $\mathcal{H}^0$ be its unit component. Then we have $\mathcal{H}^j = (\mathcal{H}^0)^j$ for any positive rational number j.*

Proof. Replacing K by a finite extension, we may assume that the maximal etale quotient $\mathcal{H}^{\mathrm{et}}$ is a constant group scheme $\underline{M}$ for some abelian group M and that we have an isomorphism of schemes over $\mathcal{O}_K$

$$\mathcal{H}^0 \times \underline{M} \to \mathcal{H}$$

which induces the natural isomorphism of group schemes $\mathcal{H}^0 \times \{0\} \to \mathcal{H}^0$. Let $\mathcal{F}^j$ be the functor of the set of geometric connected components of the j-th tubular neighborhood as in [1, Section 2]. By [1, Lemme 2.1.1], the functor $\mathcal{F}^j$ is compatible with products and we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}^0(\mathcal{O}_{\bar{K}}) \times \underline{M}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & \mathcal{H}(\mathcal{O}_{\bar{K}}) \arrow[d] \\
\mathcal{F}^j(\mathcal{H}^0) \times \mathcal{F}^j(\underline{M}) \arrow[r] & \mathcal{F}^j(\mathcal{H}),
\end{tikzcd}$$

where the vertical arrows are homomorphisms and the horizontal arrows are bijections preserving zero elements. For $j > 0$, the natural map $\underline{M}(\mathcal{O}_{\bar{K}}) \to \mathcal{F}^j(\underline{M})$ is an isomorphism and the kernel of the left vertical arrow is the subgroup $(\mathcal{H}^0)^j(\mathcal{O}_{\bar{K}}) \times \{0\}$. Thus the kernel of the right vertical arrow is the subgroup $(\mathcal{H}^0)^j(\mathcal{O}_{\bar{K}})$ and the lemma follows. $\square$

Now Lemma 3.3 (1) implies the equality $\mathcal{D} = ((\mathcal{G}^0)^\vee)_b$ for $b = (1-w)/(p-1)$. By Lemma 3.6 and a theorem of Tian and Fargues ([20, Theorem 1.6] or [5, Proposition 6]), we have the equalities $\mathcal{C} = (\mathcal{G}^0)^{j+} = \mathcal{G}^{j+}$ for $j = p/(p-1) - pb = pw/(p-1)$. From this and the part (b), we also obtain the equality $\mathcal{C} = \mathcal{G}_b$ and the part (c) follows. The part (3) can be shown similarly, by using Lemma 3.3 (3).

Finally we show the assertion (2). By replacing $\mathcal{G}$ by $\mathcal{G}^\vee$, it is enough to show the assertion for the canonical subgroup $\mathcal{C}'$ of $\mathcal{G}^\vee$. Put $\mathcal{T} = (\mathcal{G}^{\mathrm{et}})^\vee$. Let $\mathcal{D}$ be the conjugate Hodge subgroup of $(\mathcal{G}^0)^\vee$ as above and $\tilde{\mathcal{D}}$ be the inverse image of $\mathcal{D}$ by the natural epimorphism $\iota^\vee : \mathcal{G}^\vee \to (\mathcal{G}^0)^\vee$. We claim the equality $\mathcal{C}' = \tilde{\mathcal{D}}$. Indeed, by Lemma 3.3 (1), we have $\mathcal{D} = ((\mathcal{G}^0)^\vee)_{(1-w)/(p-1)}$. By the part (c) of the theorem, this implies that $\mathcal{D}$ is the canonical subgroup of $(\mathcal{G}^0)^\vee$ and also coincides with $((\mathcal{G}^0)^\vee)^{pw/(p-1)+}$. Since $\mathcal{C}' = (\mathcal{G}^\vee)^{pw/(p-1)+}$, the natural map $\iota^\vee$ induces the surjection $\mathcal{C}'(\mathcal{O}_{\bar{K}}) \to \mathcal{D}(\mathcal{O}_{\bar{K}})$. In particular, the subgroup $\mathcal{C}'(\mathcal{O}_{\bar{K}})$ is contained in $\tilde{\mathcal{D}}(\mathcal{O}_{\bar{K}})$. On the other hand, since the group scheme $\mathcal{T}$ is isomorphic to a direct sum of μ_p after a finite extension, we have the inclusions

$$\mathcal{T} = \mathcal{T}_{1/(p-1)} \subseteq (\mathcal{G}^\vee)_{1/(p-1)} \subseteq (\mathcal{G}^\vee)_{(1-w)/(p-1)} = \mathcal{C}',$$

from which the claim follows.

Therefore, we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d, equal] & \mathcal{C}'(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & \mathcal{D}(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{T}(\mathcal{O}_{\bar{K}}) \arrow[r] & \mathcal{G}^\vee(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & (\mathcal{G}^0)^\vee(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & 0 \\
 & & \omega_{\mathcal{G}} \otimes \mathcal{O}_{\bar{K},b} \arrow[r, "\sim"] & \omega_{\mathcal{G}^0} \otimes \mathcal{O}_{\bar{K},b}, &
\end{tikzcd}$$

where the lowest vertical arrows are the Hodge-Tate maps HT_b and the lowest horizontal arrow is an isomorphism. By Lemma 3.3 (2), the subgroup $\mathcal{D}(\mathcal{O}_{\bar{K}})$ coincides with the kernel of the lowest right vertical arrow for any b satisfying $w/(p-1) < b \leq 1 - w$. This implies that the subgroup $\tilde{\mathcal{D}}(\mathcal{O}_{\bar{K}}) = \mathcal{C}'(\mathcal{O}_{\bar{K}})$ is the kernel of the lowest left vertical arrow for any such b and the assertion (2) follows. This concludes the proof of Theorem 3.5. □

Since the arguments in the proof of [9, Theorem 1.1] work verbatim also for $p = 2$, Theorem 1.1 follows from Theorem 3.5.

Moreover, we can show the following proposition on anti-canonical subgroups, by modifying the proof of [9, Proposition 4.3].

Proposition 3.7. *Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level two, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$ and Hodge height $w = \mathrm{Hdg}(\mathcal{G})$. Suppose $w < 1/2$ and let $\mathcal{C}$ be the canonical subgroup of $\mathcal{G}[p]$ as in Theorem 3.5. Let $\mathcal{D}$ be a finite flat closed subgroup scheme of $\mathcal{G}[p]$ over $\mathcal{O}_K$ such that the natural map $\mathcal{C}(\mathcal{O}_{\bar{K}}) \oplus \mathcal{D}(\mathcal{O}_{\bar{K}}) \to \mathcal{G}[p](\mathcal{O}_{\bar{K}})$ is an isomorphism.*

(1) The truncated Barsotti-Tate group $p^{-1}\mathcal{D}/\mathcal{D}$ of level one has Hodge height $\mathrm{Hdg}(p^{-1}\mathcal{D}/\mathcal{D}) = p^{-1}w$.
(2) The subgroup scheme $\mathcal{G}[p]/\mathcal{D}$ is the canonical subgroup of $p^{-1}\mathcal{D}/\mathcal{D}$.
(3) $\deg(\mathcal{D}) = p^{-1}w$.

Proof. By a base change argument as before, we may assume that the residue field k of K is perfect. Note that the truncated Barsotti-Tate group $p^{-1}\mathcal{D}/\mathcal{D}$ of level one is also of height h and dimension d. The natural homomorphism $\mathcal{C} \to \mathcal{G}[p]/\mathcal{D}$ induces an isomorphism between the generic fibers of both sides. Since the group scheme $\mathcal{C}$ is connected, the connected-etale sequence implies that the group scheme $\mathcal{G}[p]/\mathcal{D}$ is also connected. Now we claim that the group scheme $(\mathcal{G}[p]/\mathcal{D}) \times \mathscr{S}_{1-w}$ is killed by the Frobenius. For this, let $\mathfrak{L}$ and $\mathfrak{L}'$ be the objects of the category $\mathrm{Mod}^{1,\varphi,V}_{/S}$ corresponding to the unipotent finite flat group schemes $\mathcal{C}^\vee$ and $(\mathcal{G}[p]/\mathcal{D})^\vee$ via the anti-equivalence $\mathcal{G}(-)$, respectively. By [17, Corollary 2.2.2], the generic isomorphism $(\mathcal{G}[p]/\mathcal{D})^\vee \to$

$\mathcal{C}^\vee$ corresponds to an injection $\mathfrak{L} \to \mathfrak{L}'$. Then the $\mathfrak{S}_1$-modules $\wedge^d \mathfrak{L}$ and $\wedge^d \mathfrak{L}'$ are free of rank one and this injection induces an injection $\wedge^d \mathfrak{L} \to \wedge^d \mathfrak{L}'$. Hence we obtain the inequality $v_R(\det \varphi_{\mathfrak{L}'}) \leq v_R(\det \varphi_{\mathfrak{L}}) = w$. Since the group scheme $\mathcal{G}[p]/\mathcal{D}$ is a finite flat closed subgroup scheme of the truncated Barsotti-Tate group $(p^{-1}\mathcal{D}/\mathcal{D})^0$ of level one over $\mathcal{O}_K$, a similar argument to the proof of Lemma 3.2 implies the claim. Then the proposition follows as in the proof of [9, Proposition 4.3]. □

We also have the following generalization of [9, Proposition 4.4] to the case of $p = 2$.

Proposition 3.8. *Let $\mathcal{G}$ be a truncated Barsotti-Tate group of level n, height h and dimension d over $\mathcal{O}_K$ with $0 < d < h$ and Hodge height $w < p(p-1)/(p^{n+1}-1)$. Let $\mathcal{C}_n$ be the level n canonical subgroup of $\mathcal{G}$, which is defined by Theorem 1.1. Let $\mathcal{D}_n$ be a finite flat closed subgroup scheme of $\mathcal{G}$ over $\mathcal{O}_K$ such that $\mathcal{D}_n(\mathcal{O}_{\bar{K}}) \simeq (\mathbb{Z}/p^n\mathbb{Z})^d$ and the group scheme $\mathcal{D}_n \times \mathscr{S}_{1-p^{n-1}w}$ is killed by the n-th iterated Frobenius F^n. Then we have $\mathcal{C}_n = \mathcal{D}_n$.*

Proof. Suppose $n \geq 2$. Let $\mathcal{C}_1$ be the scheme-theoretic closure of $\mathcal{C}_n(\mathcal{O}_{\bar{K}})[p]$ in $\mathcal{C}_n$ and define $\mathcal{D}_1$ similarly. The group scheme $\mathcal{C}_1$ coincides with the canonical subgroup of $\mathcal{G}[p]$ by Theorem 1.1 (e). We claim $\mathcal{C}_1 = \mathcal{D}_1$, which implies the proposition by an induction as in the proof of [9, Proposition 4.4]. For this, by a base change argument as before, we may assume that the residue field k of K is perfect and $\mathcal{G}(\mathcal{O}_{\bar{K}}) = \mathcal{G}(\mathcal{O}_K)$. Suppose $\mathcal{C}_1 \neq \mathcal{D}_1$. Then we can find a finite flat closed subgroup scheme $\mathcal{E}$ of $\mathcal{G}[p]$ such that $\mathcal{C}_1(\mathcal{O}_{\bar{K}}) \oplus \mathcal{E}(\mathcal{O}_{\bar{K}}) = \mathcal{G}[p](\mathcal{O}_{\bar{K}})$ and $\mathcal{D}_1(\mathcal{O}_{\bar{K}}) \cap \mathcal{E}(\mathcal{O}_{\bar{K}}) \neq 0$. Let $\mathcal{F}$ be the scheme-theoretic closure of the latter intersection in $\mathcal{G}[p]$. As in the proof of Theorem 3.5 (2), we see that the group scheme $\mathcal{C}_1$ contains the multiplicative part of $\mathcal{G}[p]$. Thus the quotient $\mathcal{G}[p]/\mathcal{C}$ is unipotent. Since the natural map $\mathcal{E} \to \mathcal{G}[p]/\mathcal{C}$ is a generic isomorphism, the connected-etale sequence shows that $\mathcal{E}$, and thus also $\mathcal{F}$, are unipotent. Let $\mathfrak{F}$ be the object of the category $\mathrm{Mod}^{1,\varphi,V}_{/\mathfrak{S}_1}$ corresponding to $\mathcal{F}$. Then Lemma 2.2 (3) and Proposition 3.7 (3) implies

$$v_R(\det(\varphi_{\mathfrak{F}})) = \deg(\mathcal{F}) \leq \deg(\mathcal{E}) = p^{-1}w.$$

On the other hand, the group scheme $\mathcal{F} \times \mathscr{S}_{1-p^{n-1}w}$ is killed by the n-th iterated Frobenius. Put $i = 1 - p^{n-1}w$. Evaluating $\mathbb{D}^*(\mathcal{F})$ on the divided power thickening $R_{pi} \to \mathcal{O}_{\bar{K},i}$ as before, we see that the map

$$1 \otimes \varphi \otimes \varphi_{\mathfrak{F}} : R_{pi} \otimes_{\varphi, R_{pi}} (R_{pi} \otimes_{\varphi,\mathfrak{S}} \mathfrak{F}) \to R_{pi} \otimes_{\varphi,\mathfrak{S}} \mathfrak{F}$$

is the one induced by the Frobenius

$$\mathbb{D}^*(F) : \mathbb{D}^*((\mathcal{F} \times \mathscr{S}_i)^{(p)}) \to \mathbb{D}^*(\mathcal{F} \times \mathscr{S}_i)$$

and that the composite of its pull-backs

$$\varphi^{n*}(R_{pi} \otimes_{\varphi,\mathfrak{S}} \mathfrak{F}) \to \varphi^{n-1*}(R_{pi} \otimes_{\varphi,\mathfrak{S}} \mathfrak{F}) \to \cdots \to R_{pi} \otimes_{\varphi,\mathfrak{S}} \mathfrak{F}$$

is the map induced by the n-th iterated Frobenius. Taking the valuation of the determinant of this map, we obtain the inequalities

$$p(1-p^{n-1}w) \leq v_R(\det \varphi_{\mathfrak{F}})p(p^n-1)/(p-1) \leq w(p^n-1)/(p-1),$$

which contradict the assumption on w and the equality $\mathcal{C}_1 = \mathcal{D}_1$ follows. □

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