ON A RAMIFICATION BOUND OF SEMI-STABLE MOD $p$ REPRESENTATIONS OVER A LOCAL FIELD

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Abstract. Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors, $K$ be a finite totally ramified extension of $\text{Frac}(W)$ of degree $e$ and $r$ be a non-negative integer satisfying $r < p - 1$. In this paper, we prove the upper numbering ramification group $G^{(j)}_K$ for $j > u(K, r)$ acts trivially on the mod $p$ representations associated to the semi-stable $p$-adic $G_K$-representations with Hodge-Tate weights in $\{0, \ldots, r\}$, where we put $u(K, 1) = 1 + e(1 + 1/(p - 1))$ and $u(K, r) = 1 - 1/p + e(1 + r/(p - 1))$ for $r \geq 2$.

1. Introduction

Let $p$ be a rational prime, $k$ be a perfect field of characteristic $p$, $W = W(k)$ be the ring of Witt vectors and $K$ be a finite totally ramified extension of $K_0 = \text{Frac}(W)$ of degree $e = e(K)$. Let the maximal ideal of $K$ be denoted by $m_K$, an algebraic closure of $K$ by $\bar{K}$ and the absolute Galois group of $K$ by $G_K = \text{Gal}(\bar{K}/K)$. We normalize the valuation $v_K$ of $K$ as $v_K(p) = e$ and extend this to $\bar{K}$. Let $G^{(j)}_K$ denote the $j$-th upper numbering ramification group in the sense of [6]. Namely, we put $G^{(j)}_K = G^{j-1}_K$, where the latter is the upper numbering ramification group defined in [12].

Let $X_K$ be a proper smooth scheme over $K$ and put $\bar{X}_K = X_K \times_K \bar{K}$. Consider the $r$-th $p$-adic étale cohomology group $H^r_{\text{ét}}(X_K, \mathbb{Q}_p)$ and its $G_K$-stable $\mathbb{Z}_p$-lattices $L \supset L'$. In [6], Fontaine conjectured the upper numbering ramification group $G^{(j)}_K$ acts trivially on the $G_K$-module $L/L'$ for $j > e(n + r/(p - 1))$ if $X_K$ has good reduction and this module is killed by $p^n$. For $e = 1$ and $r < p - 1$, this conjecture was solved independently by himself ([7], for $n = 1$) and Abrashkin ([1], for any $n$), using the theory of Fontaine-Laffaille ([8]) and the comparison theorem of Fontaine-Messing ([9]) between the $p$-adic étale cohomology groups of $X_K$ and the crystalline cohomology groups of the reduction of $X_K$. From this result, Fontaine also showed some rareness of a proper smooth scheme over $\mathbb{Q}$ with everywhere good reduction ([7, Théorème 1]). In fact, they proved this ramification bound for the torsion representations of the crystalline $p$-adic representations of $G_K$ with Hodge-Tate weights in $\{0, \ldots, r\}$ in the case where $K$ is absolutely unramified.

On the other hand, as for a semi-stable $p$-adic representation $V$ with Hodge-Tate weights in the same range, a similar ramification bound for $e = 1$ and $n = 1$ is obtained by Breuil (see [4, Proposition 9.2.2.2]). He showed, assuming Griffiths transversality which in general does not hold, that if $e = 1$ and $r < p - 1$, then

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the ramification group $G^{(j)}_K$ acts trivially on the mod $p$ representations of $V$ for $j > 2 + 1/(p-1)$.

In this paper, we prove a version of the result of Breuil for the case where $K$ is absolutely ramified, under the condition $r < p - 1$ and $n = 1$. Our main theorem is the following.

**Theorem 1.1.** Let $r$ be a non-negative integer such that $r < p - 1$. Let $V$ be a semi-stable $p$-adic $G_K$-representation with Hodge-Tate weights in $\{0, \ldots, r\}$ and $\mathcal{L} \supset \mathcal{L}'$ be $G_K$-stable $\mathbb{Z}_p$-lattices in $V$. Suppose that the quotient $\mathcal{L}/\mathcal{L}'$ is killed by $p$. Then the $j$-th upper numbering ramification group $G^{(j)}_K$ acts trivially on the $G_K$-module $\mathcal{L}/\mathcal{L}'$ for $j > u(K, r)$, where we put $u(K, 1) = 1 + e(1 + 1/(p-1))$ and $u(K, r) = 1 - 1/p + e(1 + r/(p - 1))$ for $r \geq 2$.

Note that this ramification bound is sharp at least for $r = 1$, since the upper bound $1 + e(1 + 1/(p-1))$ of the ramification is obtained by a mod $p$ representation associated to a Tate curve over $K$.

For the proof of the theorem, we follow the same lines as in [7]. Thanks to Liu’s theorem ([11]) on $G_K$-stable $\mathbb{Z}_p$-lattices in semi-stable $p$-adic representations, it is enough to bound the ramification of the $G_K$-module

$$T^*_n(\mathcal{M}) = \text{Hom}_{\mathcal{S}, \text{Fil}^r, (\phi, N)}(\mathcal{M}, \hat{A}_{st, \infty}),$$

where $\mathcal{M}$ is a $p$-torsion object of the category $\mathcal{M}^r$ of filtered $(\phi, N)$-modules over $S$ defined by Breuil ([2]) and $\hat{A}_{st, \infty}$ is a $p$-adic period ring. Put $\hat{S}_1 = k[u]/(u^p) \cong S/(p, \text{Fil}^p S)$, $\hat{\mathcal{M}} = \mathcal{M} \otimes_S \hat{S}_1$, $\pi_1 = \pi^1/p$ and $K_1 = K(\pi_1)$. We also have a $G_{K_1}$-linear bijection

$$T^*_n(\mathcal{M})|_{G_{K_1}} \cong T_{\text{crys}, \pi_1}(\mathcal{M}) = \text{Hom}_{\hat{S_1}, \text{Fil}^r, \phi_1}(\hat{\mathcal{M}}, \mathcal{O}_K/p\mathcal{O}_K)$$

with the natural filtered $\phi_1$-module structure on $\mathcal{O}_K/p\mathcal{O}_K$. Then the same argument as in [7] gives a ramification bound of the $G_{K_1}$-module $T_{\text{crys}, \pi_1}(\mathcal{M})$ and careful use of [6, Proposition 1.5] shows the theorem.

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## 2. Filtered $(\phi, N)$-modules of Breuil

In this section, we recall the theory of filtered $(\phi, N)$-modules of Breuil, which is developed by himself and most recently by Caruso and Liu. In what follows, we always take the divided power envelope of a $W$-algebra with the compatibility condition with the natural divided power structure on $pW$.

Let $\sigma$ be the Frobenius homomorphism of $W$. We fix once and for all a uniformizer $\pi$ of $K$ and a system $\{\pi_n\}_{n \in \mathbb{Z}_{\geq 0}}$ of $p$-power roots of $\pi$ such that $\pi_0 = \pi$ and $\pi_n = \pi_{n+1}^p$. Let $E(u)$ be the Eisenstein polynomial of $\pi$ over $W$ and set $S = (W[u]^{\text{PD}})^\wedge$, where the divided power envelope of $W[u]$ is taken with respect to the ideal $(E(u))$ and $\wedge$ means the $p$-adic completion. The ring $S$ is endowed with the $\sigma$-semilinear map $\phi : u \mapsto u^p$ and the natural filtration induced by the divided power structure. We set $\phi_t = p^{-t}\phi|_{\text{Fil}^t S}$ for any non-negative integer $t$ and
\[ c = \phi_1(E(u)) \in S^\times. \] The \( W \)-linear derivation \( N \) on \( S \) is defined by \( N(u) = -u. \) We also define a filtration, \( \phi, \phi_1, N \) on \( S_u = S/p^nS \) similarly.

Let \( r \in \{0, \ldots, p-2\} \) be an integer. Set \( 'M' \) to be the category consisting of the following data:

- an \( S \)-module \( M \) and its \( S \)-submodule \( \text{Fil}^r M \) containing \( \text{Fil}^r S \cdot M, \)
- a \( \phi \)-semilinear map \( \phi_r : \text{Fil}^r M \to M \) satisfying
  \[ \phi_r(s_r m) = \phi_r(s_r) \phi(m) \]
  for any \( s_r \in \text{Fil}^r S \) and \( m \in M, \)
- a \( W \)-linear map \( N : M \to M \) such that
  - \( N(s_m) = N(s)m + sN(m) \) for any \( s \in S \) and \( m \in M, \)
  - \( E(u)N(\text{Fil}^r M) \subseteq \text{Fil}^r M, \)
- the following diagram is commutative:
  \[
  \begin{array}{ccc}
  \text{Fil}^r M & \xrightarrow{\phi_r} & M \\
  E(u)N \downarrow & & \downarrow eN \\
  \text{Fil}^r M & \xrightarrow{\phi_r} & M,
  \end{array}
  \]

and the morphisms of \( 'M' \) are defined to be the \( S \)-linear maps preserving \( \text{Fil}^r \) and commuting with \( \phi_r \) and \( N. \) The category defined in the same way but dropping the data \( N \) is denoted by \( 'M'_0. \) These categories have obvious notions of exact sequences. The category \( 'M' \) (resp. \( 'M'_0. \)) is defined as the smallest full subcategory of \( 'M' \) (resp. \( 'M'_0. \)) stable under extensions and containing the objects satisfying the following condition:

- \( M \) is free of finite rank over \( S_1 \) and generated as an \( S_1 \)-module by the image of \( \phi_r. \)

For \( p \)-torsion objects, we have the following categories. Consider the algebra \( k[u]/(u^p) \cong S_1/\text{Fil}^p S_1 \) and let this be denoted by \( \tilde{S}_1. \) The algebra \( \tilde{S}_1 \) has the natural filtration, \( \phi \) and \( N \) induced by those of \( S. \) Namely, \( \text{Fil}^0 \tilde{S}_1 = u^0, \phi(u) = u^p \) and \( N(u) = -u. \) Then the category \( ' \tilde{M} ' \) consists of the following data:

- an \( \tilde{S}_1 \)-module \( \tilde{M} \) and its \( \tilde{S}_1 \)-submodule \( \text{Fil}^r \tilde{M} \) containing \( u^r \tilde{M}, \)
- a \( \phi \)-semilinear map \( \phi_r : \text{Fil}^r \tilde{M} \to \tilde{M}, \)
- a \( k \)-linear map \( N : \tilde{M} \to \tilde{M} \) such that
  - \( N(s_m) = N(s)m + sN(m) \) for any \( s \in \tilde{S}_1 \) and \( m \in \tilde{M}, \)
  - \( u^rN(\text{Fil}^r \tilde{M}) \subseteq \text{Fil}^r \tilde{M}, \)
- the following diagram is commutative:
  \[
  \begin{array}{ccc}
  \text{Fil}^r \tilde{M} & \xrightarrow{\phi_r} & \tilde{M} \\
  u^rN \downarrow & & \downarrow eN \\
  \text{Fil}^r \tilde{M} & \xrightarrow{\phi_r} & \tilde{M},
  \end{array}
  \]

and the morphisms are defined as before. Its full subcategory \( ' \tilde{M} ' \) is defined by the following condition:

- As an \( \tilde{S}_1 \)-module, \( \tilde{M} \) is free of finite rank and generated by the image of \( \phi_r. \)
The categories $\breve{\mathcal{M}}_0$ and $\breve{\mathcal{M}}'_0$ are also defined similarly.

Let $D$ be a weakly admissible filtered $(\phi, N)$-module over $K$ satisfying $\text{Fil}^0 D_K = D_K$ and $\text{Fil}^{r+1} D_K = 0$. Set $S_{K_0} = S \otimes_W K_0$ and $D = D \otimes_{K_0} S_{K_0}$. Then the $S_{K_0}$-module $D$ is endowed with the natural $\phi$-semilinear map $\phi \otimes \sigma$ and $K_0$-linear derivation $N \otimes 1 + 1 \otimes N$, which are denoted by $\phi$ and $N$, respectively. The filtration on $D$ is defined inductively by $\text{Fil}^0 D = D$ and

$$\text{Fil}^{r+1} D = \{ x \in D | N(x) \in \text{Fil}^r D \text{ and } f_\pi(x) \in \text{Fil}^{r+1} D_K \},$$

where $f_\pi : D \to D_K$ is induced by the map $S \to \mathcal{O}_K$ sending $u$ to $\pi$. An $S$-submodule $\hat{M}$ of $D$ is said to be a strongly divisible lattice of $D$ if the following conditions are satisfied:

- the $S$-module $\hat{M}$ is free of finite rank,
- $\hat{M} \otimes_W K_0 = D$,
- $\hat{M}$ is stable under $\phi$ and $N$,
- $\phi(\text{Fil}^r \hat{M}) \subseteq \text{Fil}^r \hat{M}$, where we set $\text{Fil}^r \hat{M} = \hat{M} \cap \text{Fil}^r D$.

We put $\phi_\pi = p^{-r} \phi |_{\text{Fil}^r \hat{M}}$. Then the $S$-module $\hat{M}$ is generated by $\phi_\pi(\text{Fil}^r \hat{M}) (\{2, \text{Proposition 2.1.3}\})$.

Let $A_{\text{crys}}$ and $A_{\text{st}}$ be $p$-adic period rings. These are constructed as follows. Set

$$R = \lim_{\longrightarrow} (\mathcal{O}_K/p\mathcal{O}_K \leftarrow \mathcal{O}_K/p\mathcal{O}_K \leftarrow \cdots),$$

where every arrow is the $p$-power map. For an element $x = (x_i)_{i \in \mathbb{Z}_{\geq 1}} \in R$, we set

$$x^{(n)} = \lim_{m \to \infty} x_{n+m}^p \in \mathcal{O}_C\text{ for } u \geq 0,$$

where $\hat{x}_i$ is a lift of $x_i$ in $\mathcal{O}_K$. The natural ring homomorphism $\theta$ is defined by

$$\theta : W(R) \to \mathcal{O}_C$$

$$(r_0, r_1, \ldots) \mapsto \sum_{n \geq 0} p^n r^{(n)}_n.$$

Then $A_{\text{crys}}$ is the $p$-adic completion of the divided power envelope of $W(R)$ with respect to $\text{Ker}(\theta)$ and $A_{\text{st}}$ is the $p$-adic completion of the divided power polynomial ring $A_{\text{crys}}(X)$ over $A_{\text{crys}}$. We set $A_{\text{crys}, \infty} = A_{\text{crys}} \otimes_W K_0/W$ and $A_{\text{st}, \infty} = A_{\text{st}} \otimes_W K_0/W$. Put $\pi = (\pi_1 \mod p, \pi_2 \mod p, \ldots) \in R$. These rings are considered as $S$-modules by the ring homomorphisms $S \to \hat{A}_{\text{st}}$ and $S \to A_{\text{crys}}$, which are defined by $u \mapsto [x]/(1 + X)$ and $X \mapsto 0$, respectively. The ring $A_{\text{crys}}$ is endowed with the natural filtration induced by the divided power structure, the natural Frobenius $\phi$ and the $\phi$-semilinear map $\hat{\phi}_\pi = p^{-r} \phi |_{\text{Fil}^r A_{\text{crys}}}$. With these structures, $A_{\text{crys}}$ and $A_{\text{crys}, \infty}$ are considered as objects of $\breve{\mathcal{M}}'_0$. Moreover, the absolute Galois group $G_K$ acts naturally on these two rings. As for $A_{\text{st}}$, its filtration is defined by

$$\text{Fil}^f \hat{A}_{\text{st}} = \left\{ \sum_{i \geq 0} a_i \frac{X^i}{i!} | a_i \in \text{Fil}^{f-i} A_{\text{crys}}, \lim_{i \to \infty} a_i = 0 \right\}$$

and the Frobenius structure of $A_{\text{crys}}$ extends to $\hat{A}_{\text{st}}$ by

$$\hat{\phi}(X) = (1 + X)^p - 1,$$

$$\hat{\phi}_\pi = p^{-r} \hat{\phi} |_{\text{Fil}^f \hat{A}_{\text{st}}}.$$
The $A_{\text{crys}}$-linear derivation $N$ on $\hat{A}_\text{st}$ is defined by $N(X) = 1 + X$. The rings $\hat{A}_\text{st}$ and $\hat{A}_{\text{st},\infty}$ are objects of $\mathcal{M}'$. The $G_K$-action on $A_{\text{crys}}$ naturally extends to the action on $\hat{A}_\text{st}$. Indeed, the action of $g \in G_K$ on $\hat{A}_\text{st}$ is defined by the formula

$$g(X) = [\hat{\varepsilon}(g)][X + 1] - 1,$$

where $g(\pi_n) = \varepsilon_n(g)\pi_n$ and $\hat{\varepsilon}(g) = (\varepsilon_1(g) \mod p, \varepsilon_2(g) \mod p, \ldots) \in R$.

Put $K_n = K(\pi_n)$ and $K_\infty = \cup_n K_n$. For $\mathcal{M} \in \mathcal{M}'$, the $G_K$-module $T_{\text{st},\hat{\varepsilon}}(\mathcal{M})$ and $G_{K_\infty}$-module $T_{\text{crys},\hat{\varepsilon}}(\mathcal{M})$ are defined to be

$$T_{\text{st},\hat{\varepsilon}}(\mathcal{M}) = \text{Hom}_{S,\text{Fil},\phi_v,N}(\mathcal{M}, \hat{A}_{\text{st},\infty}),$$

$$T_{\text{crys},\hat{\varepsilon}}(\mathcal{M}) = \text{Hom}_{S,\text{Fil},\phi_v}(\mathcal{M}, A_{\text{crys},\infty}).$$

Then we see as in the proof of [2, Lemme 2.3.1.1] that the natural map

$$T_{\text{st},\hat{\varepsilon}}(\mathcal{M}) \to T_{\text{crys},\hat{\varepsilon}}(\mathcal{M})$$

is bijective and $G_{K_\infty}$-linear.

For $p$-torsion objects in $\mathcal{M}'$ and $\mathcal{M}'_0$, we have a simpler description of the functors $T_{\text{st},\hat{\varepsilon}}$ and $T_{\text{crys},\hat{\varepsilon}}$ ([5]). We consider $\hat{A} = (\mathcal{O}_K/p\mathcal{O}_K)(X)$ as an object of $\mathcal{M}'$ by the natural surjection

$$\hat{A}_\text{st}/p\hat{A}_\text{st} \to \hat{A}_\text{st}/(p, \text{Fil}^p A_{\text{crys}}) \cong \hat{A},$$

where the last isomorphism is defined by $X \mapsto X$. This surjection also induces the natural $G_{K_\infty}$-action on $\hat{A}$. Similarly, the algebra $\mathcal{O}_K/p\mathcal{O}_K$ is considered as an object of $\mathcal{M}'_0$. The filtration of $\mathcal{O}_K/p\mathcal{O}_K$ is given by

$$\text{Fil}^r(\mathcal{O}_K/p\mathcal{O}_K) = \{ \hat{x} \in \mathcal{O}_K \mid v_K(\hat{x}) \geq \frac{r}{p}, \mathcal{O}_K \}$$

and the Frobenius structure by

$$\phi_v(x) = \frac{\hat{x}^p}{(-p)^r} \mod p,$$

where $\hat{x}$ denotes a lift of $x$ in $\mathcal{O}_K$. For a $p$-torsion object $\mathcal{M} \in \mathcal{M}'$, we set $T(\mathcal{M}) = \mathcal{M} \otimes S \hat{S}_1$. This $\hat{S}_1$-module is naturally considered as an object of $\hat{M}'$. By [5, Lemme 2.3.4], we have a $G_K$-linear isomorphism

$$T_{\text{st},\hat{\varepsilon}}(\mathcal{M}) \to \text{Hom}_{\hat{S}_1,\text{Fil},\phi_v,N}(T(\mathcal{M}), \hat{A}).$$

Similarly, for a $p$-torsion object $\mathcal{M} \in \mathcal{M}'_0$, the $\hat{S}_1$-module $T(\mathcal{M}) = \mathcal{M} \otimes S \hat{S}_1$ has a natural structure as an object of $\hat{M}'_0$. Then we have a $G_{K_\infty}$-linear isomorphism

$$T_{\text{crys},\hat{\varepsilon}}(\mathcal{M}) \to \text{Hom}_{\hat{S}_1,\text{Fil},\phi_v}(T(\mathcal{M}), \mathcal{O}_K/p\mathcal{O}_K),$$

where the module on the right-hand side is in fact a $G_{K_\infty}$-module. By definition, the action of $g \in G_K$ on $\hat{A}$ is defined by

$$g(X) = \hat{\varepsilon}_1(g)(1 + X) - 1.$$

Thus we see that, for a $p$-torsion object $\mathcal{M} \in \mathcal{M}'$, the natural map

$$\text{Hom}_{\hat{S}_1,\text{Fil},\phi_v,N}(T(\mathcal{M}), \hat{A}) \to \text{Hom}_{\hat{S}_1,\text{Fil},\phi_v}(T(\mathcal{M}), \mathcal{O}_K/p\mathcal{O}_K)$$

is bijective and $G_{K_\infty}$-linear. Both sides are independent of the choice of $\pi_n$ ($n \geq 2$). We refer these modules as $T_{\text{st},\pi_n}(\mathcal{M})$ and $T_{\text{crys},\pi_n}(\mathcal{M})$. 
Finally, let $D$ and $\mathcal{D}$ be as above and $\mathcal{M}$ be a strongly divisible lattice in $\mathcal{D}$. Then the $G_K$-module
\[ \hat{T}_{\text{st},z}(\mathcal{M}) = \text{Hom}_{S,\text{Fil}^r,\phi_\ast,N}(\mathcal{M}, \hat{A}_{st}) \]
is naturally considered as a $G_K$-stable $\mathbb{Z}_p$-lattice in $V_{\text{st}}^r(D)$. By Liu’s theorem ([11, Theorem 2.3.5]), the functor $\hat{T}_{\text{st},z}$ gives an anti-equivalence of categories between the category of strongly divisible lattices in $\mathcal{D}$ and the category of $G_K$-stable $\mathbb{Z}_p$-lattices in $V_{\text{st}}^r(D)$. The $S$-module $\mathcal{M} = \hat{M}/p^n\hat{M}$ has a natural structure as an object of $\mathcal{M}'$. There is an exact sequence of $G_K$-modules
\[ 0 \to \hat{T}_{\text{st},z}(\mathcal{M}) \xrightarrow{p^n} \hat{T}_{\text{st},z}(\mathcal{M}) \to T_{\text{st},z}(\mathcal{M}) \to 0. \]

3. GALOIS ACTION ON $T_{\text{crys},\pi}(\mathcal{M})$ OVER $K_1$

Let us fix a $p$-th root $\pi_1$ of $\pi$ and set $K_1 = K(\pi_1)$, as before. For an algebraic extension $F$ of $K$, we put
\[ b_F = \{ x \in \mathcal{O}_F \mid v_K(x) > \frac{cr}{p-1} \}. \]

Let $E$ be an algebraic extension of $K_1$. We define on the $\mathcal{O}_{K_1}/p\mathcal{O}_{K_1}$-algebra $\mathcal{O}_{E}/b_{E}$ a structure of a filtered $\phi_\ast$-module over $\hat{S}_1$, as follows. We consider the algebra $\mathcal{O}_{E}/b_{E}$ as an $\hat{S}_1$-algebra by $u \mapsto \pi_1$. Define an $\hat{S}_1$-module $\text{Fil}^r(\mathcal{O}_{E}/b_{E})$ of $\mathcal{O}_{E}/b_{E}$ by
\[ \text{Fil}^r(\mathcal{O}_{E}/b_{E}) = u^{cr}(\mathcal{O}_{E}/b_{E}) = \{ \hat{x} \in \mathcal{O}_{E} \mid v_K(\hat{x}) \geq \frac{cr}{p} \}/b_{E} \]
and a $\phi$-semilinear map $\phi_r$ by
\[ \phi_r : \text{Fil}^r(\mathcal{O}_{E}/b_{E}) \to \mathcal{O}_{E}/b_{E} \]
\[ x \mapsto \hat{x}^p \mod b_{E}, \]
where $\hat{x}$ is a lift of $x$ in $\mathcal{O}_{E}$. We see that $\phi_r$ is independent of the choice of a lift $\hat{x}$ and $\phi$-semilinear.

Let $\mathcal{M}$ be a $p$-torsion object in $\mathcal{M}'$. Put $\hat{\mathcal{M}} = T(\mathcal{M})$ and
\[ T_{\text{crys},\pi_1,E}(\mathcal{M}) = \text{Hom}_{\hat{S}_1,\text{Fil}^r,\phi_\ast}(\hat{\mathcal{M}}, \mathcal{O}_{E}/b_{E}). \]
For finite extensions $E \subseteq E'$ of $K_1$, we have a natural injection of filtered $\phi_\ast$-modules over $\hat{S}_1$
\[ \mathcal{O}_{E}/b_{E} \to \mathcal{O}_{E'}/b_{E'}, \]
and this induces an injection of abelian groups
\[ T_{\text{crys},\pi_1,E}(\mathcal{M}) \to T_{\text{crys},\pi_1,E'}(\mathcal{M}). \]
Thus we have a natural identification of abelian groups
\[ T_{\text{crys},\pi_1,K}(\mathcal{M}) = \bigcup_{E/K_1: \text{finite}} T_{\text{crys},\pi_1,E}(\mathcal{M}). \]

Take an adapted basis $e_1, \ldots, e_d$ of $\hat{\mathcal{M}}$ ([2, Proposition 2.2.1.3]) such that
\[ \hat{\mathcal{M}} = \hat{S}_1e_1 \oplus \cdots \oplus \hat{S}_1e_d \]
\[ \text{Fil}'\hat{\mathcal{M}} = u'^r\hat{S}_1e_1 \oplus \cdots \oplus u'^r\hat{S}_1e_d \]
with some integers \( r_1, \ldots, r_d \) satisfying \( 0 \leq r_i \leq e_E \) for any \( i \). Define \( G(u) \in GL_d(\mathcal{O}_E) \) by

\[
\phi_r \left( \begin{array}{c} u^{r_1} e_1 \\
\vdots \\
\vdots \\
u^{r_d} e_d 
\end{array} \right) = G(u) \left( \begin{array}{c} e_1 \\
\vdots \\
\vdots \\
e_d 
\end{array} \right)
\]

and choose its lift \( \hat{G}(u) \in GL_d(W[[u]]) \). For an ideal \( I \subseteq b_E \), consider the equation in \( \mathcal{O}_E/I \)

\[
(1) \quad \left( \begin{array}{c} \frac{p^{r_1}}{(-p)^{r_1}} \hat{x}_1^p \\
\vdots \\
\vdots \\
\frac{p^{r_d}}{(-p)^{r_d}} \hat{x}_d^p 
\end{array} \right) \mod I = \hat{G}(\pi_1) \left( \begin{array}{c} \hat{x}_1 \\
\vdots \\
\vdots \\
\hat{x}_d 
\end{array} \right), \text{ with } x_i \in \pi_1^{e_x-\tau}(\mathcal{O}_E/I).
\]

Here \( \hat{x}_i \) denotes a lift of \( x_i \) in \( \mathcal{O}_E \) and this equation is independent of the choice of these lifts. The elements of

\[
T_{\text{crys},x_1,E}(\mathcal{M}) = \text{Hom}_{\bar{\mathcal{O}}_E,\mathcal{O}_E}(\mathcal{M}, \mathcal{O}_E/b_E)
\]
correspond bijectively to the solutions of this equation for \( I = b_E \).

**Lemma 3.1.** Let \( E \) be a finite extension of \( K_1 \) and \( l > e_E r/(p-1) \). Then, every solution \( (x_1, \ldots, x_d) \) of the equation (1) for \( I = m_{E}^{l+1} \) lifts uniquely to a solution of this equation for \( I = m_{E}^{l+1} \).

**Proof.** Let \( (x_1, \ldots, x_d) \) be such a solution and take a lift \( \hat{x}_i \) of \( x_i \) in \( \mathcal{O}_E \). Let \( \pi_E \) denote a uniformizer of \( E \). Then, for some \( c_1, \ldots, c_d \in \mathcal{O}_E \), we have

\[
\left( \begin{array}{c} \frac{p^{r_1}}{(-p)^{r_1}} \hat{x}_1^p \\
\vdots \\
\vdots \\
\frac{p^{r_d}}{(-p)^{r_d}} \hat{x}_d^p 
\end{array} \right) = \hat{G}(\pi_1) \left( \begin{array}{c} \hat{x}_1 + \pi_E^l \hat{y}_1 \\
\vdots \\
\vdots \\
\hat{x}_d + \pi_E^l \hat{y}_d 
\end{array} \right) \mod \pi_{E}^{l+1}.
\]

For \( \hat{y}_1, \ldots, \hat{y}_d \in \mathcal{O}_E \), consider the equation

\[
\left( \begin{array}{c} \frac{p^{r_1}}{(-p)^{r_1}} \hat{x}_1 + \pi_E^l \hat{y}_1 \n\vdots \\
\vdots \\
\frac{p^{r_d}}{(-p)^{r_d}} \hat{x}_d + \pi_E^l \hat{y}_d
\end{array} \right) \equiv \hat{G}(\pi_1) \left( \begin{array}{c} \hat{x}_1 + \pi_E^l \hat{y}_1 \\
\vdots \\
\vdots \\
\hat{x}_d + \pi_E^l \hat{y}_d 
\end{array} \right) \mod \pi_{E}^{l+1}.
\]

This is equivalent to

\[
\left( \begin{array}{c} \hat{y}_1 \\
\vdots \\
\hat{y}_d 
\end{array} \right) \equiv \hat{G}(\pi_1)^{-1} \left( \begin{array}{c} c_1 \\
\vdots \\
\vdots \\
c_d 
\end{array} \right) + \left( \begin{array}{c} \frac{p^{r_1}}{(-p)^{r_1}} (\sum_{t=1}^{p} \binom{p}{t} \hat{x}_1^{p-t} \pi_E^l \hat{y}_1^t) \\
\vdots \\
\vdots \\
\frac{p^{r_d}}{(-p)^{r_d}} (\sum_{t=1}^{p} \binom{p}{t} \hat{x}_d^{p-t} \pi_E^l \hat{y}_d^t) 
\end{array} \right) \mod \pi_{E},
\]

where \( \binom{p}{t} \) denotes the binomial coefficient \( p!/(t!(p-t)!)) \).

We claim that every entry of the last term of this equation is a polynomial of \( \hat{y}_1, \ldots, \hat{y}_d \) whose coefficients are in the maximal ideal \( m_{E} \). Indeed, the coefficient of \( \hat{y}_i^t \)

\[
\frac{p^{r_i}}{(-p)^{r_i}} \pi_{E}^{l(p-1)}
\]

has the valuation

\[
e(E/K)r_i - e(E)r + l(p-1) > 0.
\]
Similarly, the coefficient of $y_t^l$ ($1 \leq t \leq p - 1$)
$$\frac{\pi_{t_0}^p}{(-p)^r} \frac{p}{t} \tilde{x}_{t_0} - \pi_t^{(t-1)}$$
has the valuation
$$e(E/K)r_i - e(E)(r - 1) + (p - t)v_E(\hat{x}_i) + l(t - 1)$$
$$= e(E/K)r_i - e(E)(r - 1) + pv_E(\hat{x}_i) - l + t(l - v_E(\hat{x}_i)).$$
If $l \geq v_E(\hat{x}_i)$, then the minimum of this value for $t = 1, \ldots, p - 1$ is obtained by $t = 1$. This minimum value is equal to
$$e(E/K)r_i - e(E)(r - 1) + (p - 1)v_E(\hat{x}_i)$$
$$\geq \frac{e(E)(p - r)}{p} + \frac{e(E/K)r_i}{p} > 0.$$ If $l < v_E(\hat{x}_i)$, then the minimum is obtained by $t = p - 1$ and is equal to
$$e(E/K)r_i - e(E)(r - 1) + (p - 2)l + v_E(\hat{x}_i)$$
$$> e(E/K)r_i - e(E)(r - 1) + \frac{e(E)(p - 2)}{p - 1} + v_E(\hat{x}_i)$$
$$= e(E/K)r_i + e(E)(1 - \frac{r}{p - 1}) + v_E(\hat{x}_i) > 0.$$ Thus we can solve uniquely this equation. \hfill \Box

**Corollary 3.2.** There is a $G_{K_1}$-linear isomorphism
$$T_{\text{crys}}^*\pi_1(M) \cong T_{\text{crys}}^*\pi_1, K(M).$$ In particular, we have $\#T_{\text{crys}}^*\pi_1, K(M) = p^d$ for $d = \text{dim}_{S_1} M$.

**Proof.** Applying Lemma 3.1 for a sufficiently large finite extension $E$ of $K_1$, we see that the natural map
$$T_{\text{crys}}^*\pi_1(M) \to T_{\text{crys}}^*\pi_1, K(M)$$
induced by $\mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_K/b_M$ is bijective. The last assertion follows from [2, Lemme 2.3.1.2]. \hfill \Box

**Lemma 3.3.** The fixed part $T_{\text{crys}}^*\pi_1, K(M)^G_E$ is equal to $T_{\text{crys}}^*\pi_1, E(M)$.

**Proof.** From Lemma 3.1, we see that the elements of $T_{\text{crys}}^*\pi_1, K(M)$ correspond bijectively to the solutions of the equation (1) for $I = 0$ in $\mathcal{O}_K$. The uniqueness of the lift shows that $g \in G_{K_1}$ fixes a solution in $\mathcal{O}_K$ if and only if $g$ fixes its image in $\mathcal{O}_K/b_M$. This concludes the proof. \hfill \Box

**Corollary 3.4.** Let $L_1$ be the finite Galois extension of $K_1$ corresponding to the kernel of the map $G_{K_1} \to \text{Aut}(T_{\text{crys}}^*\pi_1(M))$. Then an algebraic extension $E$ of $K_1$ contains $L_1$ if and only if $\#T_{\text{crys}}^*\pi_1, K(M) = p^d$, where $d = \text{dim}_{S_1} M$.

**Proof.** An algebraic extension $E$ of $K_1$ contains $L_1$ if and only if the action of $G_E$ on $T_{\text{crys}}^*\pi_1, K(M)$ is trivial. By Lemma 3.3, this is equivalent to $T_{\text{crys}}^*\pi_1, K(M) = T_{\text{crys}}^*\pi_1, E(M)$. Then the corollary follows. \hfill \Box
4. Ramification bound

In this section, we prove Theorem 1.1. Take $G_K$-stable $\mathbb{Z}_p$-lattices $\mathcal{L} \supseteq \mathcal{L}'$ in $V$ such that $\mathcal{L}' \supseteq p\mathcal{L}$. Since the $G_K$-module $\mathcal{L}/\mathcal{L}'$ is a quotient of $\mathcal{L}/p\mathcal{L}$, we may assume $\mathcal{L}' = p\mathcal{L}$. For $r = 0$, we see that the $G_K$-module $V$ is unramified and the theorem is trivial. Thus we may assume $r > 0$. Then, by Liu’s theorem ([11, Theorem 2.3.5]), it suffices to show the following.

**Theorem 4.1.** Let $r > 0$ be an integer with $r < p - 1$. For $\mathcal{M} \in \mathcal{M}^r$ which is killed by $p$, $G_K^{(j)}$ acts trivially on $T_{st,\pi_1}(\mathcal{M})$ for $j > u(K, r)$.

Let $L$ and $L_1$ be the finite Galois extensions of $K$ and $K_1$ corresponding to the kernels of the maps

$$G_K \to \text{Aut}(T_{st,\pi_1}(\mathcal{M})) \text{ and } G_K \to \text{Aut}(T_{st,\pi_1}(\mathcal{M})),$$

respectively. Similarly, put $\bar{K}_1 = K_1(\zeta_p)$ and let $\bar{L}_1$ denote the finite extension of $\bar{K}_1$ corresponding to the kernel of the map

$$G_{\bar{K}_1} \to \text{Aut}(T_{st,\pi_1}(\mathcal{M})).$$

Then we have $L_1 = L\bar{K}_1$ and $\bar{L}_1 = L\bar{K}_1 = L_1\bar{K}_1$, where the latter is a Galois extension of $K$. In the following, we bound the ramification of $\bar{L}_1$ over $K$.

For an algebraic extension $E$ of $K$, we put

$$a_{E/K}^m = \{x \in O_E \mid v_E(x) \geq m\}.$$

**Proposition 4.2.** Let $E$ be an algebraic extension of $K$. If $m > u(K, r)$ and there exists an $O_K$-algebra homomorphism

$$\eta : O_{L_1} \to O_E/a_{E/K}^m,$$

then there exists a $K$-algebra injection $\bar{L}_1 \to E$.

**Proof.** We mimic the proof of [7, Théorème 2]. By assumption, we have $m > er/(p - 1)$ and $b_E \supseteq a_{E/K}^m$. Thus $\eta$ induces an $O_K$-algebra homomorphism

$$O_{L_1} \to O_E/b_E.$$

Note that $\bar{K}_1 = K_1(\zeta_p)$ is a Galois extension of $K$ and its greatest upper ramification break $u_{\bar{K}_1/K}$ ([6]) is

$$u_{\bar{K}_1/K} = 1 + e(1 + \frac{1}{p - 1}) < m.$$

As $\eta$ induces an $O_K$-algebra homomorphism $O_{\bar{K}_1} \to O_E/a_{E/K}^m$, from [6, Proposition 1.5] we get a $K$-linear injection $\bar{K}_1 \to E$. Thus we see that $E$ contains $\pi_1$ and $\zeta_p$. More precisely, we have the following lemma.

**Lemma 4.3.** $\eta(\pi_1) \equiv \pi_1\zeta_i^i \mod b_E$ for some $i$.

**Proof.** As $\eta$ is $O_K$-linear, the equality $\eta(\pi_1)^p = \pi$ holds in $O_E/a_{E/K}^m$. Set $\hat{x}$ to be a lift of $\eta(\pi_1)$ in $O_E$. Then we have

$$v_K(\hat{x}^p - \pi) = \sum_{i=0}^{p-1} v_K(\hat{x} - \pi_1\zeta_i^i) \geq m.$$
Take $i$ such that $v_K(\hat{x} - \pi_1\zeta'_p) \geq v_K(\hat{x} - \pi_1\zeta'_p)$ holds for any $i'$. From the equality
\[ v_K(\hat{x} - \pi_1\zeta'_p) = v_K(\hat{x} - \pi_1\zeta'_p + \pi_1(\zeta'_p - \zeta'_p)), \]
we see that $v_K(\hat{x} - \pi_1\zeta'_p) \leq 1/p + e/(p-1)$ and
\[ v_K(\hat{x} - \pi_1\zeta'_p) \geq m - e - \frac{p-1}{p}. \]
Since
\[ m - e - \frac{p-1}{p} > e\frac{r}{p-1} \]
by assumption, we have $\hat{x} \equiv \pi_1\zeta'_p \mod b_E$ and the lemma follows. \hfill \Box

**Lemma 4.4.** The $O_K$-algebra homomorphism $\eta$ induces an $O_K$-algebra injection
\[ \bar{\eta}: O_{L_1}/b_{L_1} \to O_E/b_E. \]

**Proof.** We write the Eisenstein polynomial of a uniformizer $\pi_{L_1}$ of $\tilde{L}_1$ over $O_K$ as
\[ P(T) = T^\tilde{\epsilon} + c_1T^{\tilde{\epsilon}-1} + \cdots + c_{\tilde{\epsilon}-1}T + c_{\tilde{\epsilon}}, \]
where $\tilde{\epsilon} = e(\tilde{L}_1/K)$. Then $t = \eta(\pi_{L_1})$ satisfies
\[ t^\tilde{\epsilon} = -(c_1\tilde{\epsilon}^{-1} + \cdots + c_{\tilde{\epsilon}-1}T + c_{\tilde{\epsilon}}) \]
in $O_E/a_{E/K}^m$. Let $\tilde{t}$ be a lift of $t$ in $O_E$. Since $m > 1$, we have $v_E(\tilde{t}) = e(E/K)/\tilde{\epsilon}$.

The condition $n > e(\tilde{L}_1)r/(p-1)$ is equivalent to the condition
\[ v_E(\tilde{t}^n) > \frac{e(\tilde{L}_1)r}{p-1} \cdot \frac{e(E/K)}{\tilde{\epsilon}} = \frac{e(E)r}{p-1}. \]
Thus the lemma follows. \hfill \Box

We give the $O_K$-algebras $O_{L_1}/b_{L_1}$ and $O_E/b_E$ natural structures of filtered $\phi_r$-modules over $\tilde{S}_1$ as follows. The $O_K$-algebra $O_{L_1}/b_{L_1}$ is considered as an $\tilde{S}_1$-module by $u \mapsto \pi_1$. Put
\[ \text{Fil}^r(O_{L_1}/b_{L_1}) = \{ \tilde{x} \in O_{L_1} \mid v_K(\tilde{x}) \geq \frac{er}{p} \}/b_{L_1} \]
and for $x \in \text{Fil}^r(O_{L_1}/b_{L_1})$, set
\[ \phi_r(x) = \frac{\tilde{x}^p}{(-p)^r} \mod b_{L_1}, \]
as in Section 3. On the other hand, we consider $O_E/b_E$ as an $\tilde{S}_1$-module by $u \mapsto \pi_1\zeta_p^i$ and define $\text{Fil}^r$ and $\phi_r$ of $O_E/b_E$ in the same way as $O_{L_1}/b_{L_1}$. From the definition, we see that $\bar{\eta}$ is $\tilde{S}_1$-linear. We can also check that $\bar{\eta}$ preserves $\text{Fil}^r$ and $\phi_r$ of both sides, just as in the proof of [7, Théorème 2].

Then the injection $\bar{\eta}$ induces an injection of abelian groups
\[ T_{\text{crys}, L_1, \pi_1}^*(M) \to T_{\text{crys}, E, \pi_1\zeta_p^i}^*(M). \]

Since $\tilde{L}_1$ contains $L_1$, the abelian group $T_{\text{crys}, L_1, \pi_1}^*(M)$ is of order $p^d$ by Corollary 3.4, where $d = \dim_{S_1} M$. This implies $\#T_{\text{crys}, E, \pi_1\zeta_p^i}^*(M) = p^d$. Let $L'_1$ denote the finite Galois extension of $K'_1 = K(\pi_1\zeta_p^i)$ corresponding to the kernel of the map $G_{K'_1} \to \text{Aut}(T_{\text{crys}, E, \pi_1\zeta_p^i}^*(M))$. 


Then, again by Corollary 3.4, we see that $E \supset \tilde{L}_1$. However, by definition, the finite extension $L_1$ is Galois over $K$, the Galois group $G_{L_1} = G_{L_1} \cap G_{\tilde{K}_1}$ is equal to
$$gG_{L_1}g^{-1} \cap G_{\tilde{K}_1} = g(G_{L_1} \cap G_{\tilde{K}_1})g^{-1} = gG_{L_1}g^{-1} = G_{L_1}.$$ Hence $\tilde{L}_1$ is equal to $L_1$. Since $E$ contains $\tilde{K}_1$, we conclude that $E$ also contains $\tilde{L}_1$.

\begin{corollary}
we have the inequality
$$u_{L/K} \leq u_{\tilde{L}_1/K} \leq u(\tilde{K}, r).$$
\end{corollary}

\begin{proof}
This follows from [10, Proposition 5.6].
\end{proof}

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