RAMIFICATION CORRESPONDENCE OF FINITE FLAT GROUP SCHEMES OVER EQUAL AND MIXED CHARACTERISTIC LOCAL FIELDS

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Abstract. Let $p > 2$ be a rational prime, k be a perfect field of characteristic p and K be a finite totally ramified extension of the fractional field of the Witt ring of k. Let $\mathcal{G}$ and $\mathcal{H}$ be finite flat commutative group schemes killed by p over $\mathcal{O}_K$ and $k[[u]]$, respectively. In this paper, we show the ramification subgroups of $\mathcal{G}$ and $\mathcal{H}$ in the sense of Abbes-Saito are naturally isomorphic to each other when they are associated to the same Kisin module.

1. Introduction

Let p be a rational prime, k be a perfect field of characteristic p, $W = W(k)$ be the Witt ring of k and K be a finite totally ramified extension of $\mathrm{Frac}(W)$ of degree e. Let ϕ denote the Frobenius endomorphism of W. We fix once and for all an algebraic closure $\bar{K}$ of K, a uniformizer π of K and a system of its p-power roots $(\pi_n)_{n\in\mathbb{Z}_{\geq 0}}$ in $\bar{K}$ with $\pi_0 = \pi$ and $\pi_n = \pi_{n+1}^p$. Put $K_n = K(\pi_n)$ and $K_\infty = \cup_n K_n$. By the theory of norm fields ([27]), there exist a complete discrete valuation field $\mathcal{X} \simeq k((u))$ of characteristic p with residue field k and an isomorphism of groups

$$\mathrm{Gal}(\bar{K}/K_\infty) \simeq \mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X}),$$

where $\mathcal{X}^{\mathrm{sep}}$ is a separable closure of $\mathcal{X}$. A striking feature of this isomorphism is its compatibility with the upper ramification subgroups of both sides up to a shift by the Herbrand function of K_∞/K ([27, Corollaire 3.3.6]).

On the other hand, Breuil ([8]) introduced linear algebraic data over a ring S, which are now called as Breuil modules, and proved a classification of finite flat (commutative) group schemes over $\mathcal{O}_K$ via these data for $p > 2$. Based on his works, Kisin ([20]) simplified this classification by replacing Breuil's data by ϕ-modules over $W[[u]]$, which are referred as Kisin modules. Let us consider the case where finite flat group schemes are killed by p. Put $\mathfrak{S}_1 = k[[u]]$, which is isomorphic to the k-algebra $\mathcal{O}_{\mathcal{X}}$. We let ϕ also denote the absolute Frobenius endomorphism of the ring $\mathfrak{S}_1$. For a non-negative integer r, let $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ be the category of free $\mathfrak{S}_1$-modules $\mathfrak{M}$ of finite rank endowed with a ϕ-semilinear map $\phi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the

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cokernel of the map $1 \otimes \phi_{\mathfrak{M}} : \mathfrak{S}_1 \otimes_{\phi,\mathfrak{S}_1} \mathfrak{M} \to \mathfrak{M}$ is killed by u^{er}. Then there exists an anti-equivalence of categories $\mathcal{G}(-)$ from $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ to the category of finite flat group schemes over $\mathcal{O}_K$ killed by p. It is well-known that we also have an anti-equivalence $\mathcal{H}(-)$ from $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ to a category of finite flat generically etale group schemes over $\mathcal{O}_{\mathcal{X}}$ killed by their Verschiebung ([16]). Hence a correspondence between finite flat group schemes over $\mathcal{O}_K$ and $\mathcal{O}_{\mathcal{X}}$ is obtained, and if finite flat group schemes $\mathcal{G}$ over $\mathcal{O}_K$ and $\mathcal{H}$ over $\mathcal{O}_{\mathcal{X}}$ are in correspondence, then their generic fiber Galois modules $\mathcal{G}(\mathcal{O}_{\bar{K}})$ and $\mathcal{H}(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})$ are also in correspondence via the theory of norm fields. Namely, for an object $\mathfrak{M}$ of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$, we have an isomorphism

$$\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}})|_{\mathrm{Gal}(\bar{K}/K_\infty)} \to \mathcal{H}(\mathfrak{M})(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})$$

of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules ([20, Proposition 1.1.13]). From this, we can show that the Galois modules $\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}})$ and $\mathcal{H}(\mathfrak{M})(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})$ have exactly the same greatest upper ramification jump in the classical sense (see also [1]).

Besides the classical ramification theory of their generic fibers, finite flat group schemes over a complete discrete valuation ring have their own ramification theory, the fact which was discovered by Abbes-Saito ([3], [4]) and Abbes-Mokrane ([2]). Such a finite flat group scheme $\mathcal{G}$ has filtrations of upper ramification subgroups $\{\mathcal{G}^j\}_{j\in\mathbb{Q}_{>0}}$ ([2]) and lower ramification subgroups $\{\mathcal{G}_i\}_{i\in\mathbb{Q}_{\geq 0}}$ ([17], [14]), as in the classical ramification theory of local fields. For simplicity, let K be a complete discrete valuation field as above and consider a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$. Then, using the upper ramification filtration of $\mathcal{G}$, we can bound the classical greatest upper ramification jump of the generic fiber $\mathrm{Gal}(\bar{K}/K)$-module $\mathcal{G}(\mathcal{O}_{\bar{K}})$ ([17]) and also describe completely the semi-simplification of the restriction to the inertia subgroup of this Galois module ([19]). Moreover, the canonical subgroup of a possibly higher dimensional abelian scheme A over $\mathcal{O}_K$ is found in the upper ramification filtration of $A[p^n]$ ([2], [25], [26]), while the canonical subgroup is also found in the Harder-Narasimhan filtration of $A[p^n]$ defined by Fargues ([13], [14]).

In this paper, we establish the following correspondence of the ramification filtrations between finite flat group schemes over $\mathcal{O}_K$ and $\mathcal{O}_{\mathcal{X}}$ which is similar to that of the classical ramification jumps of their generic fiber Galois modules stated above.

Theorem 1.1. *Let $p > 2$ be a rational prime and K and $\mathcal{X}$ be as before. Let $\mathfrak{M}$ be an object of the category* $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$. *Then the natural* $\mathrm{Gal}(\bar{K}/K_\infty)$-*equivariant isomorphism*

$$\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}})|_{\mathrm{Gal}(\bar{K}/K_\infty)} \to \mathcal{H}(\mathfrak{M})(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})$$

induces isomorphisms of the upper and the lower ramification subgroups

$$\mathcal{G}(\mathfrak{M})^j(\mathcal{O}_{\bar{K}})|_{\mathrm{Gal}(\bar{K}/K_\infty)} \to \mathcal{H}(\mathfrak{M})^j(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}),$$
$$\mathcal{G}(\mathfrak{M})_i(\mathcal{O}_{\bar{K}})|_{\mathrm{Gal}(\bar{K}/K_\infty)} \to \mathcal{H}(\mathfrak{M})_i(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})$$

for any $j \in \mathbb{Q}_{>0}$ *and* $i \in \mathbb{Q}_{\geq 0}$.

This theorem enables us to reduce the study of ramification of finite flat group schemes over $\mathcal{O}_K$ killed by p to the case where the base is a complete discrete valuation ring of equal characteristic. This makes many calculations of ramification of finite flat group schemes over $\mathcal{O}_K$, for example as in [19, Section 5], much easier. We remark that, for the Harder-Narasimhan filtration, such a correspondence of filtrations of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{H}(\mathfrak{M})$ follows easily from the definition.

A key idea to prove the theorem is to switch from the upper ramification filtration to the lower ramification filtration via Cartier duality, which the author learned from works of Tian ([25]) and Fargues ([14]). Let $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ killed by p and $\mathcal{G}^\vee$ be its Cartier dual. Then they showed that the upper ramification subgroup $\mathcal{G}^j(\mathcal{O}_{\bar{K}})$ is the orthogonal subgroup of the lower ramification subgroup $(\mathcal{G}^\vee)_i(\mathcal{O}_{\bar{K}})$ for some i via the natural pairing

$$\mathcal{G}(\mathcal{O}_{\bar{K}}) \times \mathcal{G}^\vee(\mathcal{O}_{\bar{K}}) \to \mu_p(\mathcal{O}_{\bar{K}}).$$

We prove a version of this theorem for the group scheme $\mathcal{H}(\mathfrak{M})$ over $\mathcal{O}_\mathcal{X}$. Since we are in characteristic p, usual Cartier dual does not preserve the generic etaleness of finite flat group schemes. Instead, we use a duality theory of Caruso ([11]) and Liu ([21]) for Breuil modules and Kisin modules. This requires us to check compatibilities of various duality theories, though it is straightforward to carry out. Thus we reduce ourselves to prove the correspondence of the lower ramification subgroups of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{H}(\mathfrak{M})$. This is a consequence of the fact that, after the base changes from K and $\mathcal{X}$ to the extensions generated by p-th roots of their uniformizers π and u, the schemes $\mathcal{G}(\mathfrak{M})$ modulo π^e and $\mathcal{H}(\mathfrak{M})$ modulo u^e become isomorphic to each other, not as group schemes but as pointed schemes. We prove this fact by using Breuil's explicit computation of the affine algebra of a finite flat group scheme over $\mathcal{O}_K$ killed by p in terms of its corresponding Breuil module ([8, Section 3]), after showing that his classification of finite flat group schemes is compatible with the base change from K to K_n.

As a byproduct of the proof of the main theorem, we also prove a description of the semi-simplification of $\mathcal{H}(\mathfrak{M})(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})$ via the upper ramification filtration, as follows. Let $I_\mathcal{X}$ be the inertia subgroup of $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$, $\theta_j : I_\mathcal{X} \to \bar{\mathbb{F}}_p^\times$ be the fundamental character of level j ([24, Subsection 1.7]) and $\mathbb{F}_{(j)}$ be the finite field generated by its image. Put

$$\mathcal{H}(\mathfrak{M})^{j+}(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}) = \bigcup_{j'>j} \mathcal{H}(\mathfrak{M})^{j'}(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})$$

and set the j-th graded piece of the upper ramification filtration of $\mathcal{H}(\mathfrak{M})$ to be

$$\mathrm{gr}^j \mathcal{H}(\mathfrak{M})^\bullet(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}) = \mathcal{H}(\mathfrak{M})^j(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})/\mathcal{H}(\mathfrak{M})^{j+}(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}).$$

Then we prove in Subsection 3.3 the following theorem, which is a version of [19, Theorem 1.1] for an equal characteristic local field.

Theorem 1.2. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ and j be a positive rational number. Then the action of $I_{\mathcal{X}}$ on the j-th graded piece of the upper ramification filtration of $\mathcal{H}(\mathfrak{M})$ is tame. Moreover, there exists an action of $\mathbb{F}_{(j)}$ on this graded piece such that we have an $I_{\mathcal{X}}$-equivariant $\mathbb{F}_{(j)}$-linear isomorphism*

$$\mathrm{gr}^j \mathcal{H}(\mathfrak{M})^\bullet(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}) \simeq \mathbb{F}_{(j)}(\theta_j)^{\oplus n},$$

where n is the dimension of the graded piece over $\mathbb{F}_{(j)}$ and $\mathbb{F}_{(j)}(\theta_j)$ is the one-dimensional $\mathbb{F}_{(j)}$-vector space on which $I_{\mathcal{X}}$ acts via the character $\theta_j : I_{\mathcal{X}} \to \mathbb{F}^\times_{(j)}$.

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2. Review of Cartier duality theory of Caruso and Liu

Let $p > 2$ be a rational prime and K be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field k, as in Section 1. It is well-known that finite flat group schemes over $\mathcal{O}_K$ killed by some p-power are classified by linear algebraic data, Breuil modules ([8]) or Kisin modules ([20]). For these data, corresponding notions of duality to Cartier duality for finite flat group schemes are introduced by Caruso ([10], [11]) and Liu ([21]), which play key roles in the integral p-adic Hodge theory (see for example [22], [23]). In this section, we recall the definitions of these data and their theories of Cartier duality.

2.1. **Breuil and Kisin modules.** Let $E(u) \in W[u]$ be the Eisenstein polynomial of the uniformizer π over W. Put $F(u) = p^{-1}(u^e - E(u))$. This defines units in the rings $\mathfrak{S} = W[[u]]$ and $\mathfrak{S}_1 = k[[u]]$. The ϕ-semilinear continuous ring endomorphisms of these rings defined by $u \mapsto u^p$ are also denoted by ϕ. Let r be a non-negative integer. Then a Kisin module over $\mathfrak{S}_1$ of E-height $\leq r$ is a free $\mathfrak{S}_1$-module $\mathfrak{M}$ of finite rank endowed with a ϕ-semilinear map $\phi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that the cokernel of the map $1 \otimes \phi_{\mathfrak{M}} : \mathfrak{S}_1 \otimes_{\phi,\mathfrak{S}_1} \mathfrak{M} \to \mathfrak{M}$ is killed by $E(u)^r$. We write $\phi_{\mathfrak{M}}$ also as ϕ if there is no risk of confusion. A morphism of Kisin modules over $\mathfrak{S}_1$ is an $\mathfrak{S}_1$-linear map which is compatible with ϕ's of the source and the target. The category of Kisin modules over $\mathfrak{S}_1$ of E-height $\leq r$ is denoted by $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$.

For $r < p-1$, we have another category $\mathrm{Mod}^{r,\phi}_{/S_1}$ of Breuil modules defined as follows. Let S be the p-adic completion of the divided power envelope $W[u]^{\mathrm{DP}}$ of $W[u]$ with respect to the ideal $(E(u))$ and the compatibility condition with the canonical divided power structure on pW. The ring S has a natural filtration $\mathrm{Fil}^i S$ defined as the closure in S of the ideal generated by $E(u)^j/j!$ for integers $j \geq i$. The ϕ-semilinear continuous ring homomorphism $S \to S$ defined by $u \mapsto u^p$ is also denoted by ϕ. For $0 \leq i \leq$

$p-1$, we have $\phi(\mathrm{Fil}^i S) \subseteq p^i S$ and put $\phi_i = p^{-i}\phi|_{\mathrm{Fil}^i S}$. These filtration and ϕ_i's induce a similar structure on the ring $S_n = S/p^n S$. Put $c = \phi_1(E(u)) \in S^\times$. Then we let $'\mathrm{Mod}^{r,\phi}_{/S}$ denote the category of S-modules $\mathcal{M}$ endowed with an S-submodule $\mathrm{Fil}^r\mathcal{M}$ containing $(\mathrm{Fil}^r S)\mathcal{M}$ and a ϕ-semilinear map $\phi_{r,\mathcal{M}} : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$ satisfying $\phi_{r,\mathcal{M}}(s_r m) = c^{-r}\phi_r(s_r)\phi_{r,\mathcal{M}}(E(u)^r m)$ for any $s_r \in \mathrm{Fil}^r S$ and $m \in \mathcal{M}$. A morphism of this category is defined to be a homomorphism of S-modules compatible with Fil^r's and ϕ_r's. We drop the subscript $\mathcal{M}$ of $\phi_{r,\mathcal{M}}$ if no confusion may occur. Note that we have $\phi_r((\mathrm{Fil}^p S)\mathcal{M}) = 0$ if $p\mathcal{M} = 0$. An object $\mathcal{M}$ of this category is said as a Breuil module over S_1 of weight $\leq r$ if $\mathcal{M}$ is a free S_1-module of finite rank and the image $\phi_{r,\mathcal{M}}(\mathrm{Fil}^r\mathcal{M})$ generates the S-module $\mathcal{M}$. The full subcategory of $'\mathrm{Mod}^{r,\phi}_{/S}$ of Breuil modules over S_1 of weight $\leq r$ is denoted by $\mathrm{Mod}^{r,\phi}_{/S_1}$. Both of the categories $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ and $\mathrm{Mod}^{r,\phi}_{/S_1}$ have obvious notions of exact sequences.

The category $\mathrm{Mod}^{r,\phi}_{/S_1}$ is equivalent to a simpler category $\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ defined as follows. Put $\tilde{S}_1 = k[u]/(u^{ep})$. The natural map $k[u] \to S_1$ induces an isomorphism $\tilde{S}_1 \to S_1/\mathrm{Fil}^p S_1$ and we regard $\tilde{S}_1$ as an S-algebra by this isomorphism. Define Fil^r, ϕ and ϕ_r of $\tilde{S}_1$ to be those induced from S_1. In particular, $\mathrm{Fil}^r\tilde{S}_1 = u^{er}\tilde{S}_1$ and $\phi_r(u^{er}) = c^r$. We let $\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ denote the category of free $\tilde{S}_1$-modules $\tilde{\mathcal{M}}$ of finite rank endowed with an $\tilde{S}_1$-submodule $\mathrm{Fil}^r\tilde{\mathcal{M}}$ containing $(\mathrm{Fil}^r\tilde{S}_1)\tilde{\mathcal{M}}$ and a ϕ-semilinear map $\phi_{r,\tilde{\mathcal{M}}} : \mathrm{Fil}^r\tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$ such that the image $\phi_{r,\tilde{\mathcal{M}}}(\mathrm{Fil}^r\tilde{\mathcal{M}})$ generates the $\tilde{S}_1$-module $\tilde{\mathcal{M}}$. Then we can show that the functor $T_0 : \mathrm{Mod}^{r,\phi}_{/S_1} \to \mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ defined by $\mathcal{M} \mapsto \mathcal{M}/(\mathrm{Fil}^p S_1)\mathcal{M}$ is an equivalence of categories, just as in the proof of [6, Proposition 2.2.2.1].

The categories $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ and $\mathrm{Mod}^{r,\phi}_{/S_1}$ for $r < p-1$ are in fact equivalent. We define an exact functor $\mathcal{M}_{\mathfrak{S}} : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1} \to \mathrm{Mod}^{r,\phi}_{/S_1}$ by putting $\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) = S \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$ with

$$\begin{aligned} &\mathrm{Fil}^r\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Ker}(S \otimes_{\phi,\mathfrak{S}} \mathfrak{M} \overset{1\otimes\phi}{\to} (S_1/\mathrm{Fil}^r S_1) \otimes_{\mathfrak{S}} \mathfrak{M}), \\ &\phi_r : \mathrm{Fil}^r\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \overset{1\otimes\phi}{\to} \mathrm{Fil}^r S_1 \otimes_{\mathfrak{S}} \mathfrak{M} \overset{\phi_r\otimes 1}{\to} S_1 \otimes_{\phi,\mathfrak{S}_1} \mathfrak{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}). \end{aligned}$$

Then the functor $\mathcal{M}_{\mathfrak{S}}$ is an equivalence of categories ([12, Theorem 2.3.1]).

2.2. **Cartier duality theory for Breuil and Kisin modules.** The category $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ has a natural notion of duality ([21, Section 3.1]). For an object $\mathfrak{M}$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$, put $\mathfrak{M}^\vee = \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_1)$. Choose a basis $e_1, \ldots, e_d$ of the free $\mathfrak{S}_1$-module $\mathfrak{M}$ and let $e_1^\vee, \ldots, e_d^\vee$ denote its dual basis. Define a matrix $A \in M_d(\mathfrak{S}_1)$ by

$$\phi_{\mathfrak{M}}(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A.$$

Since the matrix $(-E(u)/F(u))^r({}^{\mathrm{t}}A)^{-1}$ is contained in $M_d(\mathfrak{S}_1)$, we can define a ϕ-semilinear map $\phi_{\mathfrak{M}^\vee}:\mathfrak{M}^\vee \to \mathfrak{M}^\vee$ by

$$\phi_{\mathfrak{M}^\vee}(e_1^\vee,\ldots,e_d^\vee)=(e_1^\vee,\ldots,e_d^\vee)(-E(u)/F(u))^r({}^{\mathrm{t}}A)^{-1}.$$

Then the map $\phi_{\mathfrak{M}^\vee}$ is independent of the choice of a basis $e_1,\ldots,e_d$ and with this map, the $\mathfrak{S}_1$-module $\mathfrak{M}^\vee$ is an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. The correspondence $\mathfrak{M}\mapsto\mathfrak{M}^\vee$ defines a contravariant functor from $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ to itself. This functor is an anti-equivalence of categories, since we have a natural isomorphism of the double duality $\mathfrak{M}\to(\mathfrak{M}^\vee)^\vee$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. Note that here we adopt a slightly different normalization from that of [21] where the representing matrix of $\phi_{\mathfrak{M}^\vee}$ is defined to be $(pE(u)/E(0))^r({}^{\mathrm{t}}A)^{-1}$.

For $0\le r\le p-2$, a notion of duality was also introduced for the category $\mathrm{Mod}^{r,\phi}_{/S_1}$ ([10], [11], [22]). Let $\mathcal{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/S_1}$. Put $\mathcal{M}^\vee=\mathrm{Hom}_S(\mathcal{M},S_1)$ and $\mathrm{Fil}^r\mathcal{M}^\vee=\{f\in\mathcal{M}^\vee\mid f(\mathrm{Fil}^r\mathcal{M})\subseteq\mathrm{Fil}^rS_1\}$. For an element $f\in\mathrm{Fil}^r\mathcal{M}^\vee$, we define $\phi_{r,\mathcal{M}^\vee}(f)$ to be the unique S-linear map $g:\mathcal{M}\to S_1$ which makes the following diagram commutative.

$$\begin{tikzcd}
\mathrm{Fil}^r\mathcal{M} \arrow[r, "\phi_r"] \arrow[d, "f"'] & \mathcal{M} \arrow[d, "g"] \\
\mathrm{Fil}^r S_1 \arrow[r, "\phi_r"'] & S_1
\end{tikzcd}$$

Since $\phi_r(\mathrm{Fil}^r\mathcal{M})$ generates the S_1-module $\mathcal{M}$, such a map g is unique if it exists. We can show the existence of g just as in [11, Section 2.1] using an adapted basis of $\mathcal{M}$ over S_1 ([7, Définition 2.2.1.4]). This duality defines a contravariant functor from $\mathrm{Mod}^{r,\phi}_{/S_1}$ to itself. Choose an adapted basis $e_1,\ldots,e_d$ of $\mathcal{M}$ such that

$$\mathrm{Fil}^r\mathcal{M}=u^{r_1}S_1e_1\oplus\cdots\oplus u^{r_d}S_1e_d+(\mathrm{Fil}^pS)\mathcal{M}$$

for some r_i satisfying $0\le r_i\le er$. Let $e_1^\vee,\ldots,e_d^\vee$ be its dual basis. Let $G\in GL_d(S_1)$ be the matrix defined by

$$\phi_r(u^{r_1}e_1,\ldots,u^{r_d}e_d)=(e_1,\ldots,e_d)G.$$

Since the map

$$(k[u]/(u^{ep}))[Y_1,Y_2,\ldots]/(Y_1^p,Y_2^p,\ldots)\to S_1$$

which sends Y_i to the image of $E(u)^{p^i}/p^i$ is an isomorphism of k-algebras, we have

$$\mathrm{Fil}^r\mathcal{M}^\vee=u^{er-r_1}S_1e_1^\vee\oplus\cdots\oplus u^{er-r_d}S_1e_d^\vee+(\mathrm{Fil}^pS)\mathcal{M}^\vee,$$
$$\phi_r(u^{er-r_1}e_1^\vee,\ldots,u^{er-r_d}e_d^\vee)=(e_1^\vee,\ldots,e_d^\vee)c^r({}^{\mathrm{t}}G)^{-1}.$$

From this wee see that the natural map $\mathcal{M}\to(\mathcal{M}^\vee)^\vee$ is also an isomorphism of $\mathrm{Mod}^{r,\phi}_{/S_1}$ and the duality functor from $\mathrm{Mod}^{r,\phi}_{/S_1}$ to itself is an anti-equivalence of categories.

Lemma 2.1. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. Then we have an isomorphism $\tau_{\mathfrak{M}} : \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}^\vee) \to \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})^\vee$ of the category $\mathrm{Mod}^{r,\phi}_{/S_1}$ such that $T_0(\tau_{\mathfrak{M}})$ coincides with the natural isomorphism*

$$\tilde{S}_1 \otimes_{\phi,\mathfrak{S}_1} \mathrm{Hom}_{\mathfrak{S}_1}(\mathfrak{M}, \mathfrak{S}_1) \to \tilde{S}_1 \otimes_{S_1} \mathrm{Hom}_{S_1}(S_1 \otimes_{\phi,\mathfrak{S}} \mathfrak{M}, S_1).$$

These isomorphisms define a natural isomorphism of functors

$$\tau : \mathcal{M}_{\mathfrak{S}}((-)^\vee) \to \mathcal{M}_{\mathfrak{S}}(-)^\vee.$$

Proof. Since T_0 is an equivalence of categories, we only have to check that the natural isomorphism of the lemma is compatible with Fil^r's and ϕ_r's of both sides. For this, choose a basis $e_1, \dots, e_d$ of $\mathfrak{M}$ over $\mathfrak{S}_1$ and let $e_1^\vee, \dots, e_d^\vee$ denote its dual basis of $\mathfrak{M}^\vee$. Put $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) = S \otimes_{\phi,\mathfrak{S}} \mathfrak{M}$. Let us consider its basis $1 \otimes e_1, \dots, 1 \otimes e_d$ and the dual basis $(1 \otimes e_1)^\vee, \dots, (1 \otimes e_d)^\vee$ of $\mathcal{M}^\vee$. Let $A \in M_d(\mathfrak{S}_1)$ be the matrix defined by

$$\phi(e_1, \dots, e_d) = (e_1, \dots, e_d)A.$$

By definition, there exists a matrix $B \in M_d(\mathfrak{S}_1)$ such that $AB = BA = E(u)^r I$, where I is the identity matrix. Since the map $\phi_{\mathfrak{M}^\vee}$ is given by

$$\phi_{\mathfrak{M}^\vee}(e_1^\vee, \dots, e_d^\vee) = (e_1^\vee, \dots, e_d^\vee)(-F(u))^{-r}\,{}^{\mathrm{t}}B,$$

we can write Fil^r and ϕ_r of $\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}^\vee)$ as

$$\mathrm{Fil}^r \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}^\vee) = \mathrm{Span}_S((1 \otimes e_1^\vee, \dots, 1 \otimes e_d^\vee)\,{}^{\mathrm{t}}A) + (\mathrm{Fil}^p S)\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}^\vee),$$
$$\phi_r((1 \otimes e_1^\vee, \dots, 1 \otimes e_d^\vee)\,{}^{\mathrm{t}}A) = (1 \otimes e_1^\vee, \dots, 1 \otimes e_d^\vee)c^r \phi(-F(u))^{-r}.$$

On the other hand, we have

$$\mathrm{Fil}^r \mathcal{M} = \mathrm{Span}_S((1 \otimes e_1, \dots, 1 \otimes e_d)B) + (\mathrm{Fil}^p S)\mathcal{M},$$
$$\phi_r((1 \otimes e_1, \dots, 1 \otimes e_d)B) = (1 \otimes e_1, \dots, 1 \otimes e_d)c^r,$$

and from this we see that

$$\mathrm{Fil}^r \mathcal{M}^\vee = \mathrm{Span}_S(((1 \otimes e_1)^\vee, \dots, (1 \otimes e_d)^\vee)\,{}^{\mathrm{t}}A) + (\mathrm{Fil}^p S)\mathcal{M}^\vee,$$
$$\phi_r(((1 \otimes e_1)^\vee, \dots, (1 \otimes e_d)^\vee)\,{}^{\mathrm{t}}A) = ((1 \otimes e_1)^\vee, \dots, (1 \otimes e_d)^\vee).$$

The natural isomorphism in the lemma sends the image of $1 \otimes e_i^\vee$ to that of $(1 \otimes e_i)^\vee$. Since the elements c and $\phi(-F(u))$ have the same image in $\tilde{S}_1$, this isomorphism is compatible with Fil^r's and ϕ_r's. $\square$

2.3. **The associated Galois representations and duality.** Next we recall a construction of the associated Galois representations to Breuil or Kisin modules and their duality theories. Let v_K be the valuation on K which is normalized as $v_K(\pi) = 1$ and we extend it naturally to $\bar{K}$. Set $\tilde{\mathcal{O}}_{\bar{K}} = \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}}$ and $\mathbb{C}$ to be the completion of $\bar{K}$. Consider the ring

$$R = \varprojlim(\tilde{\mathcal{O}}_{\bar{K}} \leftarrow \tilde{\mathcal{O}}_{\bar{K}} \leftarrow \cdots),$$

where the transition maps are defined by $x \mapsto x^p$. For an element $x = (x_0, x_1, \dots) \in R$ with $x_i \in \tilde{\mathcal{O}}_{\bar{K}}$, we put $x^{(m)} = \lim_{n\to\infty} \hat{x}_{n+m}^{p^n} \in \mathcal{O}_{\mathbb{C}}$, where

$\hat{x}_i$ is a lift of x_i in $\mathcal{O}_{\bar{K}}$. This is independent of the choice of lifts. Then the ring R is a complete valuation ring of characteristic p with valuation $v_R(x) = v_K(x^{(0)})$. We put $m_R^{\geq i} = \{x \in R \mid v_R(x) \geq i\}$ and similarly for $m_R^{>i}$. Consider a natural W-algebra surjection $\theta : W(R) \to \mathcal{O}_{\mathbb{C}}$ defined as

$$\theta((z_0, z_1, \ldots)) = \sum_{i=0}^{\infty} p^i z_i^{(i)},$$

where z_i is an element of R. The p-adic completion of the divided power envelope of $W(R)$ with respect to the ideal $\mathrm{Ker}(\theta)$ is denoted by A_{crys}. The ring A_{crys} has a Frobenius endomorphism ϕ and a $\mathrm{Gal}(\bar{K}/K)$-action induced from those of R, and also a filtration induced by the divided power structure. Define an element $\underline{\pi}$ of R by $\underline{\pi} = (\pi, \pi_1, \pi_2, \ldots)$, where we abusively write π_n also for its image in $\tilde{\mathcal{O}}_{\bar{K}}$. The W-algebras R, $W(R)$ and A_{crys} have natural $\mathfrak{S}$-algebra structures defined by the continuous map $\mathfrak{S} \to W(R)$ which sends u to the Teichmüller lift $[\underline{\pi}]$ of the element $\underline{\pi}$. We consider the ring A_{crys} as an S-algebra by the induced map $S \to A_{\mathrm{crys}}$ from a similar map $W[u] \to W(R)$ defined by $u \mapsto [\underline{\pi}]$. For $0 \leq r \leq p-2$, the ring A_{crys} has a natural structure as an object of $'\mathrm{Mod}_{/S}^{r,\phi}$ by putting $\phi_r = p^{-r}\phi|_{\mathrm{Fil}^r A_{\mathrm{crys}}}$. Note that the identification $\mathrm{Gal}(\bar{K}/K_\infty) \simeq \mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$ stated in Section 1 is given by the action of $\mathrm{Gal}(\bar{K}/K_\infty)$ on the ring R and the inclusion $k[[u]] \to R$ defined by $u \mapsto \underline{\pi}$ ([9, Subsection 3.3]).

We have another description of the ring A_{crys} as follows. Consider the Witt ring $W_n(\tilde{\mathcal{O}}_{\bar{K}})$ as a W-algebra by twisting the usual W-action by ϕ^{-n}. Then the map

$$\begin{aligned} \theta_n : W_n(\tilde{\mathcal{O}}_{\bar{K}}) &\to \mathcal{O}_{\bar{K}}/p^n\mathcal{O}_{\bar{K}} \\ (a_0, \ldots, a_{n-1}) &\mapsto \sum_{i=0}^{n-1} p^i \hat{a}_i^{p^{n-i}}, \end{aligned}$$

where $\hat{a}_i$ is a lift of a_i to $\mathcal{O}_{\bar{K}}/p^n\mathcal{O}_{\bar{K}}$, is a well-defined W-algebra surjection. We let $W_n^{\mathrm{DP}}(\tilde{\mathcal{O}}_{\bar{K}})$ denote the divided power envelope of the ring $W_n(\tilde{\mathcal{O}}_{\bar{K}})$ with respect to the ideal $\mathrm{Ker}(\theta_n)$. We consider this ring as an S-algebra by $u \mapsto [\pi_n]$ and give it a structure as an object of $'\mathrm{Mod}_{/S}^{r,\phi}$ similar to that of A_{crys} ([7, Subsection 2.2.2]). Then the map $\mathrm{pr}_n : R \to \tilde{\mathcal{O}}_{\bar{K}}$ defined by $x = (x_0, x_1, \ldots) \mapsto x_n$ induces an isomorphism $A_{\mathrm{crys}}/p^n A_{\mathrm{crys}} \to W_n^{\mathrm{DP}}(\tilde{\mathcal{O}}_{\bar{K}})$ of $'\mathrm{Mod}_{/S}^{r,\phi}$.

Let $\mathfrak{M}$ be an object of $\mathrm{Mod}_{/\mathfrak{S}_1}^{r,\phi}$. The associated $\mathrm{Gal}(\bar{K}/K_\infty)$-module $T_{\mathfrak{S}}^*(\mathfrak{M})$ is defined as

$$T_{\mathfrak{S}}^*(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\phi}(\mathfrak{M}, R).$$

Similarly, for an object $\mathcal{M}$ of $\mathrm{Mod}_{/S_1}^{r,\phi}$, we also have the associated $\mathrm{Gal}(\bar{K}/K_\infty)$-module

$$T_{\mathrm{crys}}^*(\mathcal{M}) = \mathrm{Hom}_{S,\mathrm{Fil}^r,\phi_r}(\mathcal{M}, R^{\mathrm{DP}}),$$

where R^{DP} is the divided power envelope of R with respect to the ideal $m_R^{\geq e}$ and we identify this ring with $A_{\mathrm{crys}}/pA_{\mathrm{crys}}$. Then we have an isomorphism of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules

$$\begin{aligned} T^*_{\mathfrak{S}}(\mathfrak{M}) &\to T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})) \\ f &\mapsto (s \otimes m \mapsto s\phi(f(m))). \end{aligned}$$

For an object $\tilde{\mathcal{M}}$ of $\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$, we also put

$$\tilde{T}^*_{\mathrm{crys}}(\tilde{\mathcal{M}}) = \mathrm{Hom}_{\tilde{S}_1,\mathrm{Fil}^r,\phi_r}(\tilde{\mathcal{M}}, R^{\mathrm{DP}}/\mathrm{Fil}^p R^{\mathrm{DP}}).$$

The Galois group $\mathrm{Gal}(\bar{K}/K_\infty)$ also acts on this module and for an object $\mathcal{M}$ of $\mathrm{Mod}^{r,\phi}_{/S_1}$, the natural map

$$T^*_{\mathrm{crys}}(\mathcal{M}) \to \tilde{T}^*_{\mathrm{crys}}(T_0(\mathcal{M}))$$

is an isomorphism of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules. The functors $T^*_{\mathfrak{S}}$, T^*_{crys} and $\tilde{T}^*_{\mathrm{crys}}$ from $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$, $\mathrm{Mod}^{r,\phi}_{/S_1}$ and $\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ to the category of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules over $\mathbb{F}_p$ are exact. For an object $\mathfrak{M}$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$, we have $\dim_{\mathbb{F}_p}(T^*_{\mathfrak{S}}(\mathfrak{M})) = \mathrm{rank}_{\mathfrak{S}_1}(\mathfrak{M})$ and similar assertions also hold for T^*_{crys} and $\tilde{T}^*_{\mathrm{crys}}$.

To describe a duality theory for $T^*_{\mathfrak{S}}(\mathfrak{M})$ and $T^*_{\mathrm{crys}}(\mathcal{M})$, let us also fix a system $\{\zeta_{p^n}\}_{n\in\mathbb{Z}_{\geq 0}}$ of p-power roots of unity in $\bar{K}$ such that $\zeta_1 = 1$, $\zeta_p \neq 1$ and $\zeta_{p^n} = \zeta_{p^{n+1}}^p$, and set an element $\underline{\varepsilon} \in R$ to be $\underline{\varepsilon} = (1, \zeta_p, \zeta_{p^2}, \ldots)$ with an abusive notation as before. Define an element $\bar{t}$ of R^{DP} by

$$\bar{t} = \log(\underline{\varepsilon}) = \sum_{i=1}^{\infty}(-1)^{i-1}(i-1)!\gamma_i(\underline{\varepsilon}-1),$$

where γ_i means the i-th divided power.

Let $\mathfrak{S}_1(r) = \mathfrak{S}_1\mathbf{e}$ be the free $\mathfrak{S}_1$-module of rank one with a basis $\mathbf{e}$ endowed with the ϕ-semilinear map $\phi(\mathbf{e}) = (-E(u)/F(u))^r\mathbf{e}$. Note that the infinite product

$$\prod_{i=0}^{\infty}\phi^i(\phi_r((-E(u)/F(u))^r))$$

converges p-adically to a unit α in S. Then $\mathcal{M}_{\mathfrak{S}}(\mathfrak{S}_1(r))$ is isomorphic to the Breuil module $S_1(r)$ defined as $(S_1(r) = S_1\mathbf{e}, \mathrm{Fil}^r S_1(r) = S_1(r), \phi_r(\mathbf{e}) = \mathbf{e})$ via the multiplication by α^{-1}. Thus we have isomorphisms of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules

$$T^*_{\mathfrak{S}}(\mathfrak{S}_1(r)) \to T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{S}_1(r))) \to T^*_{\mathrm{crys}}(S_1(r)) = \mathbb{F}_p(\mathbf{e} \mapsto \bar{t}^r).$$

Their composite is given by $f \mapsto (\mathbf{e} \mapsto \alpha\phi(f(\mathbf{e})))$.

Let $\mathfrak{M}$ and $\mathfrak{M}^\vee$ be as above and choose a basis $e_1, \ldots, e_d$ of $\mathfrak{M}$. Consider the pairing

$$D_{\mathfrak{M}} : T^*_{\mathfrak{S}}(\mathfrak{M}) \times T^*_{\mathfrak{S}}(\mathfrak{M}^\vee) \to T^*_{\mathfrak{S}}(\mathfrak{S}_1(r))$$
$$(f, f^\vee) \mapsto (\mathbf{e} \mapsto \sum_{i=1}^{d} f(e_i) f^\vee(e_i^\vee)),$$

where $e_1^\vee, \ldots, e_d^\vee$ is the dual basis of $e_1, \ldots, e_d$. From the definition, we can check that the map on the right-hand is compatible with $\phi_{\mathfrak{S}_1(r)}$.

Lemma 2.2. *The pairing $D_{\mathfrak{M}}$ is a $\mathrm{Gal}(\bar{K}/K_\infty)$-equivariant perfect pairing independent of the choice of a basis.*

Proof. The $\mathrm{Gal}(\bar{K}/K_\infty)$-equivariance is obvious. Since this pairing is induced by the natural pairing

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, R) \times \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}^\vee, R) \simeq R \otimes_{\mathfrak{S}_1} \mathfrak{M}^\vee \times \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}^\vee, R) \to R,$$

it is independent of the choice of a basis. We can show the perfectness of the pairing just as in [10, Théorème 4.3.1], as follows. Let $A = (a_{i,j})$ be the representing matrix of $\phi_{\mathfrak{M}}$ for a basis $e_1, \ldots, e_d$ of $\mathfrak{M}$. An element f of $T^*_{\mathfrak{S}}(\mathfrak{M})$ is identified with a solution $(f(e_1), \ldots, f(e_d))$ in R of the system of equations $X_i^p = \sum_{j=1}^d a_{j,i} X_j$ for $i = 1, \ldots, d$. Suppose that an element $f^\vee \in T^*_{\mathfrak{S}}(\mathfrak{M}^\vee)$ satisfies $D_{\mathfrak{M}}(f, f^\vee) = 0$ for any $f \in T^*_{\mathfrak{S}}(\mathfrak{M})$. Put $y_i = f^\vee(e_i^\vee)$. Then the set $T^*_{\mathfrak{S}}(\mathfrak{M})$ is identified with the set of solutions in R of the system of equations $X_i^p = \sum_{j=1}^d a_{j,i} X_j$ $(i = 1, \ldots, d)$ and $y_1 X_1 + \cdots + y_d X_d = 0$. Since this system of equations has no more than p^{d-1} solutions in R unless $y_i = 0$ for any i, the perfectness follows. $\square$

Lemma 2.3. *For an object $\mathcal{M}$ of $\mathrm{Mod}^{r,\phi}_{/S_1}$, we have a $\mathrm{Gal}(\bar{K}/K_\infty)$-equivariant perfect pairing*

$$D_{\mathcal{M}} : T^*_{\mathrm{crys}}(\mathcal{M}) \times T^*_{\mathrm{crys}}(\mathcal{M}^\vee) \to T^*_{\mathrm{crys}}(S_1(r))$$
$$(f, f^\vee) \mapsto (\mathbf{e} \mapsto \sum_{i=1}^{d} f(e_i) f^\vee(e_i^\vee)),$$

where $e_1, \ldots, e_d$ is a basis of $\mathcal{M}$ and $e_1^\vee, \ldots, e_d^\vee$ is its dual basis. This pairing is independent of the choice of a basis $e_1, \ldots, e_d$.

Proof. This is a part of [10, Théorème 4.3.1]. Here we present a slightly different proof. First note that the map $D_{\mathcal{M}}(f, f^\vee) : \mathbf{e} \mapsto \sum_{i=1}^d f(e_i) f^\vee(e_i^\vee)$ is independent of the choice of a basis. Thus we may assume that $e_1, \ldots, e_d$ is an adapted basis and we put

$$\mathrm{Fil}^r \mathcal{M} = u^{r_1} S_1 e_1 \oplus \cdots \oplus u^{r_d} S_1 e_d + (\mathrm{Fil}^p S)\mathcal{M},$$
$$\phi_r(u^{r_1} e_1, \ldots, u^{r_d} e_d) = (e_1, \ldots, e_d) G$$

as before. Note that the ring homomorphism

$$(R/m_R^{\geq ep})[Y_1, Y_2, \ldots]/(Y_1^p, Y_2^p, \ldots) \to R^{\mathrm{DP}}$$

sending Y_i to $(p^i-1)!\gamma_{p^i}(E(\underline{\pi}))$ is an isomorphism. Take $f \in T^*_{\mathrm{crys}}(\mathcal{M})$ and $f^\vee \in T^*_{\mathrm{crys}}(\mathcal{M}^\vee)$. Since the elements $u^{r_i}f(e_i)$ and $u^{er-r_i}f^\vee(e_i^\vee)$ are contained in $\mathrm{Fil}^r R^{\mathrm{DP}}$, we can write $f(e_i) = u^{er-r_i}z_i + z'_i$ and $f^\vee(e_i^\vee) = u^{r_i}w_i + w'_i$ with some $z_i, w_i \in R/m_R^{\geq ep}$ and $z'_i, w'_i \in \mathrm{Fil}^p R^{\mathrm{DP}}$. This shows that the map $D_{\mathcal{M}}(f, f^\vee)$ preserves the filtration. Moreover, since we have

$$(f(e_1), \ldots, f(e_d))G = (\phi_r(u^{r_1}f(e_1)), \ldots, \phi_r(u^{r_d}f(e_d))),$$
$$(f^\vee(e_1^\vee), \ldots, f^\vee(e_d^\vee))c^r({}^{\mathrm{t}}G)^{-1} = (\phi_r(u^{er-r_1}f^\vee(e_1^\vee)), \ldots, \phi_r(u^{er-r_d}f^\vee(e_d^\vee))),$$

we see that the equality

$$\begin{aligned} c^r D_{\mathcal{M}}(f, f^\vee)(\mathbf{e}) &= \sum_i \phi_r(u^{r_i}f(e_i))\phi_r(u^{er-r_i}f^\vee(e_i^\vee)) \\ &= c^{2r}\sum_i \phi(z_i)\phi(w_i) = c^r\phi_r(D_{\mathcal{M}}(f, f^\vee)(\mathbf{e})) \end{aligned}$$

holds. This shows that the pairing $D_{\mathcal{M}}$ is well-defined. Then the $\mathrm{Gal}(\bar{K}/K_\infty)$-equivariance is obvious, and the following proposition and Lemma 2.2 show the perfectness of $D_{\mathcal{M}}$. □

Proposition 2.4. *We have a commutative diagram*

$$\begin{tikzcd} T^*_{\mathfrak{S}}(\mathfrak{M}) \times T^*_{\mathfrak{S}}(\mathfrak{M}^\vee) \arrow[r, "D_{\mathfrak{M}}"] \arrow[d, "\wr"'] & T^*_{\mathfrak{S}}(\mathfrak{S}_1(r)) \arrow[dd, "\wr\ \alpha\phi"] \\ T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})) \times T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}^\vee)) \arrow[d, "{= \times T^*_{\mathrm{crys}}(\tau_{\mathfrak{M}})^{-1}}"'] & \\ T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})) \times T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})^\vee) \arrow[r, "D_{\mathcal{M}}"] & T^*_{\mathrm{crys}}(S_1(r)), \end{tikzcd}$$

where the right vertical arrow is given by $f \mapsto (\mathbf{e} \mapsto \alpha\phi(f(\mathbf{e})))$ as before.

Proof. For an object $\mathcal{M}$ of $\mathrm{Mod}^{r,\phi}_{/S_1}$, we have a commutative diagram

$$\begin{tikzcd} T^*_{\mathrm{crys}}(\mathcal{M}) \times T^*_{\mathrm{crys}}(\mathcal{M}^\vee) \arrow[r, "D_{\mathcal{M}}"] \arrow[d, "\wr"'] & T^*_{\mathrm{crys}}(S_1(r)) \arrow[d, "\wr"'] \\ \tilde{T}^*_{\mathrm{crys}}(T_0(\mathcal{M})) \times \tilde{T}^*_{\mathrm{crys}}(T_0(\mathcal{M}^\vee)) \arrow[r, "D_{\tilde{\mathcal{M}}}"] & \tilde{T}^*_{\mathrm{crys}}(\tilde{S}_1(r)), \end{tikzcd}$$

where the object $\tilde{S}_1(r)$ of $\mathrm{Mod}^{r,\phi}_{/\tilde{S}_1}$ and the pairing $D_{\tilde{\mathcal{M}}}$ are defined similarly to the case of $\mathrm{Mod}^{r,\phi}_{/S_1}$. Composing these two diagrams, it suffices to show the commutativity of the diagram

$$\begin{tikzcd} T^*_{\mathfrak{S}}(\mathfrak{M}) \times T^*_{\mathfrak{S}}(\mathfrak{M}^\vee) \arrow[r, "D_{\mathfrak{M}}"] \arrow[d, "\wr"'] & T^*_{\mathfrak{S}}(\mathfrak{S}_1(r)) \arrow[d, "\wr"', "\alpha\phi"] \\ \tilde{T}^*_{\mathrm{crys}}(T_0(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}))) \times \tilde{T}^*_{\mathrm{crys}}(T_0(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})^\vee)) \arrow[r, "D_{\tilde{\mathcal{M}}}"] & \tilde{T}^*_{\mathrm{crys}}(\tilde{S}_1(r)), \end{tikzcd}$$

where the middle vertical arrow is defined as

$$f^\vee \mapsto ((1 \otimes e_i)^\vee \mapsto \phi(f^\vee(e_i^\vee))).$$

Since the image of α in $\tilde{S}_1$ is equal to 1, the commutativity follows from the definition. □

3. Cartier duality for upper and lower ramification subgroups

Let $\mathcal{G}$ be a finite flat generically etale (commutative) group scheme over the ring of integers of a complete discrete valuation field. Abbes-Mokrane ([2]) initiated a study of ramification of $\mathcal{G}$ using a ramification theory of Abbes-Saito ([3], [4]). As in the classical ramification theory of local fields, $\mathcal{G}$ has upper and lower ramification subgroups ([17], [14]). When the base field is of mixed characteristic, Tian proved that the upper and lower ramification subgroups correspond to each other via usual Cartier duality if $\mathcal{G}$ is killed by p ([25]), and Fargues found a much simpler proof of this theorem ([14]). In this section, after briefly recalling the ramification theory of finite flat group schemes, we show a variant of Tian's theorem for a complete discrete valuation field of equal characteristic p with perfect residue field, using the duality techniques presented in the previous section instead of Cartier duality of finite flat group schemes.

3.1. **Ramification theory of Abbes-Saito and finite flat group schemes.** In this subsection, we let K denote a complete discrete valuation field, π a uniformizer of K, $\mathcal{O}_K$ the ring of integers, F the residue field and K^{sep} a separable closure of K. Let v_K be the valuation of K normalized as $v_K(\pi) = 1$ and extend it naturally to K^{sep}. We put

$$m^{\geq i}_{K^{\mathrm{sep}}} = \{x \in \mathcal{O}_{K^{\mathrm{sep}}} \mid v_K(x) \geq i\}$$

and similarly for $m^{>i}_{K^{\mathrm{sep}}}$.

An $\mathcal{O}_K$-algebra $\mathbb{B}$ is said to be formally of finite type if $\mathbb{B}$ is a complete Noetherian semi-local ring and $\mathbb{B}/\mathrm{rad}(\mathbb{B})$ is finite over F. An $\mathcal{O}_K$-algebra surjection $\mathbb{B} \to B$ from an $\mathcal{O}_K$-algebra $\mathbb{B}$ formally of finite type and formally smooth to a finite flat $\mathcal{O}_K$-algebra B is said as an embedding of B if it induces an isomorphism $\mathbb{B}/\mathrm{rad}(\mathbb{B}) \to B/\mathrm{rad}(B)$. The embeddings of B form a category in an obvious way. Then the j-th tubular neighborhood $X^j_K(\mathbb{B} \to B)$ of an embedding $(\mathbb{B} \to B)$ for $j \in \mathbb{Q}_{>0}$ is the affinoid variety over K defined as

$$X^j_K(\mathbb{B} \to B) = \mathrm{Sp}(K \otimes_{\mathcal{O}_K} (\mathbb{B}[I^l/\pi^k])^\wedge),$$

where $j = k/l$ with $k, l \in \mathbb{Z}_{>0}$, $I = \mathrm{Ker}(\mathbb{B} \to B)$, $\mathbb{B}[I^l/\pi^k]$ is the subring of $K \otimes_{\mathcal{O}_K} \mathbb{B}$ generated by the elements f/π^k with $f \in I^l$ and $\wedge$ means the π-adic completion. This affinoid variety is independent of the choice of a presentation $j = k/l$. For an affinoid variety X over K, the set of geometric connected components of X is denoted by $\pi_0(X_{K^{\mathrm{sep}}})$. We put abusively $\mathcal{F}_K(B) = \mathcal{F}_K(\mathrm{Spec}(B)) = \mathrm{Hom}_{\mathcal{O}_K\text{-alg.}}(B, \mathcal{O}_{K^{\mathrm{sep}}})$ and

$$\mathcal{F}_K^j(B) = \mathcal{F}_K^j(\mathrm{Spec}(B)) = \varprojlim \pi_0(X_K^j(\mathbb{B} \to B)_{K^{\mathrm{sep}}}),$$

where the projective limit is taken along the category of embeddings of B. In fact, this projective system is constant. These two define functors from the category of finite flat $\mathcal{O}_K$-algebras (or $\mathcal{O}_K$-schemes) to that of finite $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-sets and we also have a natural map $\mathcal{F}_K(B) \to \mathcal{F}_K^j(B)$ of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-sets. If B is relatively complete intersection over $\mathcal{O}_K$ and $K \otimes_{\mathcal{O}_K} B$ is etale over K, then this map is a surjection and the conductor of B (or of $\mathrm{Spec}(B)$)

$$c_K(B) = c_K(\mathrm{Spec}(B)) = \inf\{j \in \mathbb{Q}_{>0} \mid \mathcal{F}_K(B) \to \mathcal{F}_K^j(B) \text{ is a bijection}\}$$

is shown to be an element of $\mathbb{Q}_{\geq 0}$. Moreover, for a finite separable extension L/K in K^{sep} of relative ramification index e', we have a commutative diagram of $\mathrm{Gal}(K^{\mathrm{sep}}/L)$-sets

$$\begin{array}{ccc}
\mathcal{F}_L(\mathcal{O}_L \otimes_{\mathcal{O}_K} B) & \longrightarrow & \mathcal{F}_K(B)|_{\mathrm{Gal}(K^{\mathrm{sep}}/L)} \\
\downarrow & & \downarrow \\
\mathcal{F}_L^{je'}(\mathcal{O}_L \otimes_{\mathcal{O}_K} B) & \longrightarrow & \mathcal{F}_K^j(B)|_{\mathrm{Gal}(K^{\mathrm{sep}}/L)}
\end{array}$$

whose horizontal arrows are natural isomorphisms.

Let $\mathcal{G} = \mathrm{Spec}(B)$ be a finite flat generically etale group scheme over $\mathcal{O}_K$. Then $\mathcal{G}$ is relatively complete intersection over $\mathcal{O}_K$ ([8, Proposition 2.2.2]). By a functoriality of the functors $\mathcal{F}_K$ and $\mathcal{F}_K^j$, the $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-sets $\mathcal{F}_K(\mathcal{G})$ and $\mathcal{F}_K^j(\mathcal{G})$ have natural $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-module structures and the map $\mathcal{F}_K(\mathcal{G}) \to \mathcal{F}_K^j(\mathcal{G})$ is a surjection of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-modules. The scheme-theoretic closure in $\mathcal{G}$ of the kernel of this surjection is denoted by $\mathcal{G}^j$ and called the j-th upper ramification subgroup of $\mathcal{G}$. On the other hand, for $i \in \mathbb{Q}_{\geq 0}$, the scheme-theoretic closure in $\mathcal{G}$ of $\mathrm{Ker}(\mathcal{G}(\mathcal{O}_{K^{\mathrm{sep}}}) \to \mathcal{G}(\mathcal{O}_{K^{\mathrm{sep}}}/m_{K^{\mathrm{sep}}}^{\geq i}))$ is denoted by $\mathcal{G}_i$ and called the i-th lower ramification subgroup of $\mathcal{G}$. We also put

$$\mathcal{G}^{j+}(\mathcal{O}_{K^{\mathrm{sep}}}) = \bigcup_{j'>j} \mathcal{G}^{j'}(\mathcal{O}_{K^{\mathrm{sep}}}),\ \mathcal{G}_{i+}(\mathcal{O}_{K^{\mathrm{sep}}}) = \bigcup_{i'>i} \mathcal{G}_{i'}(\mathcal{O}_{K^{\mathrm{sep}}}).$$

As in the classical case, the upper (*resp.* lower) ramification subgroups are compatible with quotients (*resp.* subgroups). Namely, for a faithfully flat homomorphism $\mathcal{G} \to \mathcal{G}''$ of finite flat group schemes over $\mathcal{O}_K$, the image of $\mathcal{G}^j(\mathcal{O}_{K^{\mathrm{sep}}})$ in $\mathcal{G}''(\mathcal{O}_{K^{\mathrm{sep}}})$ coincides with $(\mathcal{G}'')^j(\mathcal{O}_{K^{\mathrm{sep}}})$ ([2, Lemme 2.3.2]). From the definition, we also see that for a closed immersion $\mathcal{G}' \to \mathcal{G}$ of finite flat group schemes over $\mathcal{O}_K$, the subgroup $\mathcal{G}'(\mathcal{O}_{K^{\mathrm{sep}}}) \cap \mathcal{G}_i(\mathcal{O}_{K^{\mathrm{sep}}})$ coincides with $(\mathcal{G}')_i(\mathcal{O}_{K^{\mathrm{sep}}})$. In addition, for a finite separable extension L/K in K^{sep} of relative ramification index e', we have natural isomorphisms

$$(\mathcal{O}_L \times_{\mathcal{O}_K} \mathcal{G})^{je'} \to \mathcal{O}_L \times_{\mathcal{O}_K} \mathcal{G}^j,\ (\mathcal{O}_L \times_{\mathcal{O}_K} \mathcal{G})_{ie'} \to \mathcal{O}_L \times_{\mathcal{O}_K} \mathcal{G}_i$$

of finite flat group schemes over $\mathcal{O}_L$.

Suppose that K is of mixed characteristic $(0,p)$. If $\mathcal{G}$ is killed by p^n, then we have $c_K(\mathcal{G}) \leq e(n+1/(p-1))$, where e is the absolute ramification index of K ([17, Theorem 7]). When $\mathcal{G}$ is killed by p, the upper and lower ramification subgroups of $\mathcal{G}$ switch to each other via Cartier duality, as follows. Let $\mathcal{G}^\vee$ be the Cartier dual of $\mathcal{G}$, and consider the perfect pairing of Cartier duality

$$\langle\ ,\ \rangle_{\mathcal{G}} : \mathcal{G}(\mathcal{O}_{\bar{K}}) \times \mathcal{G}^\vee(\mathcal{O}_{\bar{K}}) \to \mu_p(\mathcal{O}_{\bar{K}}).$$

Then we have the following duality theorem for the upper and lower ramification subgroups of $\mathcal{G}$ ([25, Theorem 1.6], [14, Proposition 6]).

Theorem 3.1. *(Tian, Fargues) Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ with absolute ramification index e and $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ killed by p. For $j \leq pe/(p-1)$, we have an equality*

$$\mathcal{G}^j(\mathcal{O}_{\bar{K}})^\perp = (\mathcal{G}^\vee)_{l(j)+}(\mathcal{O}_{\bar{K}})$$

of subgroups of $\mathcal{G}^\vee(\mathcal{O}_{\bar{K}})$, where $\perp$ means the orthogonal subgroup with respect to the pairing $\langle\ ,\ \rangle_{\mathcal{G}}$ and $l(j) = e/(p-1) - j/p$.

3.2. **Kisin modules and finite flat group schemes of equal characteristic.** Consider the complete discrete valuation field $\mathcal{X} = k((u))$ with uniformizer u and perfect residue field k. We embed $\mathcal{O}_{\mathcal{X}} = \mathfrak{S}_1 = k[[u]]$ into R by $u \mapsto \underline{\pi}$ as before. Then R is the completion of the ring of integers of an algebraic closure of $\mathcal{O}_{\mathcal{X}}$. We let $\mathcal{X}^{\mathrm{sep}}$ denote the separable closure of $\mathcal{X}$ in $\mathrm{Frac}(R)$.

Let $\mathcal{Y}$ be a finite extension of $\mathcal{X}$ in R of relative ramification index e' and let ϕ also denote the absolute Frobenius endomorphism of $\mathcal{Y}$. We identify $\mathcal{O}_{\mathcal{Y}}$ with $l[[v]]$ for a finite extension l of k. Define a category $\mathrm{Mod}^{r,\phi}_{/\mathcal{O}_{\mathcal{Y}}}$ to be the category of free $\mathcal{O}_{\mathcal{Y}}$-modules $\mathfrak{N}$ of finite rank endowed with a ϕ-semilinear map $\phi_{\mathfrak{N}} : \mathfrak{N} \to \mathfrak{N}$ such that the cokernel of the map $1 \otimes \phi_{\mathfrak{N}} : \mathcal{O}_{\mathcal{Y}} \otimes_{\phi, \mathcal{O}_{\mathcal{Y}}} \mathfrak{N} \to \mathfrak{N}$ is killed by u^{er}.

For an object $\mathfrak{M}$ of $\mathrm{Mod}^{r,\phi}_{/\mathcal{O}_{\mathcal{X}}} = \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$, we consider the $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathfrak{M}$ as an object of $\mathrm{Mod}^{r,\phi}_{/\mathcal{O}_{\mathcal{Y}}}$ by giving it the ϕ-semilinear map $\phi \otimes \phi_{\mathfrak{M}}$. This defines a base change functor $\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} - : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1} \to \mathrm{Mod}^{r,\phi}_{/\mathcal{O}_{\mathcal{Y}}}$. We also have a duality theory for the category $\mathrm{Mod}^{r,\phi}_{/\mathcal{O}_{\mathcal{Y}}}$ similar to that of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. Let $\mathfrak{N}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathcal{O}_{\mathcal{Y}}}$ and put $\mathfrak{N}^\vee = \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\mathfrak{N}, \mathcal{O}_{\mathcal{Y}})$. Take a basis $e_1, \ldots, e_d$ of $\mathfrak{N}$ and let $e_1^\vee, \ldots, e_d^\vee$ be its dual basis of $\mathfrak{N}^\vee$. Define a matrix $A \in M_d(\mathcal{O}_{\mathcal{Y}})$ by

$$\phi(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A.$$

Then we put

$$\phi_{\mathfrak{N}^\vee}(e_1^\vee, \dots, e_d^\vee) = (e_1^\vee, \dots, e_d^\vee)(-E(u)/F(u))^r ({}^{\mathrm{t}}A)^{-1}.$$

We can show that this is independent of the choice of a basis and we have an isomorphism of the double duality as before. Moreover, for an object $\mathfrak{M}$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$, we have a natural isomorphism

$$\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathfrak{M}^\vee \to (\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathfrak{M})^\vee$$

of $\mathrm{Mod}^{r,\phi}_{/\mathcal{O}_{\mathcal{Y}}}$, by which we identify both sides.

For a finite flat group scheme $\mathcal{H}$ over a base of characteristic p, we let $F_{\mathcal{H}}$ and $V_{\mathcal{H}}$ denote the Frobenius and Verschiebung of $\mathcal{H}$, respectively. We say that a finite flat group scheme $\mathcal{H}$ over $\mathcal{O}_{\mathcal{Y}}$ is v-height $\leq s$ if its Verschiebung $V_{\mathcal{H}}$ is zero and the cokernel of the natural map

$$F_{\mathcal{H}}^* : \mathcal{O}_{\mathcal{Y}} \otimes_{\phi, \mathcal{O}_{\mathcal{Y}}} \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}\text{-grp.}}(\mathcal{H}, \mathbb{G}_{\mathrm{a}}) \to \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}\text{-grp.}}(\mathcal{H}, \mathbb{G}_{\mathrm{a}})$$

is killed by v^s. The category of finite flat group schemes over $\mathcal{O}_{\mathcal{Y}}$ of v-height $\leq s$ is denoted by $\mathcal{C}^{\leq s}_{\mathcal{O}_{\mathcal{Y}}}$. Then we have an anti-equivalence of categories

$$\mathcal{H}_{\mathcal{Y}}(-) : \mathrm{Mod}^{r,\phi}_{/\mathcal{O}_{\mathcal{Y}}} \to \mathcal{C}^{\leq ee'r}_{\mathcal{O}_{\mathcal{Y}}}$$

([16, Théorème 7.4]). The group scheme $\mathcal{H}_{\mathcal{Y}}(\mathfrak{N})$ is defined as a functor over $\mathcal{O}_{\mathcal{Y}}$ by

$$\mathfrak{A} \mapsto \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}, \phi}(\mathfrak{N}, \mathfrak{A}),$$

where we consider an $\mathcal{O}_{\mathcal{Y}}$-algebra $\mathfrak{A}$ as a ϕ-module over $\mathcal{O}_{\mathcal{Y}}$ with the absolute Frobenius endomorphism of $\mathfrak{A}$. If we choose a basis $e_1, \dots, e_d$ of $\mathfrak{N}$ and a matrix $A = (a_{i,j}) \in M_d(\mathcal{O}_{\mathcal{Y}})$ as above, then $\mathcal{H}_{\mathcal{Y}}(\mathfrak{N})$ is isomorphic to the additive group scheme over $\mathcal{O}_{\mathcal{Y}}$ defined by the system of equations

$$X_i^p - \sum_{j=1}^{d} a_{j,i} X_j = 0 \quad (i = 1, \dots, d).$$

For an object $\mathfrak{M}$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$, we also have a natural isomorphism

$$\mathcal{O}_{\mathcal{Y}} \times_{\mathcal{O}_{\mathcal{X}}} \mathcal{H}_{\mathcal{X}}(\mathfrak{M}) \to \mathcal{H}_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathfrak{M})$$

of finite flat group schemes over $\mathcal{O}_{\mathcal{Y}}$. We drop the subscript $\mathcal{Y}$ of $\mathcal{H}_{\mathcal{Y}}$ if there is no risk of confusion.

The following lemma is a variant of the scheme-theoretic closure for finite flat group schemes.

Lemma 3.2. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ and L be a $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$-stable subgroup of $T^*_{\mathfrak{S}}(\mathfrak{M})$. Then there exists a surjection $\mathfrak{M} \to \mathfrak{M}''$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ such that the image of the corresponding injection $T^*_{\mathfrak{S}}(\mathfrak{M}'') \to T^*_{\mathfrak{S}}(\mathfrak{M})$ coincides with L. A surjection $\mathfrak{M} \to \mathfrak{M}''$ satisfying this property is unique up to a unique isomorphism.*

Proof. This follows from the same argument as in [21, Lemma 2.3.6]. Indeed, let $\mathrm{Mod}^{\phi}_{/\mathcal{X}}$ denote the category of etale ϕ-modules over $\mathcal{X}$ ([15, Section A1]). We have an equivalence of categories T_* from $\mathrm{Mod}^{\phi}_{/\mathcal{X}}$ to the category of finite $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$-modules over $\mathbb{F}_p$ defined by $T_*(M) = (\mathcal{X}^{\mathrm{sep}} \otimes_{\mathcal{X}} M)^{\phi=1}$. For an object M of $\mathrm{Mod}^{\phi}_{/\mathcal{X}}$, we also put $T^*(M) = \mathrm{Hom}_{\mathcal{X},\phi}(M, \mathcal{X}^{\mathrm{sep}})$. Then the natural map

$$T_*(M) \to \mathrm{Hom}_{\mathbb{F}_p}(T^*(M), \mathbb{F}_p)$$

is an isomorphism of $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$-modules. Set $M = \mathcal{X} \otimes_{\mathfrak{S}_1} \mathfrak{M}$. We have a canonical isomorphism $T^*_{\mathfrak{S}}(\mathfrak{M}) \to T^*(M)$ of $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$-modules and let M'' be the quotient of M corresponding to the surjection

$$T_*(M) \to \mathrm{Hom}_{\mathbb{F}_p}(T^*(M), \mathbb{F}_p) \to \mathrm{Hom}_{\mathbb{F}_p}(L, \mathbb{F}_p).$$

Then the Kisin module $\mathfrak{M}'' = \mathrm{Im}(\mathfrak{M} \to M \to M'')$ satisfies the desired property. □

Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ and L be a $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$-stable subgroup of $\mathcal{H}(\mathfrak{M})(R) \simeq T^*_{\mathfrak{S}}(\mathfrak{M})$. Take a surjection $\mathfrak{M} \to \mathfrak{M}''$ corresponding to the image of L in $T^*_{\mathfrak{S}}(\mathfrak{M})$ as in the previous lemma. We call the subgroup scheme $\mathcal{H}(\mathfrak{M}'')$ of $\mathcal{H}(\mathfrak{M})$ as the scheme-theoretic closure of L.

Since the finite flat group scheme $\mathcal{H}(\mathfrak{M})$ is generically etale, the group $\mathcal{H}(\mathfrak{M})(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})$ can be identified with the group $\mathcal{H}(\mathfrak{M})(R)$. Moreover, we have a surjection of $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$-modules

$$\mathcal{F}_{\mathcal{X}}(\mathcal{H}(\mathfrak{M})) = \mathcal{H}(\mathfrak{M})(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}) = \mathcal{H}(\mathfrak{M})(R) \to \mathcal{F}^j_{\mathcal{X}}(\mathcal{H}(\mathfrak{M}))$$

and its kernel is the j-th upper ramification subgroup $\mathcal{H}(\mathfrak{M})^j(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}) = \mathcal{H}(\mathfrak{M})^j(R)$. We also have the i-th lower ramification subgroup

$$\mathcal{H}(\mathfrak{M})_i(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}) = \mathrm{Ker}(\mathcal{H}(\mathfrak{M})(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}) \to \mathcal{H}(\mathfrak{M})(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}/m^{\geq i}_{\mathcal{X}^{\mathrm{sep}}})),$$

which we identify with

$$\mathcal{H}(\mathfrak{M})_i(R) = \mathrm{Ker}(\mathcal{H}(\mathfrak{M})(R) \to \mathcal{H}(\mathfrak{M})(R/m^{\geq i}_R))$$

by using the injection $\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}}/m^{\geq i}_{\mathcal{X}^{\mathrm{sep}}} \to R/m^{\geq i}_R$. Since $\mathcal{H}(\mathfrak{M})(R) = T^*_{\mathfrak{S}}(\mathfrak{M})$, the natural pairing $D_{\mathfrak{M}}$ of Lemma 2.2 induces a perfect pairing

$$\langle\ ,\ \rangle_{\mathfrak{M}} : \mathcal{H}(\mathfrak{M})(R) \times \mathcal{H}(\mathfrak{M}^\vee)(R) \to R.$$

Then the main theorem of this section is the following.

Theorem 3.3. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. Then the conductor of the finite flat group scheme $\mathcal{H}(\mathfrak{M})$ over $\mathcal{O}_{\mathcal{X}}$ satisfies the inequality $c_{\mathcal{X}}(\mathcal{H}(\mathfrak{M})) \leq per/(p-1)$. Moreover, for $j \leq per/(p-1)$, we have an equality*

$$\mathcal{H}(\mathfrak{M})^j(R)^\perp = \mathcal{H}(\mathfrak{M}^\vee)_{l_r(j)+}(R)$$

of subgroups of $\mathcal{H}(\mathfrak{M}^\vee)(R)$, where $\perp$ means the orthogonal subgroup with respect to the pairing $\langle\ ,\ \rangle_{\mathfrak{M}}$ and $l_r(j) = er/(p-1) - j/p$.

Proof. We proceed as in the proof of [14, Proposition 6]. Let $\mathcal{Y}$ be a finite separable extension of $\mathcal{X}$ in $\mathcal{X}^{\mathrm{sep}}$ of relative ramification index e' and put $\mathfrak{N} = \mathcal{O}_{\mathcal{Y}} \otimes_{\mathfrak{S}_1} \mathfrak{M}$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}(\mathfrak{M})(R) \times \mathcal{H}(\mathfrak{M}^\vee)(R) \arrow[r, "\langle\ ,\ \rangle_{\mathfrak{M}}"] \arrow[d, "\wr"'] & R \arrow[d, equal] \\
\mathcal{H}_{\mathcal{Y}}(\mathfrak{N})(R) \times \mathcal{H}_{\mathcal{Y}}(\mathfrak{N}^\vee)(R) \arrow[r, "\langle\ ,\ \rangle_{\mathfrak{N}}"'] & R,
\end{tikzcd}$$

where $\langle\ ,\ \rangle_{\mathfrak{N}}$ is a perfect pairing defined similarly to the pairing $\langle\ ,\ \rangle_{\mathfrak{M}}$ and the vertical arrows are isomorphisms. Since $\mathcal{H}(\mathfrak{M}^\vee)$ is generically etale, after making a finite separable base change and replacing e by ee', we may assume that the $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$-action on $\mathcal{H}(\mathfrak{M}^\vee)(\mathcal{O}_{\mathcal{X}^{\mathrm{sep}}})$ is trivial.

Let $f^\vee$ be an element of $\mathcal{H}(\mathfrak{M}^\vee)(R)$ and $\lambda : \mathfrak{M}^\vee \to \mathfrak{N}$ be the surjection of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ corresponding to the subspace $\mathbb{F}_p f^\vee \subseteq \mathcal{H}(\mathfrak{M}^\vee)(R)$ by Lemma 3.2. The natural isomorphism $\mathfrak{M} \to (\mathfrak{M}^\vee)^\vee$ induces a morphism

$$\mathcal{H}(\mathfrak{M}) \simeq \mathcal{H}((\mathfrak{M}^\vee)^\vee) \to \mathcal{H}(\mathfrak{N}^\vee)$$

of finite flat group schemes over $\mathcal{O}_{\mathcal{X}}$, where the right arrow is a faithfully flat homomorphism.

Lemma 3.4. *We have a commutative diagram*

$$\begin{tikzcd}
T^*_{\mathfrak{S}}(\mathfrak{M}) \arrow[dr, "D_{\mathfrak{M}}"'] & T^*_{\mathfrak{S}}((\mathfrak{M}^\vee)^\vee) \arrow[l, "\sim"'] \arrow[r, "T^*_{\mathfrak{S}}(\lambda^\vee)"] \arrow[d, "\wr", "D_{\mathfrak{M}^\vee}"'] & T^*_{\mathfrak{S}}(\mathfrak{N}^\vee) \arrow[d, "\wr", "D_{\mathfrak{N}}"'] \\
 & \mathrm{Hom}_{\mathbb{F}_p}(T^*_{\mathfrak{S}}(\mathfrak{M}^\vee), T^*_{\mathfrak{S}}(\mathfrak{S}_1(r))) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{F}_p}(T^*_{\mathfrak{S}}(\mathfrak{N}), T^*_{\mathfrak{S}}(\mathfrak{S}_1(r))) \arrow[d] \\
 & \mathrm{Hom}_{\mathbb{F}_p}(T^*_{\mathfrak{S}}(\mathfrak{M}^\vee), R) \arrow[r, "T^*_{\mathfrak{S}}(\lambda)^*"'] & \mathrm{Hom}_{\mathbb{F}_p}(T^*_{\mathfrak{S}}(\mathfrak{N}), R),
\end{tikzcd}$$

where $T^*_{\mathfrak{S}}(\lambda)^*$ is the natural map induced by $T^*_{\mathfrak{S}}(\lambda) : T^*_{\mathfrak{S}}(\mathfrak{N}) \to T^*_{\mathfrak{S}}(\mathfrak{M}^\vee)$ and the lower vertical arrows are induced by the map $T^*_{\mathfrak{S}}(\mathfrak{S}_1(r)) \to R$ defined as $f \mapsto f(\mathbf{e})$.

Proof. Choose basis $e_1, \ldots, e_d$ of $\mathfrak{M}$ and $\mathbf{n}$ of $\mathfrak{N}$. The commutativity of the lower square is obvious. The top left triangle is also commutative, since the double duality isomorphism $\mathfrak{M} \to (\mathfrak{M}^\vee)^\vee$ maps e_i to $(e_i^\vee)^\vee$. It suffices to show that the right outer square is commutative. Put

$$\lambda(e_1^\vee, \ldots, e_d^\vee) = (\lambda_1 \mathbf{n}, \ldots, \lambda_d \mathbf{n})$$

with some $\lambda_i \in \mathfrak{S}_1$. Then the commutativity follows from the fact that, for $f^{\vee\vee} \in T^*_{\mathfrak{S}}((\mathfrak{M}^\vee)^\vee)$, both of its two images in $\mathrm{Hom}_{\mathbb{F}_p}(T^*_{\mathfrak{S}}(\mathfrak{N}), R)$ send $\psi \in T^*_{\mathfrak{S}}(\mathfrak{N})$ to $\sum_{i=1}^d \lambda_i \psi(\mathbf{n}) f^{\vee\vee}((e_i^\vee)^\vee)$. $\square$

Let f be an element of $T^*_{\mathfrak{S}}(\mathfrak{M})$. Its image in $\mathrm{Hom}_{\mathbb{F}_p}(T^*_{\mathfrak{S}}(\mathfrak{N}), R)$ of the diagram in the lemma is $\psi \mapsto \langle f, \psi \circ \lambda \rangle_{\mathfrak{M}}$. From the definition of the scheme-theoretic closure, we see that the equality $\mathbb{F}_p f^\vee = \{\psi \circ \lambda \mid \psi \in T^*_{\mathfrak{S}}(\mathfrak{N})\}$ holds in $T^*_{\mathfrak{S}}(\mathfrak{M}^\vee)$. Thus the previous lemma implies that an element $f^\vee$ of $T^*_{\mathfrak{S}}(\mathfrak{M}^\vee)$ is contained in the orthogonal subgroup $(\mathcal{H}(\mathfrak{M})^j(R))^\perp$ if and only if the image of $\mathcal{H}(\mathfrak{M})^j(R)$ by the surjection $\mathcal{H}(\mathfrak{M})(R) \to \mathcal{H}(\mathfrak{N}^\vee)(R)$ is zero. This is the same as $\mathcal{H}(\mathfrak{N}^\vee)^j(R) = 0$. By Lemma 3.5 below, this holds if and only if $j > per/(p-1)$ or $j \leq per/(p-1)$ and $\mathcal{H}(\mathfrak{N})_{l_r(j)+}(R) = \mathcal{H}(\mathfrak{N})(R)$. Since the last equality occurs if and only if $f^\vee \in \mathcal{H}(\mathfrak{M}^\vee)_{l_r(j)+}(R)$, the theorem follows. □

Lemma 3.5. *Let $\mathfrak{N}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ which is free of rank one over $\mathfrak{S}_1$. Then we have $\mathcal{H}(\mathfrak{N}^\vee)^j(R) = 0$ if $j > per/(p-1)$. For $j \leq per/(p-1)$, the subgroup $\mathcal{H}(\mathfrak{N}^\vee)^j(R)$ is zero if and only if $\mathcal{H}(\mathfrak{N})_{l_r(j)+}(R) = \mathcal{H}(\mathfrak{N})(R)$.*

Proof. Let $\mathbf{n}$ be a basis of $\mathfrak{N}$ and $\mathbf{n}^\vee$ be its dual basis of $\mathfrak{N}^\vee$. Put $\phi_{\mathfrak{N}}(\mathbf{n}) = u^s a\mathbf{n}$ with $0 \leq s \leq er$ and $a \in \mathfrak{S}_1^\times$. Then we have $\phi_{\mathfrak{N}^\vee}(\mathbf{n}^\vee) = u^{er-s}a'\mathbf{n}^\vee$ with some $a' \in \mathfrak{S}_1^\times$. Hence the defining equations of $\mathcal{H}(\mathfrak{N})$ and $\mathcal{H}(\mathfrak{N}^\vee)$ are $X^p - u^s aX = 0$ and $X^p - u^{er-s}a'X = 0$, respectively. By a calculation as in [17, Section 3], we see that

$$\mathcal{H}(\mathfrak{N}^\vee)^j(R) = \begin{cases} \mathcal{H}(\mathfrak{N}^\vee)(R) & (j \leq p(er-s)/(p-1)) \\ 0 & (j > p(er-s)/(p-1)), \end{cases}$$

$$\mathcal{H}(\mathfrak{N})_i(R) = \begin{cases} \mathcal{H}(\mathfrak{N})(R) & (i \leq s/(p-1)) \\ 0 & (i > s/(p-1)) \end{cases}$$

and the first assertion follows. Moreover, for $j \leq per/(p-1)$, we have $l_r(j) \geq 0$ and

$$\begin{aligned} \mathcal{H}(\mathfrak{N}^\vee)^j(R) = 0 &\Leftrightarrow j > p(er-s)/(p-1) \\ &\Leftrightarrow l_r(j) < s/(p-1) \Leftrightarrow \mathcal{H}(\mathfrak{N})_{l_r(j)+}(R) = \mathcal{H}(\mathfrak{N})(R). \end{aligned}$$

□

By the previous lemma, we also have the following corollary.

Corollary 3.6. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$. Then the i-th lower ramification subgroup $\mathcal{H}(\mathfrak{M})_i$ vanishes for $i > er/(p-1)$.*

Proof. As in the proof of Theorem 3.3, we may assume that the $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X})$-module $\mathcal{H}(\mathfrak{M})(R)$ is trivial. For $i > er/(p-1)$, take an element $x \in \mathcal{H}(\mathfrak{M})_i(R)$ and consider the scheme-theoretic closure $\mathcal{H}(\mathfrak{N})$ of the subspace $\mathbb{F}_p x \subseteq \mathcal{H}(\mathfrak{M})(R)$. By Lemma 3.5, we have $\mathcal{H}(\mathfrak{N})_i(R) = 0$ and thus $x = 0$. □

Remark 3.7. Let K be a complete discrete valuation field of mixed characteristic $(0,p)$ and $\mathcal{G}$ be a finite flat group scheme over $\mathcal{O}_K$ killed by p. Then, by using the usual scheme-theoretic closure of finite group schemes, we can easily see that the i-th lower ramification subgroup $\mathcal{G}_i$ vanishes for $i > e/(p-1)$, just as in the proof of Corollary 3.6.

3.3. **Appendix: a description of the tame characters of $T^*_{\mathfrak{S}}(\mathfrak{M})$.** In this subsection, we prove Theorem 1.2, though we do not use it in the rest of the paper. Let $\bar{k}$ be the residue field of $\mathcal{X}^{\mathrm{sep}}$ and consider the one-dimensional $\bar{k}$-vector space $\Theta_j = m^{\geq j}_{\mathcal{X}^{\mathrm{sep}}}/m^{>j}_{\mathcal{X}^{\mathrm{sep}}} \simeq m^{\geq j}_{R}/m^{>j}_{R}$ for a non-negative rational number j. Then the inertia subgroup $I_{\mathcal{X}}$ naturally acts on Θ_j and this action is $\bar{k}$-linear. Moreover, the multiplication map induces an $I_{\mathcal{X}}$-equivariant isomorphism

$$\Theta_j \otimes_{\bar{k}} \Theta_{j'} \to \Theta_{j+j'}.$$

We let θ_j denote the character $I_{\mathcal{X}} \to \bar{k}^{\times}$ defined by the $I_{\mathcal{X}}$-action on Θ_j and call it the fundamental character of level j ([24, Subsection 1.7 and 1.8]). From the definition, we see that the equalities $\theta_{j+j'} = \theta_j\theta_{j'}$ and $\theta_{1/p} = 1$ hold. Put $j = p^{-m}a/b$ with $a, m \in \mathbb{Z}_{\geq 0}$, $b \in \mathbb{Z}_{>0}$ and $p \nmid b$. Then the j-th fundamental character θ_j is also given by

$$\theta_j(\sigma) = (\sigma(u^{1/b})/u^{1/b})^{p^{-m}a} \mod m^{>0}_{\mathcal{X}^{\mathrm{sep}}}.$$

Hence the image $\theta_j(I_{\mathcal{X}})$ generates a finite extension over $\mathbb{F}_p$ in $\bar{k}$ and we let this extension be denoted by $\mathbb{F}_{(j)}$.

Lemma 3.8. *Via the identification* $\mathrm{Gal}(\mathcal{X}^{\mathrm{sep}}/\mathcal{X}) \simeq \mathrm{Gal}(\bar{K}/K_\infty)$*, the mod p cyclotomic character* $\bar{\chi}_p : \mathrm{Gal}(\bar{K}/K_\infty) \to \mathbb{F}_p^{\times}$ *corresponds to the fundamental character* $\theta_{e/(p-1)}$ *of* $I_{\mathcal{X}}$.

Proof. Recall that the element

$$\bar{t} = \sum_{i=1}^{\infty} (-1)^{i-1}(i-1)!\gamma_i(\underline{\varepsilon} - 1)$$

of the ring R^{DP} satisfies $\sigma(\bar{t}) = \bar{\chi}_p(\sigma)\bar{t}$ for any $\sigma \in \mathrm{Gal}(\bar{K}/K)$. Therefore, by considering the image of $\bar{t}$ in $R^{\mathrm{DP}}/\mathrm{Fil}^p R^{\mathrm{DP}} \simeq R/m^{\geq ep}_R$, we see that $\bar{\chi}_p$ corresponds to the fundamental character $\theta_{pe/(p-1)}$ of $I_{\mathcal{X}}$. The latter is equal to $\theta_{e/(p-1)}$. $\square$

For a field $\mathbb{F}$ and a character $\theta : I_{\mathcal{X}} \to \mathbb{F}^{\times}$, we let $\mathbb{F}(\theta)$ denote the one-dimensional $\mathbb{F}$-vector space on which $I_{\mathcal{X}}$ acts by θ. For an object $\mathfrak{M}$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ and a non-negative rational number i, we also put

$$\mathrm{gr}_i\mathcal{H}(\mathfrak{M})_\bullet(R) = \mathcal{H}(\mathfrak{M})_i(R)/\mathcal{H}(\mathfrak{M})_{i+}(R).$$

By Theorem 3.3 and the previous lemma, the perfect pairing $D_{\mathfrak{M}}$ induces an isomorphism of $I_{\mathcal{X}}$-modules

$$\mathrm{gr}^j\mathcal{H}(\mathfrak{M})^\bullet(R) \to \mathrm{Hom}_{\mathbb{F}_p}(\mathrm{gr}_{l_r(j)}\mathcal{H}(\mathfrak{M}^\vee)_\bullet(R), \mathbb{F}_p(\theta_{er/(p-1)})).$$

Lemma 3.9. *Let $\mathfrak{M}$ be an object of* $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$ *and i be a non-negative rational number. Then the $I_{\mathcal{X}}$-action on the i-th graded piece* $V_i = \mathrm{gr}_i\mathcal{H}(\mathfrak{M})_\bullet(R)$ *is tame. Moreover, there exists an $\mathbb{F}_{(i)}$-action on V_i such that the $I_{\mathcal{X}}$-action on V_i factors through* $\theta_i : I_{\mathcal{X}} \to \mathbb{F}_{(i)}^{\times}$.

Proof. Choose a basis $e_1, \ldots, e_d$ of $\mathfrak{M}$. Then we have an injection of $I_{\mathcal{X}}$-modules $\iota : V_i \to \Theta_i^{\oplus d}$ defined by $f \mapsto (f(e_1), \ldots, f(e_d))$. Hence we see that if $\sigma \in I_{\mathcal{X}}$ acts trivially on Θ_i, then it acts trivially also on V_i. This implies that the $I_{\mathcal{X}}$-action on V_i is tame and factors through $\theta_i(I_{\mathcal{X}})$. Let x be a generator of the finite cyclic group $\theta_i(I_{\mathcal{X}}) \subseteq \mathbb{F}_{(i)}^{\times}$. By definition, the finite extension $\mathbb{F}_{(i)}/\mathbb{F}_p$ is also generated by x. We let the minimal polynomial of x over $\mathbb{F}_p$ be denoted by $H_x(T) \in \mathbb{F}_p[T]$. Take an element $\sigma \in I_{\mathcal{X}}$ satisfying $\theta_i(\sigma) = x$. Since $H_x(\sigma)$ acts on Θ_i as the zero map, it acts on V_i also as the zero map. Thus we have a ring homomorphism

$$\begin{aligned} \mathbb{F}_p[T]/(H_x(T)) &\to \mathrm{End}_{\mathbb{F}_p}(V_i) \\ T &\mapsto \sigma. \end{aligned}$$

We give V_i a structure of an $\mathbb{F}_{(i)}$-vector space via the composite of this homomorphism and the inverse of the isomorphism $\mathbb{F}_p[T]/(H_x(T)) \simeq \mathbb{F}_{(i)}$ sending T to x. Then the action of the element $x = \theta_i(\sigma) \in \mathbb{F}_{(i)}$ on V_i coincides with the action of σ. Thus this action has the desired property. $\square$

Proof of Theorem 1.2. Note that the field $\mathbb{F}_{(l_r(j))}$ coincides with $\mathbb{F}_{(j)}$. Write V^j for the j-th graded piece in the theorem. By the previous lemma, we have an $I_{\mathcal{X}}$-equivariant $\mathbb{F}_p$-linear isomorphism $V^j \to \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_{(j)}(\theta_{-j/p})^{\oplus n}, \mathbb{F}_p)$ and thus the $I_{\mathcal{X}}$-action on V^j is tame. Moreover, an action of $\mathbb{F}_{(j)}$ as in the theorem is given by twisting the induced action of $\mathbb{F}_{(j)}$ from the right-hand side of this isomorphism by the p-th power map. This concludes the proof. $\square$

Remark 3.10. We can show Theorem 1.2 also as in the proof of the main theorem of [19], since the proof carries over verbatim for $\mathcal{H}(\mathfrak{M})$.

4. Comparison of ramification

Let $p > 2$ be a rational prime and K be a complete discrete valuation field of mixed characteristic $(0, p)$ with perfect residue field k, as in Section 2. Then it is known that there exists an anti-equivalence $\mathcal{G}(-)$ from the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ to the category of finite flat group schemes over $\mathcal{O}_K$ killed by p. On the other hand, we also have the anti-equivalence $\mathcal{H}(-) : \mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1} \to \mathcal{C}^{\leq e}_{\mathcal{O}_{\mathcal{X}}}$ defined in Section 3 and an isomorphism of $\mathrm{Gal}(\bar{K}/K_\infty)$-module

$$\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})|_{\mathrm{Gal}(\bar{K}/K_\infty)} \to \mathcal{H}(\mathfrak{M})(R).$$

In this section, we prove that this isomorphism is compatible with the upper and the lower ramification subgroups of both sides. For the proof, after recalling the construction of the anti-equivalence $\mathcal{G}(-)$ ([8], [20]) and checking its compatibility with the base change to $K_n = K(\pi_n)$, we show that the isomorphism $\varepsilon_{\mathfrak{M}}$ is also compatible with dualities on both sides. Then, by the duality theorems presented in Section 3, we reduce ourselves to compare the lower ramification subgroups of both sides. This comparison of lower ramification subgroups relies on the fact that reductions of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{H}(\mathfrak{M})$

are isomorphic to each other, not as group schemes but merely as pointed schemes.

4.1. **Breuil's classification.** In this subsection, we briefly recall Breuil's classification of finite flat group schemes over $\mathcal{O}_K$ ([8]). Here we restrict ourselves to the case of p-torsion group schemes.

Put $E_1 = \mathrm{Spec}(S_1)$ and $\mathscr{S}_1 = \mathrm{Spec}(\mathcal{O}_K/p\mathcal{O}_K)$. For a scheme X over $\mathscr{S}_1$, the big crystalline site with the syntomic topology for X/E_1 is denoted by $\mathrm{CRYS}(X/E_1)$ and its topos by $(X/E_1)_{\mathrm{CRYS}}$. We have the structure sheaf $\mathcal{O}_{X/E_1}$ and the ideal sheaf $\mathcal{J}_{X/E_1}$ on this site.

Let $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$ be the category of syntomic p-adic formal schemes over $\mathrm{Spf}(\mathcal{O}_K)$ with the syntomic topology ([8, Subsection 2.2]) and $(\mathrm{Ab}/\mathcal{O}_K)$ be the category of abelian sheaves on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$. We consider a finite flat group scheme over $\mathcal{O}_K$ naturally as an object of $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$ or $(\mathrm{Ab}/\mathcal{O}_K)$. For a p-adic formal scheme $\mathfrak{X}$ over $\mathrm{Spf}(\mathcal{O}_K)$, we put $\mathfrak{X}_1 = \mathfrak{X}\times_{\mathcal{O}_K}(\mathcal{O}_K/p\mathcal{O}_K)$. Then the sheaf of rings $\mathcal{O}^{\mathrm{crys}}_{1,\pi}$ and its ideal sheaf $\mathcal{J}^{\mathrm{crys}}_{1,\pi}$ on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$ are defined as

$$\mathfrak{X} \mapsto \mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{X}) = \Gamma((\mathfrak{X}_1/E_1)_{\mathrm{CRYS}}, \mathcal{O}_{\mathfrak{X}_1/E_1}),$$
$$\mathfrak{X} \mapsto \mathcal{J}^{\mathrm{crys}}_{1,\pi}(\mathfrak{X}) = \Gamma((\mathfrak{X}_1/E_1)_{\mathrm{CRYS}}, \mathcal{J}_{\mathfrak{X}_1/E_1}).$$

Let $\phi : \mathcal{O}^{\mathrm{crys}}_{1,\pi} \to \mathcal{O}^{\mathrm{crys}}_{1,\pi}$ be the map induced by the crystalline Frobenius map. Then there exists a ϕ-semilinear map $\phi_1 : \mathcal{J}^{\mathrm{crys}}_{1,\pi} \to \mathcal{O}^{\mathrm{crys}}_{1,\pi}$ of $(\mathrm{Ab}/\mathcal{O}_K)$ and with these structures the S-module $\mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{X})$ can be considered as an object of ${}'\mathrm{Mod}^{1,\phi}_{/S}$.

Let $\mathcal{C}_{\mathcal{O}_K,1}$ be the category of finite flat group schemes over $\mathcal{O}_K$ killed by p and $\mathcal{G}$ be an object of this category. Then the S_1-module

$$\mathrm{Mod}(\mathcal{G}) = \mathrm{Hom}_{(\mathrm{Ab}/\mathcal{O}_K)}(\mathcal{G}, \mathcal{O}^{\mathrm{crys}}_{1,\pi})$$

has a natural structure as an object of $\mathrm{Mod}^{1,\phi}_{/S_1}$ with $\mathrm{rank}_{S_1}(\mathrm{Mod}(\mathcal{G})) = \log_p(\mathrm{rank}(\mathcal{G}))$. On the other hand, for an object $\mathcal{M}$ of $\mathrm{Mod}^{1,\phi}_{/S_1}$, we define an abelian sheaf $\mathrm{Gr}(\mathcal{M})$ on $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$ by

$$\mathfrak{X} \mapsto \mathrm{Gr}(\mathcal{M})(\mathfrak{X}) = \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M}, \mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{X})).$$

This sheaf is shown to be represented by a finite flat group scheme over $\mathcal{O}_K$ killed by p, which is denoted also by $\mathrm{Gr}(\mathcal{M})$ ([8, Corollaire 3.1.8]). Then the functor $\mathrm{Mod}(-) : \mathcal{C}_{\mathcal{O}_K,1} \to \mathrm{Mod}^{1,\phi}_{/S_1}$ is an anti-equivalence of categories and the functor $\mathrm{Gr}(-)$ is a quasi-inverse of $\mathrm{Mod}(-)$ ([8, Théorème 3.3.7]).

Now we put

$$\mathcal{G}(-) = \mathrm{Gr}(\mathcal{M}_{\mathfrak{S}}(-)) : \mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1} \to \mathcal{C}_{\mathcal{O}_K,1},$$

which is also an anti-equivalence of categories. Then, for an object $\mathfrak{M}$ of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$, we have a natural isomorphism of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules

$$\varepsilon_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \to T^*_{\mathfrak{S}}(\mathfrak{M}) = \mathcal{H}(\mathfrak{M})(R)$$

([20, Proposition 1.1.13]). This isomorphism is constructed as follows. Let H_1 be the divided power polynomial ring $R^{\mathrm{DP}}\langle u-\underline{\pi}\rangle$ over R^{DP}. The natural surjection

$$\begin{aligned} H_1 = R^{\mathrm{DP}}\langle u-\underline{\pi}\rangle &\to \tilde{\mathcal{O}}_{\bar{K}} \\ r=(r_0,r_1,\ldots) &\mapsto r_0 \\ u &\mapsto \pi \end{aligned}$$

is a divided power thickening of $\tilde{\mathcal{O}}_{\bar{K}}$ and we can show that this gives the final object of $\mathrm{CRYS}(\mathrm{Spec}(\tilde{\mathcal{O}}_{\bar{K}})/E_1)$. Thus the ring $\mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathcal{O}_{\bar{K}})$ is naturally identified with H_1 and the map $H_1\to R^{\mathrm{DP}}$ defined by $u\mapsto\underline{\pi}$ induces an isomorphism of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules

$$\varepsilon_{\mathcal{M}}:\mathrm{Gr}(\mathcal{M})(\mathcal{O}_{\bar{K}})=\mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M},H_1)\to\mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M},R^{\mathrm{DP}})$$

(see the proof of [8, Lemme 5.3.1]). The isomorphism $\varepsilon_{\mathfrak{M}}$ is the composite of $\varepsilon_{\mathcal{M}_{\mathfrak{S}}(\mathfrak{M})}$ and the inverse of the natural isomorphism $T^*_{\mathfrak{S}}(\mathfrak{M})\to T^*_{\mathrm{crys}}(\mathcal{M}_{\mathfrak{S}}(\mathfrak{M}))$.

Let $\mathcal{M}$ be an object of $\mathrm{Mod}^{1,\phi}_{/S_1}$. In [8, Section 3], Breuil gave an explicit description of the affine algebra $R_{\mathcal{M}}$ of the finite flat group scheme $\mathrm{Gr}(\mathcal{M})$ in terms of $\mathcal{M}$. Let $e_1,\ldots,e_d$ be an adapted basis of $\mathcal{M}$ such that

$$\begin{aligned} &\mathrm{Fil}^1\mathcal{M}=u^{r_1}S_1e_1\oplus\cdots\oplus u^{r_d}S_1e_d+(\mathrm{Fil}^pS)\mathcal{M}, \\ &\phi_1(u^{r_1}e_1,\ldots,u^{r_d}e_d)=(e_1,\ldots,e_d)G \end{aligned}$$

with a matrix $G\in GL_d(S_1)$ (note that we adopt the transpose of the notation in [8]). Assume that the image $\tilde{G}$ of G in $GL_d(\tilde{S}_1)$ is contained in the subgroup $GL_d(k[u^p]/(u^{ep}))$. Consider the ring homomorphisms

$$k[u]/(u^{ep})\to\mathcal{O}_{K_1}/p\mathcal{O}_{K_1}\leftarrow\mathcal{O}_{K_1},$$

where the first map is the ϕ^{-1}-semilinear isomorphism defined by $u\mapsto\pi_1$. We choose a lift $G_\pi=(a_{i,j})$ of $\tilde{G}$ by this map to $GL_d(\mathcal{O}_K)$ using the assumption on G and put

$$R_{\mathcal{M}}=\frac{\mathcal{O}_K[X_1,\ldots,X_d]}{\left(X_1^p+\frac{\pi^{e-r_1}}{F(\pi)}(\sum_{j=1}^d a_{j,1}X_j),\ldots,X_d^p+\frac{\pi^{e-r_d}}{F(\pi)}(\sum_{j=1}^d a_{j,d}X_j)\right)}.$$

Then we have an isomorphism of p-adic formal schemes $\mathrm{Spf}(R_{\mathcal{M}})\simeq\mathrm{Gr}(\mathcal{M})$. The induced map

$$\Psi:\mathrm{Hom}_{\mathcal{O}_K\text{-alg.}}(R_{\mathcal{M}},\mathcal{O}_{\bar{K}})\to\mathrm{Gr}(\mathcal{M})(\mathcal{O}_{\bar{K}})=\mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M},H_1)$$

is defined as follows. Let $f:R_{\mathcal{M}}\to\mathcal{O}_{\bar{K}}$ be an element of the left-hand side. Set $x_i=f(X_i)$ and $\bar{x}_i$ to be its image in $\tilde{\mathcal{O}}_{\bar{K}}$. We let y_i denote the inverse image of $\bar{x}_i$ by the isomorphism $\mathrm{pr}_1:R^{\mathrm{DP}}\to(\tilde{\mathcal{O}}_{\bar{K}})^{\mathrm{DP}}$ recalled in Subsection 2.3. Then $\Psi(f)$ is the unique element of the right-hand side which is congruent to the S-linear map $(e_i\mapsto y_i)$ modulo Fil^pH_1 (see the proof of [8, Proposition 3.1.5]). From this description, we see that the zero section of the group scheme $\mathrm{Spec}(R_{\mathcal{M}})$ is defined by $X_1=\cdots=X_d=0$.

4.2. **Compatibility with the base change to K_n.** Next we investigate a compatibility of the map $\varepsilon_{\mathfrak{M}}$ with the base change from K to $K_n = K(\pi_n)$. Put $\mathfrak{S}' = W[[v]]$ and define a Frobenius endomorphism of $\mathfrak{S}'$ by $\phi(v) = v^p$. Then the ring homomorphism $\mathfrak{S}_1 \to \mathfrak{S}'_1 = k[[v]]$ defined by $u \mapsto v^{p^n}$ is compatible with ϕ's. Let $E'(v) = E(v^{p^n})$ be the Eisenstein polynomial of π_n over W. Then we see that a base change functor $\mathfrak{S}' \otimes_{\mathfrak{S}} - : \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1} \to \mathrm{Mod}^{r,\phi}_{/\mathfrak{S}'_1}$ is defined by $\mathfrak{M} \mapsto \mathfrak{M}' = \mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M}$ with $\phi_{\mathfrak{M}'} = \phi \otimes \phi_{\mathfrak{M}}$.

As in [18, Section 3], we can define a similar base change functor for $\mathrm{Mod}^{r,\phi}_{/S_1}$. Let S' be the p-adic completion of the divided power envelope $W[v]^{\mathrm{DP}}$ of $W[v]$ with respect to the ideal $(E'(v))$ and we define ϕ, Fil^r and ϕ_r of the ring S' similarly to the ring S. Then the ring homomorphism $W[u] \to W[v]$ sending u to v^{p^n} induces a ring homomorphism $W[u]^{\mathrm{DP}} \to W[v]^{\mathrm{DP}}$ (*resp.* $S \to S'$) and this makes the $W[u]^{\mathrm{DP}}$-algebra $W[v]^{\mathrm{DP}}$ (*resp.* the S-algebra S') free of rank p^n. From this we can show that the natural map $S' \otimes_S \mathrm{Fil}^r S \to \mathrm{Fil}^r S'$ is an isomorphism and the ϕ-semilinear map $\phi_r : \mathrm{Fil}^r S' \to S'$ corresponds via this isomorphism to $\phi \otimes \phi_r$. Then the functor $S' \otimes_S - : \mathrm{Mod}^{r,\phi}_{/S_1} \to \mathrm{Mod}^{r,\phi}_{/S'_1}$ is defined by $\mathcal{M} \mapsto \mathcal{M}' = S' \otimes_S \mathcal{M}$ with

$$\mathrm{Fil}^r \mathcal{M}' = S' \otimes_S \mathrm{Fil}^r \mathcal{M},\ \ \phi_{r,\mathcal{M}'} = \phi \otimes \phi_{r,\mathcal{M}}.$$

By definition, we have a functorial isomorphism $S' \otimes_S \mathcal{M}_{\mathfrak{S}}(\mathfrak{M}) \to \mathcal{M}_{\mathfrak{S}'}(\mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M})$ for an object $\mathfrak{M}$ of $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}_1}$, by which we identify both sides.

Consider the W-algebra homomorphism $W[[v]] \to W(R)$ defined by $v \mapsto [\underline{\pi}^{1/p^n}]$. Using this map, we have a similar functor $T^*_{\mathfrak{S}'}$ (*resp.* $T^*_{\mathrm{crys}'}$) from the category $\mathrm{Mod}^{r,\phi}_{/\mathfrak{S}'_1}$ (*resp.* $\mathrm{Mod}^{r,\phi}_{/S'_1}$) to the category of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules. Moreover, we have a natural commutative diagram

$$\begin{tikzcd}
T^*_{\mathfrak{S}}(\mathfrak{M}) \arrow[r] \arrow[d] & T^*_{\mathrm{crys}}(\mathcal{M}) \arrow[d] \\
T^*_{\mathfrak{S}'}(\mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M}) \arrow[r] & T^*_{\mathrm{crys}'}(S' \otimes_S \mathcal{M})
\end{tikzcd}$$

of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules whose arrows are isomorphisms.

By the definition of the functor $\mathcal{H}(-)$, we have a natural isomorphism

$$k[[v]] \times_{k[[u]]} \mathcal{H}(\mathfrak{M}) \to \mathcal{H}(\mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M})$$

of finite flat group schemes over $k[[v]]$ which induces on the R-valued points the above isomorphism $T^*_{\mathfrak{S}}(\mathfrak{M}) \to T^*_{\mathfrak{S}'}(\mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M})$. On the other hand, the functor $\mathrm{Gr}(-)$ is also compatible with the base change functor $\mathrm{Mod}^{r,\phi}_{/S_1} \to \mathrm{Mod}^{r,\phi}_{/S'_1}$. Though a proof of this fact can be found in [18, Theorem 3.6] at least for $n = 1$, we give here a more transparent proof.

Proposition 4.1. *Let $\mathcal{M}$ be an object of $\mathrm{Mod}^{1,\phi}_{/S_1}$ and put $\mathcal{M}' = S' \otimes_S \mathcal{M}$. Then there exists a natural isomorphism*

$$\mathcal{O}_{K_n} \times_{\mathcal{O}_K} \mathrm{Gr}(\mathcal{M}) \to \mathrm{Gr}(\mathcal{M}')$$

of finite flat group schemes over $\mathcal{O}_{K_n}$.

Proof. Put $E'_1 = \mathrm{Spec}(S'_1)$. Then the pull-back $(\mathfrak{X}_1/E_1)_{\mathrm{CRYS}} \to (\mathfrak{X}_1/E'_1)_{\mathrm{CRYS}}$ for an object $\mathfrak{X} \in \mathrm{Spf}(\mathcal{O}_{K_n})_{\mathrm{syn}}$ defines a natural map of sheaves

$$\mathrm{Gr}(\mathcal{M})|_{\mathrm{Spf}(\mathcal{O}_{K_n})_{\mathrm{syn}}} \to \mathrm{Gr}(S' \otimes_S \mathcal{M}).$$

It suffices to show that this map is an isomorphism. Take a p-adic syntomic formal scheme

$$\mathfrak{A} = \mathcal{O}_{K_n}\{X_1, \ldots, X_r\}/(f_1, \ldots, f_s)$$

over $\mathcal{O}_{K_n}$, where $\mathcal{O}_{K_n}\{X_1, \ldots, X_r\}$ is the p-adic completion of the polynomial ring $\mathcal{O}_{K_n}[X_1, \ldots, X_r]$ and $f_1, \ldots, f_s$ be a transversally regular sequence of that ring over $\mathcal{O}_{K_n}$. As in [8, Lemme 2.3.2], we put

$$\mathfrak{A}_i = \mathcal{O}_{K_n}\{X_0^{p^{-i}}, X_1^{p^{-i}}, \ldots, X_r^{p^{-i}}\}/(X_0 - \pi_n, f_1, \ldots, f_s),$$

$\mathfrak{A}_\infty = \varinjlim_i \mathfrak{A}_i$ and

$$A_\infty = \mathfrak{A}_\infty/p\mathfrak{A}_\infty \simeq k[X_0^{p^{-\infty}}, X_1^{p^{-\infty}}, \ldots, X_r^{p^{-\infty}}]/(X_0^{ep^n}, f_1, \ldots, f_s).$$

Note that $\mathfrak{A}_i$ is a p-adic syntomic cover of $\mathfrak{A}$ over $\mathcal{O}_{K_n}$. We also put

$$\mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{A}_\infty) = \varinjlim_i \mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{A}_i),\ \mathcal{J}^{\mathrm{crys}}_{1,\pi}(\mathfrak{A}_\infty) = \varinjlim_i \mathcal{J}^{\mathrm{crys}}_{1,\pi}(\mathfrak{A}_i)$$

and similarly for $\mathcal{O}^{\mathrm{crys}}_{1,\pi_n}(\mathfrak{A}_\infty)$ and $\mathcal{J}^{\mathrm{crys}}_{1,\pi_n}(\mathfrak{A}_\infty)$. We give these rings the filtrations defined by the divided power structures. Then as in the proof of [8, Lemme 2.3.2], we can show that there exist isomorphisms

$$\mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{A}_\infty)/\mathrm{Fil}^p\mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{A}_\infty) \simeq A_\infty[u]/(u^p - X_0^{p^n}),$$
$$\mathcal{O}^{\mathrm{crys}}_{1,\pi_n}(\mathfrak{A}_\infty)/\mathrm{Fil}^p\mathcal{O}^{\mathrm{crys}}_{1,\pi_n}(\mathfrak{A}_\infty) \simeq A_\infty[v]/(v^p - X_0),$$

where A_∞ is considered as a k-algebra by twisting the natural k-action by ϕ^{-1}. Let B_∞ (*resp.* B'_∞) denote the former (*resp.* the latter) k-algebra. Since we have $\phi_1(\mathrm{Fil}^p\mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{A}_\infty)) = 0$, we can define a structure as an object of $'\mathrm{Mod}^{1,\phi}_{/S}$ on B_∞ induced from $\mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{A}_\infty)$ and similarly for B'_∞. Moreover, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M}, \mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{A}_\infty)) & \longrightarrow & \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M}, B_\infty) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{S',\mathrm{Fil}^1,\phi_1}(\mathcal{M}', \mathcal{O}^{\mathrm{crys}}_{1,\pi_n}(\mathfrak{A}_\infty)) & \longrightarrow & \mathrm{Hom}_{S',\mathrm{Fil}^1,\phi_1}(\mathcal{M}', B'_\infty),
\end{array}$$

where the vertical arrows are the natural pull-back maps and the horizontal arrows are isomorphisms. Thus, to prove the proposition, it is enough to show the bijectivity of the right vertical arrow. Note that this arrow is induced by the A_∞-algebra homomorphism $B_\infty \to B'_\infty$ sending u to v^{p^n}.

Let ψ_i be the element of the ring $k[X_0^{p^{-\infty}}, X_1^{p^{-\infty}}, \ldots, X_r^{p^{-\infty}}]$ such that ψ_i^p coincides with the image of f_i, and we also let it abusively denote its image in A_∞. Put $\xi = u - X_0^{p^{n-1}}$ and $\xi' = v - X_0^{1/p}$. By definition, the filtrations of the rings B_∞ and B'_∞ are given by

$$\mathrm{Fil}^1 B_\infty = (\xi, \psi_1, \ldots, \psi_s, X_0^{ep^{n-1}}),\ \mathrm{Fil}^1 B'_\infty = (\xi', \psi_1, \ldots, \psi_s, X_0^{ep^{n-1}}).$$

We can describe ϕ_1's of the rings B_∞ and B'_∞ as follows. Take a lift $\hat{\psi}_i$ of ψ_i in the ring $\mathfrak{A}_\infty$. We have $\hat{\psi}_i^p + p\hat{\psi}'_i = 0$ for some $\hat{\psi}'_i \in \mathfrak{A}_\infty$ and set ψ'_i to be the image of $\hat{\psi}'_i$ in A_∞. Then we have

$$\phi_1(\xi) = \sum_{k=1}^{p-1} (-1)^k \frac{{}_pC_k}{p} u^k (X_0^{p^{n-1}})^{p-k},\ \phi_1(X_0^{ep^{n-1}}) = \phi(-F(u))$$

and $\phi_1(\psi_i) = (\psi'_i)^p$ for B_∞. Similarly, we also have

$$\phi_1(\xi') = \sum_{k=1}^{p-1} (-1)^k \frac{{}_pC_k}{p} v^k (X_0^{1/p})^{p-k},\ \phi_1(X_0^{ep^{n-1}}) = \phi(-F(v^{p^n}))$$

and $\phi_1(\psi_i) = (\psi'_i)^p$ for B'_∞. Since we have the equality

$$\sum_{k=1}^{p-1} (-1)^k \frac{{}_pC_k}{p} = 0,$$

we can also write $\phi_1(\xi) = wX_0^{p^{n-1}}\xi$ with some $w \in B_\infty$ and $\phi_1(\xi') = w'X_0^{1/p}\xi'$ with some $w' \in B'_\infty$.

The ring surjection $B_\infty \to B_\infty/(\xi) \simeq A_\infty$ induces on the k-algebra A_∞ a structure as an object of $\mathrm{Mod}^{1,\phi}_{/S_1}$. Namely, it is defined by

$$\mathrm{Fil}^1 A_\infty = (\psi_1, \ldots, \psi_s, X_0^{ep^{n-1}}),$$

$\phi_1(\psi_i) = (\psi'_i)^p$ and $\phi_1(X_0^{ep^{n-1}}) = \phi(-F(X_0^{p^{n-1}}))$. This is the same as the induced structure by the ring surjection $B'_\infty \to B'_\infty/(\xi') \simeq A_\infty$. Moreover, since $X_0^{1/p}$ is nilpotent in A_∞, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M}, B_\infty) \arrow[r] \arrow[d] & \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M}, A_\infty) \arrow[d] \\
\mathrm{Hom}_{S',\mathrm{Fil}^1,\phi_1}(\mathcal{M}', B'_\infty) \arrow[r] & \mathrm{Hom}_{S',\mathrm{Fil}^1,\phi_1}(\mathcal{M}', A_\infty)
\end{tikzcd}$$

whose horizontal arrows are isomorphisms. From the definition of $\mathcal{M}'$, we see that the right vertical arrow is also an isomorphism. Thus we conclude the proof of the proposition. $\square$

Put $H_1' = R^{\mathrm{DP}}\langle v - \underline{\pi}^{1/p^n}\rangle$. This ring is naturally identified with the ring $\mathcal{O}^{\mathrm{crys}}_{1,\pi_n}(\mathcal{O}_{\bar{K}})$ as in the case of H_1 and the diagram

$$\begin{tikzcd} H_1 = R^{\mathrm{DP}}\langle u - \underline{\pi}\rangle \arrow[r, "u\mapsto\underline{\pi}"] \arrow[d] & R^{\mathrm{DP}} \arrow[d, equal] \\ H_1' = R^{\mathrm{DP}}\langle v - \underline{\pi}^{1/p^n}\rangle \arrow[r, "v\mapsto\underline{\pi}^{1/p^n}"'] & R^{\mathrm{DP}} \end{tikzcd}$$

is commutative, where the left vertical arrow is defined by $u \mapsto v^{p^n}$. Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ and put $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$, $\mathfrak{M}' = \mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M}$ and $\mathcal{M}' = S' \otimes_S \mathcal{M}$. Then we have a commutative diagram of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules

$$\begin{tikzcd} \mathrm{Gr}(\mathcal{M})(\mathcal{O}_{\bar{K}}) \arrow[r] \arrow[d] & \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M}, R^{\mathrm{DP}}) \arrow[r] \arrow[d] & \mathcal{H}(\mathfrak{M})(R) \arrow[d] \\ \mathrm{Gr}(\mathcal{M}')(\mathcal{O}_{\bar{K}}) \arrow[r] & \mathrm{Hom}_{S',\mathrm{Fil}^1,\phi_1}(\mathcal{M}', R^{\mathrm{DP}}) \arrow[r] & \mathcal{H}(\mathfrak{M}')(R), \end{tikzcd}$$

where the vertical arrows are isomorphisms.

Let v_{K_n} be the valuation on $\mathbb{C}$ satisfying $v_{K_n}(\pi_n) = 1$ and v_R' be the corresponding valuation $v_R'(r) = v_{K_n}(r^{(0)})$ on the ring R. Put

$$m'^{\geq i}_{\bar{K}} = \{x \in \mathcal{O}_{\bar{K}} \mid v_{K_n}(x) \geq i\},\ m'^{\geq i}_R = \{r \in R \mid v_R'(r) \geq i\}.$$

Then the lower ramification subgroups of $\mathrm{Gr}(\mathcal{M}')$ over $\mathcal{O}_{K_n}$ and of $\mathcal{H}(\mathfrak{M}')$ over $k[[v]]$ are by definition

$$\mathrm{Gr}(\mathcal{M}')_i(\mathcal{O}_{\bar{K}}) = \mathrm{Ker}(\mathrm{Gr}(\mathcal{M}')(\mathcal{O}_{\bar{K}}) \to \mathrm{Gr}(\mathcal{M}')(\mathcal{O}_{\bar{K}}/m'^{\geq i}_{\bar{K}})),$$
$$\mathcal{H}(\mathfrak{M}')_i(R) = \mathrm{Ker}(\mathcal{H}(\mathfrak{M}')(R) \to \mathcal{H}(\mathfrak{M}')(R/m'^{\geq i}_R)).$$

Hence we have the following proposition.

Proposition 4.2. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ and put $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$, $\mathfrak{M}' = \mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M}$ and $\mathcal{M}' = S' \otimes_S \mathcal{M}$. Then the natural isomorphisms of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules $\varepsilon_{\mathfrak{M}}$ and $\varepsilon_{\mathfrak{M}'}$ make the following diagram commutative.*

$$\begin{tikzcd} \mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \arrow[r, "\varepsilon_{\mathfrak{M}}"] \arrow[d] & \mathcal{H}(\mathfrak{M})(R) \arrow[d] \\ \mathcal{G}(\mathfrak{M}')(\mathcal{O}_{\bar{K}}) \arrow[r, "\varepsilon_{\mathfrak{M}'}"'] & \mathcal{H}(\mathfrak{M}')(R) \end{tikzcd}$$

Here the vertical arrows are the base change isomorphisms. Moreover, these vertical arrows induce isomorphisms from the i-th to the $p^n i$-th lower ramification subgroups.

□

4.3. **Compatibility with duality.** In this subsection, we prove that there exists a natural isomorphism $\beta : \mathrm{Gr}(-) \to \mathrm{Gr}((-)^\vee)^\vee$ of functors from $\mathrm{Mod}^{1,\phi}_{/S_1}$ to $\mathcal{C}_{\mathcal{O}_K,1}$, where the first $\vee$ means the duality of $\mathrm{Mod}^{1,\phi}_{/S_1}$ presented in Section 2 and the second $\vee$ means usual Cartier duality (a similar result is obtained independently by Abrashkin ([1])). Then we define for an object $\mathfrak{M}$ of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ an isomorphism of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules

$$\delta_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})^\vee(\mathcal{O}_{\bar{K}}) \to \mathcal{H}(\mathfrak{M}^\vee)(R)$$

which has a compatibility with the map $\varepsilon_{\mathfrak{M}}$ and the dualities on both sides. In [11], Caruso constructed a similar natural isomorphism to β more generally for the category of finite flat group schemes over $\mathcal{O}_K$ killed by some p-power. In fact, we can check that his isomorphism also yields the desired compatibility (and the two natural isomorphisms are the same) by unwinding the definition and doing a lengthy diagram chase. Here we prefer a shorter exposition by giving the following direct construction of this natural isomorphism which works only for $\mathrm{Mod}^{1,\phi}_{/S_1}$.

Proposition 4.3. *Let $\mathcal{M}$ be an object of $\mathrm{Mod}^{1,\phi}_{/S_1}$. Then there exists a natural isomorphism*

$$\beta_{\mathcal{M}} : \mathrm{Gr}(\mathcal{M}) \to \mathrm{Gr}(\mathcal{M}^\vee)^\vee$$

of finite flat group schemes over $\mathcal{O}_K$, where the first $\vee$ is the duality of $\mathrm{Mod}^{1,\phi}_{/S_1}$ and the second $\vee$ means usual Cartier duality.

Proof. Let us consider the object $S_1(1)$ of $\mathrm{Mod}^{1,\phi}_{/S_1}$ defined by ($S_1(1) = S_1\mathbf{e} = \mathrm{Fil}^1 S_1(1), \phi_1(\mathbf{e}) = \mathbf{e}$) as in Subsection 2.3. To construct the isomorphism in the proposition, we first show the following generalization of Lemma 2.3.

Lemma 4.4. *Let $\mathcal{M}$ be an object of $\mathrm{Mod}^{1,\phi}_{/S_1}$. Take a basis $e_1, \ldots, e_d$ of $\mathcal{M}$ and its dual basis $e_1^\vee, \ldots, e_d^\vee$ of $\mathcal{M}^\vee$. Let $\mathfrak{X}$ be an object of $\mathrm{Spf}(\mathcal{O}_K)_{\mathrm{syn}}$. Then we have a natural pairing*

$$\begin{aligned} D_{\mathcal{M},\mathfrak{X}} : \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M}, \mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{X})) \times \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(\mathcal{M}^\vee, \mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{X})) \\ \to \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(S_1(1), \mathcal{O}^{\mathrm{crys}}_{1,\pi}(\mathfrak{X})) \end{aligned}$$

defined by $D_{\mathcal{M},\mathfrak{X}}(f, f^\vee)(\mathbf{e}) = \sum_{i=1}^d f(e_i) f^\vee(e_i^\vee)$, which is independent of the choice of a basis and functorial on $\mathfrak{X}$.

Proof. The functoriality on $\mathfrak{X}$ is clear. We can also check the independence of the choice of a basis as before. Thus we may take an adapted basis $e_1, \ldots, e_d$ of $\mathcal{M}$ satisfying

$$\begin{aligned} \mathrm{Fil}^r\mathcal{M} &= u^{r_1}S_1e_1 \oplus \cdots \oplus u^{r_d}S_1e_d + (\mathrm{Fil}^p S)\mathcal{M}, \\ \phi_r(u^{r_1}e_1, \ldots, u^{r_d}e_d) &= (e_1, \ldots, e_d)G \end{aligned}$$

with $0 \le r_i \le e$ and $G \in GL_d(S_1)$. Then the dual basis $e_1^\vee, \ldots, e_d^\vee$ of $\mathcal{M}^\vee$ is also an adapted basis, as we have seen in Subsection 2.2.

Now we only have to show that the S-linear map $S_1(1) \to \mathcal{O}_{1,\pi}^{\mathrm{crys}}(\mathfrak{X})$ defined by $\mathbf{e} \mapsto \sum_{i=1}^{d} f(e_i) f^\vee(e_i^\vee)$ is compatible with Fil^1's and ϕ_1's. For this, we may assume that the p-adic formal scheme $\mathfrak{X}$ is affine and $\mathfrak{X} = \mathrm{Spf}(\mathfrak{A})$ with

$$\mathfrak{A} = \mathcal{O}_K\{X_1, \ldots, X_r\}/(f_1, \ldots, f_s)$$

as in Subsection 4.2. We put $\mathfrak{A}_i$, $\mathfrak{A}_\infty$ and A_∞ as in the notation there for $n = 0$. Since $\mathfrak{A}_i$ is a p-adic syntomic cover of $\mathfrak{A}$, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{J}_{1,\pi}^{\mathrm{crys}}(\mathfrak{A}) & \longrightarrow & \mathcal{O}_{1,\pi}^{\mathrm{crys}}(\mathfrak{A}) & \longrightarrow & \mathfrak{A}/p\mathfrak{A} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{J}_{1,\pi}^{\mathrm{crys}}(\mathfrak{A}_\infty) & \longrightarrow & \mathcal{O}_{1,\pi}^{\mathrm{crys}}(\mathfrak{A}_\infty) & \longrightarrow & A_\infty \end{array}$$

whose rows are exact and vertical arrows are injective (see the remark after [8, Corollaire 2.3.3]). Hence it is enough to check the compatibility on $\mathfrak{A}_\infty$.

Put $B_\infty = A_\infty[u]/(u^p - X_0)$ and $H_{1,\mathfrak{A}} = \mathcal{O}_{1,\pi}^{\mathrm{crys}}(\mathfrak{A}_\infty)$. We give the rings A_∞ and B_∞ natural structures as objects of $'\mathrm{Mod}_{/S}^{1,\phi}$ as in Subsection 4.2. Then we have a decomposition $H_{1,\mathfrak{A}} = B_\infty \oplus \mathrm{Fil}^p H_{1,\mathfrak{A}}$ ([8, Lemme 2.3.2]). Let us consider the elements $\psi_1, \ldots, \psi_s$ and $\xi = u - X_0^{1/p}$ as before. We claim that if $a \in A_\infty$ satisfies $(X_0^{1/p})^{r_i} a \in \mathrm{Fil}^1 A_\infty$, then we have $a \in (X_0^{1/p})^{e-r_i} A_\infty$. Indeed, note that the k-algebra

$$A_\infty/\mathrm{Fil}^1 A_\infty = k[X_0^{p^{-\infty}}, \ldots, X_r^{p^{-\infty}}]/((X_0^{1/p})^e, \psi_1, \ldots, \psi_s)$$

is isomorphic via ϕ to the k-algebra

$$k[X_0^{p^{-\infty}}, \ldots, X_r^{p^{-\infty}}]/(X_0^e, f_1, \ldots, f_s) = \mathfrak{A}_\infty/p\mathfrak{A}_\infty.$$

Then we have $\pi^{r_i}\phi(a) = 0$ in this k-algebra. Since the $\mathcal{O}_K$-algebra $\mathfrak{A}_\infty$ is π-torsion free, the claim follows. Thus we can write the elements $f(e_i)$ and $f^\vee(e_i^\vee)$ of the ring $H_{1,\mathfrak{A}}$ as

$$\begin{aligned} f(e_i) &= (X_0^{1/p})^{e-r_i} z_i + \xi z_i' + z_i'', \\ f^\vee(e_i^\vee) &= (X_0^{1/p})^{r_i} w_i + \xi w_i' + w_i'' \end{aligned}$$

with some $z_i, w_i \in A_\infty$, $z_i', w_i' \in B_\infty$ and $z_i'', w_i'' \in \mathrm{Fil}^p H_{1,\mathfrak{A}}$. Since we have $u^{pe} = 0$ in the ring $H_{1,\mathfrak{A}}$, we can show the compatibility with Fil^1's and ϕ_1's just as in the proof of Lemma 2.3. □

Note that there exists an isomorphism $\mu_p \simeq \mathrm{Gr}(S_1(1))$ of finite flat group schemes over $\mathcal{O}_K$. Thus we also have an isomorphism

$$\eta_{\mathfrak{X}} : \mu_p(\mathfrak{X}) \to \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(S_1(1), \mathcal{O}_{1,\pi}^{\mathrm{crys}}(\mathfrak{X}))$$

which is functorial on $\mathfrak{X}$. After a twist by the action of $\mathrm{Aut}(\mu_p) = \mathbb{F}_p^\times$, we may assume that the induced map

$$\eta_{\mathcal{O}_{\bar{K}}} : \mu_p(\mathcal{O}_{\bar{K}}) \to \mathrm{Hom}_{S,\mathrm{Fil}^1,\phi_1}(S_1(1), H_1) = \mathbb{F}_p(\mathbf{e} \mapsto \bar{t})$$

is given by $\zeta_p \mapsto (\mathbf{e} \mapsto \bar{t})$. Then we get a homomorphism

$$\mathrm{Gr}(\mathcal{M})(\mathfrak{X}) \to \mathrm{Hom}(\mathrm{Gr}(\mathcal{M}^\vee)(\mathfrak{X}), \mu_p(\mathfrak{X}))$$

which is also functorial on $\mathfrak{X}$. Since $\mathrm{Gr}(\mathcal{M}^\vee)$ is a p-adic syntomic formal scheme over $\mathrm{Spf}(\mathcal{O}_K)$, this defines a homomorphism

$$\beta_{\mathcal{M}} : \mathrm{Gr}(\mathcal{M}) \to \mathrm{Gr}(\mathcal{M}^\vee)^\vee$$

of finite flat group schemes over $\mathcal{O}_K$.

Now we prove that $\beta_{\mathcal{M}}$ is an isomorphism. By Lemma 2.3, the composite

$$\begin{aligned}\mathrm{Gr}(\mathcal{M})(\mathcal{O}_{\bar{K}}) \to \mathrm{Gr}(\mathcal{M}^\vee)^\vee(\mathcal{O}_{\bar{K}}) = \mathrm{Hom}_{\mathcal{O}_{\bar{K}}\text{-grp.}}(\mathcal{O}_{\bar{K}} \times_{\mathcal{O}_K} \mathrm{Gr}(\mathcal{M})^\vee, \mu_{p,\mathcal{O}_{\bar{K}}}) \\ \to \mathrm{Hom}(\mathrm{Gr}(\mathcal{M}^\vee)(\mathcal{O}_{\bar{K}}), \mu_p(\mathcal{O}_{\bar{K}})) \simeq \mathrm{Hom}(T^*_{\mathrm{crys}}(\mathcal{M}^\vee), T^*_{\mathrm{crys}}(S_1(1)))\end{aligned}$$

is an isomorphism. By comparing the orders of both sides, we see that the induced map

$$\mathrm{Gr}(\mathcal{M})(\mathcal{O}_{\bar{K}}) \to \mathrm{Gr}(\mathcal{M}^\vee)^\vee(\mathcal{O}_{\bar{K}})$$

of $\mathrm{Gal}(\bar{K}/K)$-modules is also an isomorphism. This shows that the map $\beta_{\mathcal{M}}$ defines an isomorphism over K. For a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, let $d(\mathcal{G}/\mathcal{O}_K)$ denote the discriminant ideal of $\mathcal{G}/\mathcal{O}_K$. Consider the base change map $S \to S'$ as in Subsection 4.2 for $n = 1$ and put $\mathcal{M}' = S' \otimes_S \mathcal{M}$. Then, by Proposition 4.1, we have the equality

$$v_K(d(\mathrm{Gr}(\mathcal{M})/\mathcal{O}_K)) = \frac{1}{p} v_K(d(\mathrm{Gr}(\mathcal{M}')/\mathcal{O}_{K_1})).$$

Take an adapted basis $e_1, \ldots, e_d$ of $\mathcal{M}$ and $G \in GL_d(S_1)$ as before. Put $\tilde{S}'_1 = k[v]/(v^{ep^2})$ and consider the natural surjection $S'_1 \to S'_1/\mathrm{Fil}^p S'_1 \simeq \tilde{S}'_1$ as in the case of S. Then the image of G by the map $GL_d(S_1) \to GL_d(S'_1) \to GL_d(\tilde{S}'_1)$ is contained in $GL_d(k[v^p]/(v^{ep^2}))$ and we can apply the explicit description of the affine algebra of $\mathrm{Gr}(\mathcal{M}')$ recalled in Subsection 4.1. Since we have $v_K(d(\mathcal{G}/\mathcal{O}_K)) + v_K(d(\mathcal{G}^\vee/\mathcal{O}_K)) = v_K(\mathrm{rank}(\mathcal{G}))$, we get the equality

$$v_K(d(\mathrm{Gr}(\mathcal{M})/\mathcal{O}_K)) = de - \sum_{i=1}^{d} r_i = v_K(d(\mathrm{Gr}(\mathcal{M}^\vee)^\vee/\mathcal{O}_K)).$$

Hence the map $\beta_{\mathcal{M}}$ is an isomorphism. $\square$

Note that we also have an isomorphism $\gamma : \mathrm{Gr}((\mathcal{M}^\vee)^\vee) \to \mathrm{Gr}(\mathcal{M})$ induced by the isomorphism of the double duality $\mathcal{M} \to (\mathcal{M}^\vee)^\vee$. Then we define a $\mathrm{Gal}(\bar{K}/K_\infty)$-equivariant isomorphism $\delta_{\mathcal{M}}$ by the composite

$$\begin{aligned}\delta_{\mathcal{M}} : \mathrm{Gr}(\mathcal{M})^\vee(\mathcal{O}_{\bar{K}}) \xrightarrow{\gamma^\vee} \mathrm{Gr}((\mathcal{M}^\vee)^\vee)^\vee(\mathcal{O}_{\bar{K}}) \\ \underset{\beta^{-1}}{\to} \mathrm{Gr}(\mathcal{M}^\vee)(\mathcal{O}_{\bar{K}}) \underset{\varepsilon_{\mathcal{M}^\vee}}{\to} \mathrm{Hom}_{S,\mathrm{Fil}_1,\phi_1}(\mathcal{M}^\vee, R^{\mathrm{DP}}),\end{aligned}$$

where we put $\beta = \beta_{\mathcal{M}^\vee}$.

Lemma 4.5. *We have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Gr}(\mathcal{M})(\mathcal{O}_{\bar{K}}) \times \mathrm{Gr}(\mathcal{M})^\vee(\mathcal{O}_{\bar{K}}) \arrow[r, "\langle\ ,\ \rangle_{\mathrm{Gr}(\mathcal{M})}"] \arrow[d, "\varepsilon_{\mathcal{M}} \times \delta_{\mathcal{M}}"] & \mu_p(\mathcal{O}_{\bar{K}}) \arrow[d, "\iota"] \\
T^*_{\mathrm{crys}}(\mathcal{M}) \times T^*_{\mathrm{crys}}(\mathcal{M}^\vee) \arrow[r, "\langle\ ,\ \rangle_{\mathcal{M}}"'] & R^{\mathrm{DP}},
\end{tikzcd}$$

where $\langle\ ,\ \rangle_{\mathcal{M}}$ is the pairing induced by $D_{\mathcal{M}}$ and ι is the homomorphism sending ζ_p to $\bar{t}$.

Proof. Take elements $x \in \mathrm{Gr}(\mathcal{M})(\mathcal{O}_{\bar{K}})$ and $x^\vee \in \mathrm{Gr}(\mathcal{M})^\vee(\mathcal{O}_{\bar{K}})$. By the functoriality of the Cartier dual, we have the equality

$$\langle x, x^\vee\rangle_{\mathrm{Gr}(\mathcal{M})} = \langle \gamma^{-1}(x), \gamma^\vee(x^\vee)\rangle_{\mathrm{Gr}((\mathcal{M}^\vee)^\vee)}.$$

From the construction of the isomorphism β and the choice of the isomorphism $\eta_{\mathfrak{X}}$, we see that the equality

$$\iota(\langle \gamma^{-1}(x), \gamma^\vee(x^\vee)\rangle_{\mathrm{Gr}((\mathcal{M}^\vee)^\vee)}) = \langle (\varepsilon_{\mathcal{M}^\vee} \circ \beta^{-1} \circ \gamma^\vee)(x^\vee), (\varepsilon_{(\mathcal{M}^\vee)^\vee} \circ \gamma^{-1})(x)\rangle_{\mathcal{M}^\vee}$$

holds. By the definition of the pairing $\langle\ ,\ \rangle_{\mathcal{M}}$, we also have the equality

$$\langle \varepsilon_{\mathcal{M}}(x), \delta_{\mathcal{M}}(x^\vee)\rangle_{\mathcal{M}} = \langle \delta_{\mathcal{M}}(x^\vee), (\varepsilon_{(\mathcal{M}^\vee)^\vee} \circ \gamma^{-1})(x)\rangle_{\mathcal{M}^\vee}$$

and the lemma follows. □

Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ and put $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$. By combining the middle vertical arrows of the diagrams in Proposition 2.4 and Lemma 4.5, we have an isomorphism of $\mathrm{Gal}(\bar{K}/K_\infty)$-modules

$$\delta_{\mathfrak{M}} : \mathcal{G}(\mathfrak{M})^\vee(\mathcal{O}_{\bar{K}}) \to T^*_{\mathfrak{S}}(\mathfrak{M}^\vee) = \mathcal{H}(\mathfrak{M}^\vee)(R).$$

Then we get the following corollary.

Corollary 4.6. *We have a commutative diagram*

$$\begin{tikzcd}
\mathcal{G}(\mathfrak{M})(\mathcal{O}_{\bar{K}}) \times \mathcal{G}(\mathfrak{M})^\vee(\mathcal{O}_{\bar{K}}) \arrow[r, "\langle\ ,\ \rangle_{\mathcal{G}(\mathfrak{M})}"] \arrow[d, "\varepsilon_{\mathfrak{M}} \times \delta_{\mathfrak{M}}"] & \mu_p(\mathcal{O}_{\bar{K}}) \arrow[d, "\alpha^{-1}\iota"] \\
\mathcal{H}(\mathfrak{M})(R) \times \mathcal{H}(\mathfrak{M}^\vee)(R) \arrow[r, "\langle\ ,\ \rangle_{\mathfrak{M}}"'] & R,
\end{tikzcd}$$

where the right vertical arrow is the injective homomorphism defined by $\zeta_p^i \mapsto i\alpha^{-1}\bar{t}$.

□

4.4. **Proof of the main theorem.** Now we prove Theorem 1.1. By Theorem 3.1, Theorem 3.3 and Corollary 4.6, it suffices to show that the isomorphism $\delta_{\mathfrak{M}}$ induces an isomorphism of lower ramification subgroups

$$(\mathcal{G}(\mathfrak{M})^\vee)_i(\mathcal{O}_{\bar{K}}) \to \mathcal{H}(\mathfrak{M}^\vee)_i(R)$$

for any $i \in \mathbb{Q}_{\geq 0}$. By the construction of the map $\delta_{\mathfrak{M}}$, it is enough to show that, for an object $\mathfrak{M}$ of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ and $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$, the isomorphism

$$\varepsilon : \mathrm{Gr}(\mathcal{M})(\mathcal{O}_{\bar{K}}) \to T^*_{\mathrm{crys}}(\mathcal{M}) \to T^*_{\mathfrak{S}}(\mathfrak{M}) = \mathcal{H}(\mathfrak{M})(R)$$

induces an isomorphism between the i-th lower ramification subgroups of $\mathrm{Gr}(\mathcal{M})(\mathcal{O}_{\bar{K}})$ and $\mathcal{H}(\mathfrak{M})(R)$ for any $i \in \mathbb{Q}_{\geq 0}$. Note that, by Corollary 3.6 and Remark 3.7, the i-th lower ramification subgroups of both sides vanish for $i > e/(p-1)$. By Proposition 4.2, it suffices to show the claim after the base change to $K_1 = K(\pi_1)$. We adopt the notation in Subsection 4.2 for $n = 1$, as in the proof of Proposition 4.3. For example, we set the base change homomorphism $\mathfrak{S} \to \mathfrak{S}'$ to be $u \mapsto v^p$. Then we are reduced to show the following proposition.

Proposition 4.7. *Let $\mathfrak{M}$ be an object of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$ and $i' \leq ep$ be a non-negative rational number. Put $\mathcal{M} = \mathcal{M}_{\mathfrak{S}}(\mathfrak{M})$, $\mathfrak{M}' = \mathfrak{S}' \otimes_{\mathfrak{S}} \mathfrak{M}$, $\mathcal{M}' = \mathcal{M}_{\mathfrak{S}'}(\mathfrak{M}')$ and $\varepsilon' = \varepsilon_{\mathfrak{M}'}$ as in Subsection 4.2 for $n = 1$. Then there exists a bijection*

$$\varepsilon'_{i'} : \mathrm{Gr}(\mathcal{M}')(\mathcal{O}_{\bar{K}}/m'^{\geq i'}_{\bar{K}}) \to \mathcal{H}(\mathfrak{M}')(R/m'^{\geq i'}_R)$$

which satisfies $\varepsilon'_{i'}(0) = 0$ and makes the following diagram commutative.

$$\begin{array}{ccc}
\mathrm{Gr}(\mathcal{M}')(\mathcal{O}_{\bar{K}}) & \xrightarrow{\varepsilon'} & \mathcal{H}(\mathfrak{M}')(R) \\
\downarrow & & \downarrow \\
\mathrm{Gr}(\mathcal{M}')(\mathcal{O}_{\bar{K}}/m'^{\geq i'}_{\bar{K}}) & \xrightarrow[\varepsilon'_{i'}]{} & \mathcal{H}(\mathfrak{M}')(R/m'^{\geq i'}_R)
\end{array}$$

Proof. Take an adapted basis $e_1, \ldots, e_d$ of $\mathcal{M}$ and the representing matrix G as before. Then we see that $\mathcal{M}'$ has an adapted basis $e'_1, \ldots, e'_d$ such that

$$\mathrm{Fil}^1\mathcal{M}' = v^{pr_1}S'_1e'_1 \oplus \cdots \oplus v^{pr_d}S'_1e'_d + (\mathrm{Fil}^pS')\mathcal{M}',$$
$$\phi_1(v^{pr_1}e'_1, \ldots, v^{pr_d}e'_d) = (e'_1, \ldots, e'_d)G',$$

where G' is the image of G in $GL_d(S'_1)$ by the map $S_1 \to S'_1$ sending u to v^p. Consider the natural ring surjection $S'_1 \to \tilde{S}'_1 = k[v]/(v^{ep^2})$. Then the image $\tilde{G}'$ of the matrix G' by this surjection is contained in $GL_d(k[v^p]/(v^{ep^2}))$. Take its lift $\hat{G}'_1$ in $GL_d(k[[v^p]])$ via the natural surjection $\mathfrak{S}'_1 = k[[v]] \to \tilde{S}'_1 = k[v]/(v^{ep^2})$. We define an object $\mathfrak{N}'$ of $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}'_1}$ to be $\mathfrak{N}' = \mathfrak{S}'_1n_1 \oplus \cdots \oplus \mathfrak{S}'_1n_d$ with

$$\phi_{\mathfrak{N}'}(n_1, \ldots, n_d) = (n_1, \ldots, n_d)\phi^{-1}(\hat{G}'_1)(-F(v^p))^{-1}\mathrm{diag}\{v^{pe-pr_1}, \ldots, v^{pe-pr_d}\}.$$

Then the associated Breuil module $\mathcal{N}' = \mathcal{M}_{\mathfrak{S}'}(\mathfrak{N}')$ is given by

$$\begin{aligned}
&\mathcal{N}' = S'_1(1\otimes n_1)\oplus\cdots\oplus S'_1(1\otimes n_d),\\
&\mathrm{Fil}^1\mathcal{N}' = v^{pr_1}S'_1(1\otimes n_1)\oplus\cdots\oplus v^{pr_d}S'_1(1\otimes n_d) + (\mathrm{Fil}^p S')\mathcal{N}',\\
&\phi_1(v^{pr_1}(1\otimes n_1),\ldots,v^{pr_d}(1\otimes n_d)) = (1\otimes n_1,\ldots,1\otimes n_d)c'\phi(-F(v^p))^{-1}\hat{G}'_1,
\end{aligned}$$

where $c' = \phi_1(E(v^p)) \in (S')^\times$. Since the element $c'\phi(-F(v^p))^{-1}$ coincides with 1 in the ring $\tilde{S}'_1$, we see that $T_0(\mathcal{M}')$ and $T_0(\mathcal{N}')$ are naturally isomorphic to each other. Hence we have an isomorphism $\tau' : \mathfrak{M}' \to \mathfrak{N}'$, and it is enough to construct a bijection

$$\varepsilon'_{i'} : \mathrm{Gr}(\mathcal{N}')(\mathcal{O}_{\bar{K}}/m'^{\geq i'}_{\bar{K}}) \to \mathcal{H}(\mathfrak{N}')(R/m'^{\geq i'}_{R})$$

satisfying $\varepsilon'_{i'}(0) = 0$ and compatible with the isomorphism $\varepsilon' : \mathrm{Gr}(\mathcal{N}')(\mathcal{O}_{\bar{K}}) \to \mathcal{H}(\mathfrak{N}')(R)$ as in the proposition.

Choose a lift $\hat{G}'$ of $\hat{G}'_1$ in $GL_d(W[[v^p]])$ and put $\phi^{-1}(\hat{G}') = (a_{i,j})$. Note that we have a commutative diagram of W-algebras

$$\begin{tikzcd}
W[[v]] \arrow[rr, "v\mapsto\pi_1"] \arrow[d] & & \mathcal{O}_{K_1} \arrow[d] \\
k[[v]] \arrow[r] & k[v]/(v^{ep}) \arrow[r, "\sim"'] & \mathcal{O}_{K_1}/p\mathcal{O}_{K_1}.
\end{tikzcd}$$

Consider the composite map

$$W[[v]] \to \mathcal{O}_{K_1}/p\mathcal{O}_{K_1} \to \mathcal{O}_{K_2}/p\mathcal{O}_{K_2} \overset{\sim}{\to} k[v]/(v^{ep^2}) = \tilde{S}'_1,$$

where the last arrow is the ϕ-semilinear isomorphism defined by $\pi_2 \mapsto v$. Then the image of $\phi^{-1}(\hat{G}')$ by this composite map coincides with $\tilde{G}'$. This implies that the affine algebra $R_{\mathcal{N}'}$ of $\mathrm{Gr}(\mathcal{N}')$ is defined by the system of equations over $\mathcal{O}_{K_1}$

$$X_i^p + \frac{\pi_1^{pe-pr_i}}{F(\pi)}(\sum_{j=1}^d a_{j,i}(\pi_1)X_j) \quad (i=1,\ldots,d),$$

where $a_{i,j}(\pi_1)$ means the image of $a_{i,j}$ by the map $W[[v]] \to \mathcal{O}_{K_1}$ defined as in the above diagram. On the other hand, the defining equations of $\mathcal{H}(\mathfrak{N}')$ over $k[[v]]$ are

$$X_i^p + \frac{v^{pe-pr_i}}{F(v^p)}(\sum_{j=1}^d \bar{a}_{j,i}X_j) \quad (i=1,\ldots,d),$$

where $\bar{a}_{i,j}$ means the image of $a_{i,j}$ by the natural map $W[[v]] \to k[[v]]$, and the zero section of $\mathcal{H}(\mathfrak{N}')$ is by definition $X_1 = \cdots = X_d = 0$. This implies that there exists an isomorphism

$$(\mathcal{O}_{K_1}/p\mathcal{O}_{K_1}) \times_{\mathcal{O}_{K_1}} \mathrm{Gr}(\mathcal{N}') \to (k[v]/(v^{ep})) \times_{k[[v]]} \mathcal{H}(\mathfrak{N}')$$

of schemes over $\mathcal{O}_{K_1}/p\mathcal{O}_{K_1} \simeq k[v]/(v^{ep})$ defined by $X_i \mapsto X_i$. Note that we also have an isomorphism of rings

$$\mathrm{pr}_0 : R/m'^{\geq ep}_R \to \mathcal{O}_{\bar{K}}/m'^{\geq ep}_{\bar{K}} = \tilde{\mathcal{O}}_{\bar{K}}$$

which lies over the map $k[v]/(v^{ep}) \to \mathcal{O}_{K_1}/p\mathcal{O}_{K_1}$. Thus, for $i' \leq ep$, we get a bijection

$$\varepsilon'_{i'} : \mathrm{Gr}(\mathcal{N}')(\mathcal{O}_{\bar{K}}/m'^{\geq i'}_{\bar{K}}) \to \mathcal{H}(\mathfrak{N}')(R/m'^{\geq i'}_R)$$

satisfying $\varepsilon'_{i'}(0) = 0$.

To prove the compatibility of ε' and $\varepsilon'_{i'}$, let us consider the diagram

$$\begin{tikzcd}
\mathcal{O}_{\bar{K}} \arrow[r] \arrow[d] & (\tilde{\mathcal{O}}_{\bar{K}})^{\mathrm{DP}} & R^{\mathrm{DP}} \arrow[l, "\mathrm{pr}_1"', "\sim"] & R \arrow[l, "\phi"'] \arrow[d] \\
\tilde{\mathcal{O}}_{\bar{K}} & & & R/m'^{\geq ep}_R. \arrow[lll, "\mathrm{pr}_0"', "\sim"]
\end{tikzcd}$$

Let $x = (x_1, \ldots, x_d)$ be an element of $\mathrm{Spec}(R_{\mathcal{N}'})(\mathcal{O}_{\bar{K}})$ and $z = (n_i \mapsto z_i)$ be the corresponding element of $T^*_{\mathfrak{S}'}(\mathfrak{N}')$ via the composite

$$\mathrm{Spec}(R_{\mathcal{N}'})(\mathcal{O}_{\bar{K}}) \simeq \mathrm{Gr}(\mathcal{N}')(\mathcal{O}_{\bar{K}}) \xrightarrow{\varepsilon'} T^*_{\mathfrak{S}'}(\mathfrak{N}').$$

Let $y_i \in R$ be the element such that $\mathrm{pr}_1(y_i)$ coincides with the image $\bar{x}_i$ of x_i in $\tilde{\mathcal{O}}_{\bar{K}}$. Then, in the ring R^{DP}, we have $\phi(z_i) - y_i \in \mathrm{Fil}^p R^{\mathrm{DP}}$. Put $y_i = (y_{i,0}, y_{i,1}, \ldots)$ and $z_i = (z_{i,0}, z_{i,1}, \ldots)$ with $y_{i,j}, z_{i,j} \in \tilde{\mathcal{O}}_{\bar{K}}$. Since the natural map $R \to R^{\mathrm{DP}}$ induces an isomorphism

$$R/m'^{\geq ep^2}_R \to R^{\mathrm{DP}}/\mathrm{Fil}^p R^{\mathrm{DP}}$$

and the kernel of the map $\mathrm{pr}_1 : R \to \tilde{\mathcal{O}}_{\bar{K}}$ coincides with the ideal $m'^{\geq ep^2}_R$, we have $y_{i,1} = z^p_{i,1} = z_{i,0}$. This implies $\bar{x}_i = z_{i,0}$ and the compatibility of ε' and $\varepsilon'_{i'}$ as in the proposition follows. Hence we conclude the proof of Theorem 1.1. □

Note that in the course of the proof of Proposition 4.7, we have also shown the following theorem.

Theorem 4.8. *Let $\mathfrak{M}$ be an object of the category $\mathrm{Mod}^{1,\phi}_{/\mathfrak{S}_1}$. Consider the k-algebra $k[[v]]$ as a $k[[u]]$-algebra by the map $u \mapsto v^p$. By the k-algebra isomorphism $k[[v]]/(v^{ep}) \to \mathcal{O}_{K_1}/p\mathcal{O}_{K_1}$ defined by $v \mapsto \pi_1$, we identify the k-algebras of both sides. Then we have an isomorphism*

$$(\mathcal{O}_{K_1}/p\mathcal{O}_{K_1}) \times_{\mathcal{O}_K} \mathcal{G}(\mathfrak{M}) \to (k[[v]]/(v^{ep})) \times_{k[[u]]} \mathcal{H}(\mathfrak{M})$$

of schemes over $k[[v]]/(v^{ep}) \simeq \mathcal{O}_{K_1}/p\mathcal{O}_{K_1}$ preserving the zero section.

Remark 4.9. The way we have proved Theorem 1.1 is based on switching from the upper to the lower ramification subgroups via duality. The author wonders if we can prove the theorem in an "upper" way, namely by constructing a natural isomorphism between the sets of geometric connected components of tubular neighborhoods of $\mathcal{G}(\mathfrak{M})$ and $\mathcal{H}(\mathfrak{M})$ using the similarity of their affine algebras, even though they are in different characteristics.

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