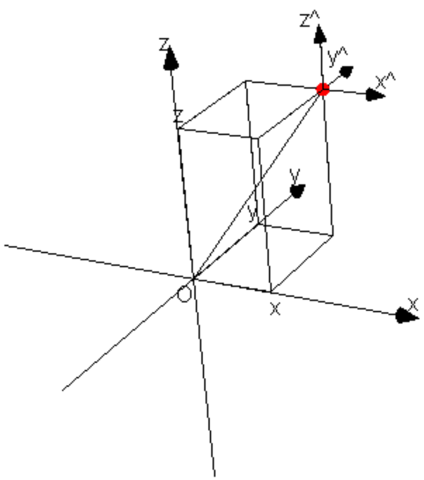
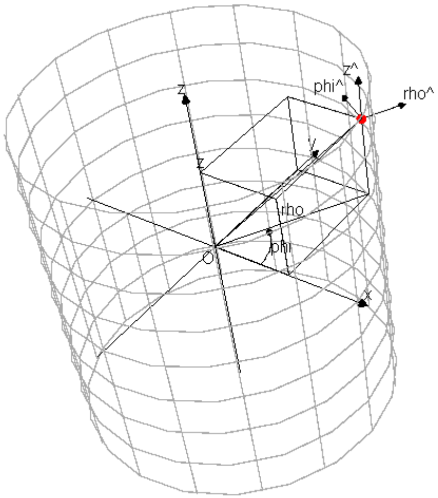
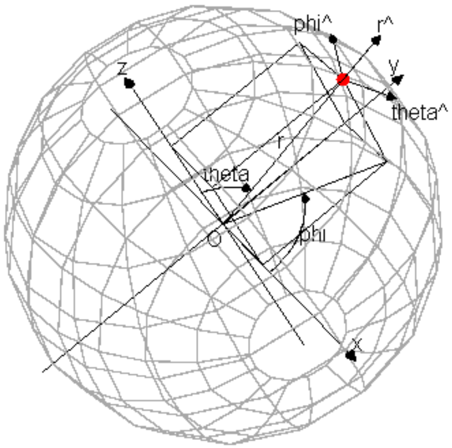
Coordinate Systems



Cartesian



Cylindrical



Spherical

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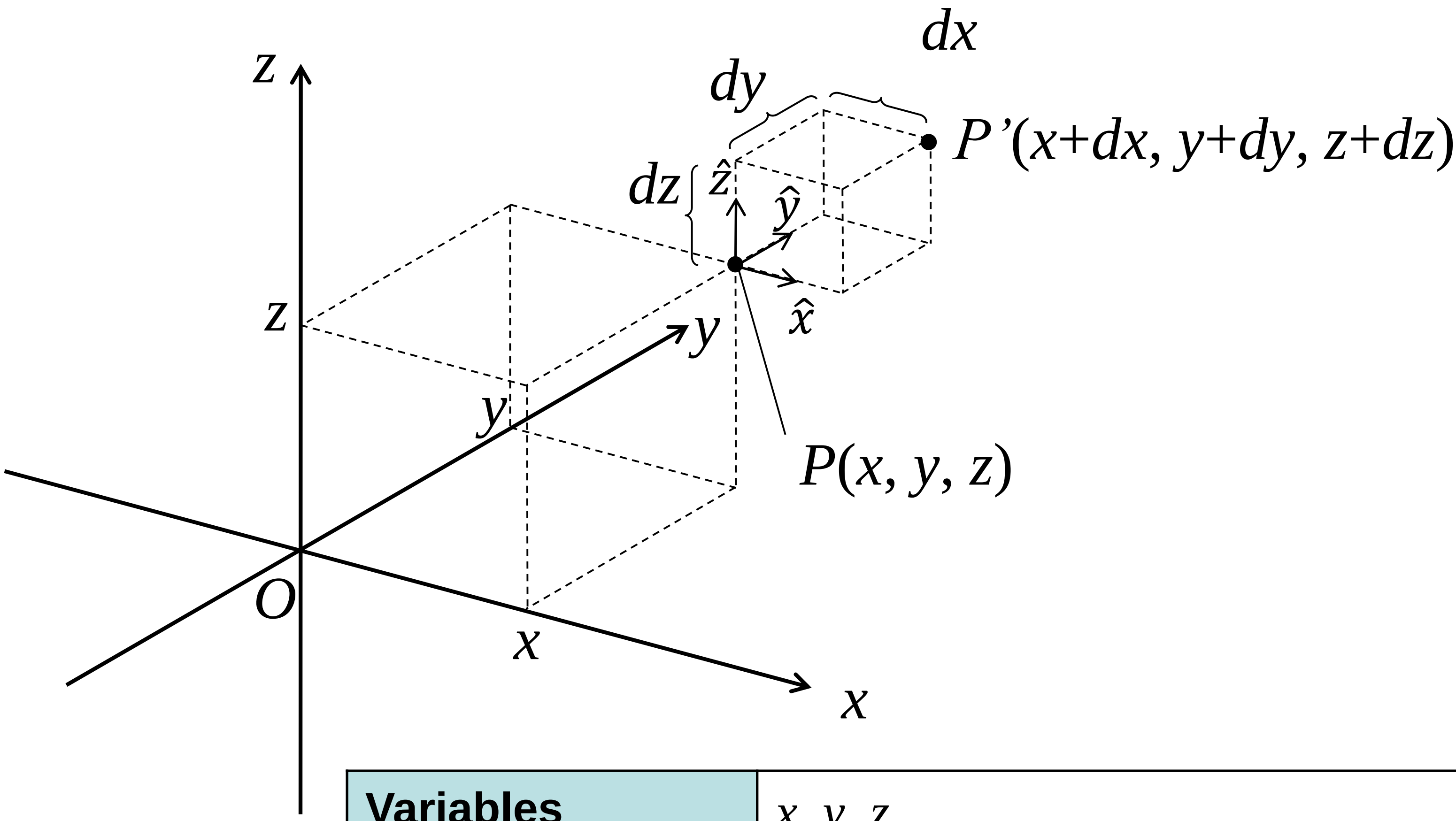
Coordinate Systems

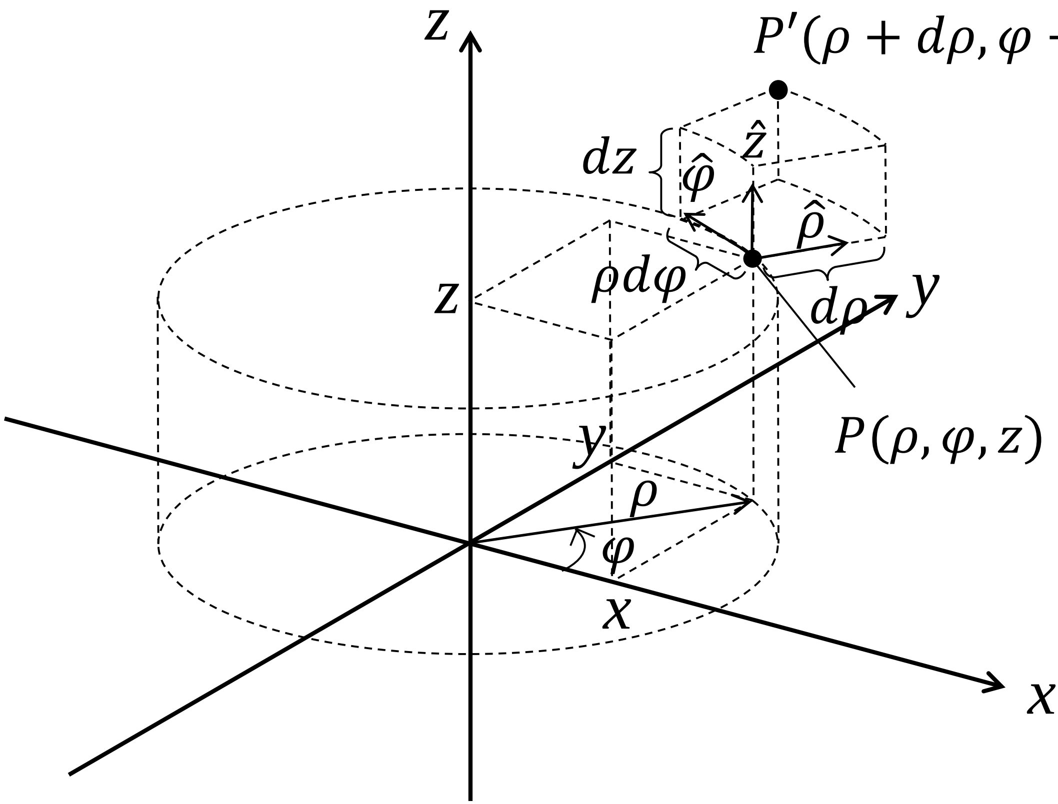
- Cartesian Coordinates
- Cylindrical Coordinates
- Spherical Coordinates

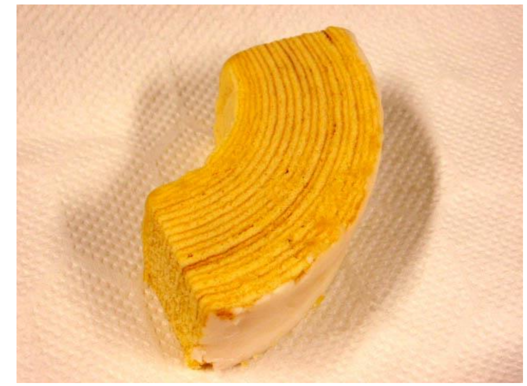
Cartesian Coordinates



Variables	x, y, z
Bases	$\hat{x}, \hat{y}, \hat{z}$
Line element	$d\boldsymbol{l} = \overrightarrow{PP'} = \hat{x}dx + \hat{y}dy + \hat{z}dz$
Surface element	$d\boldsymbol{S}_x = \hat{x}dydz,\ d\boldsymbol{S}_y = \hat{y}dxdz,\ d\boldsymbol{S}_z = \hat{z}dxdy$
Volume element	$dV = dxdydz$

Cylindrical Coordinates

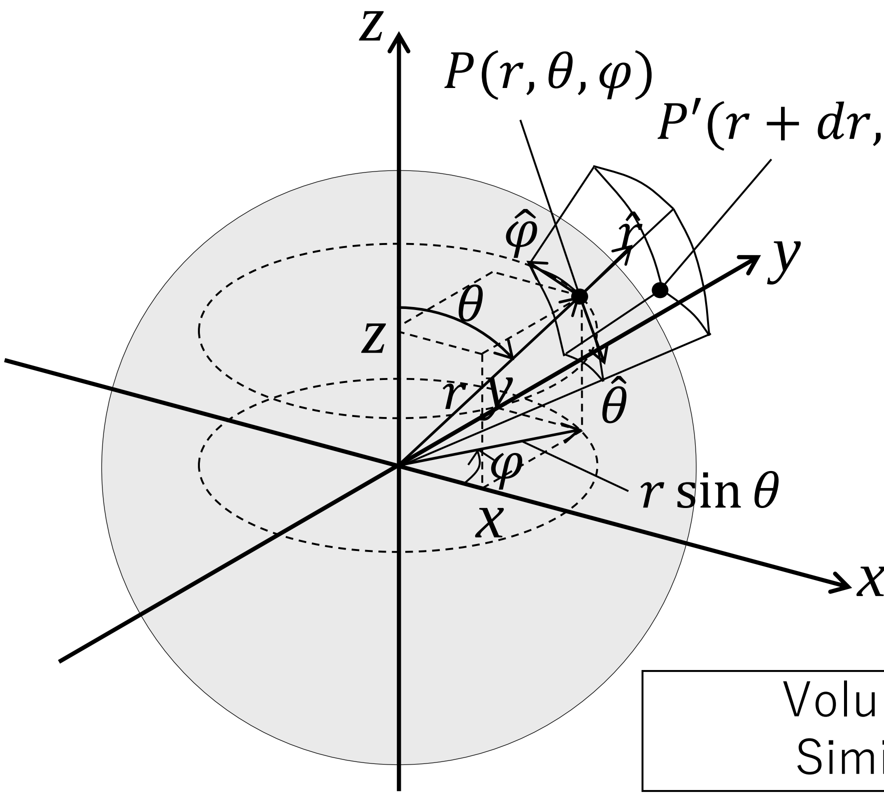


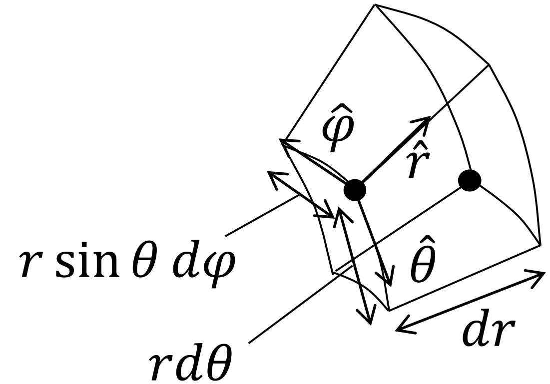


Volume element:
Similar to Baumkuchen

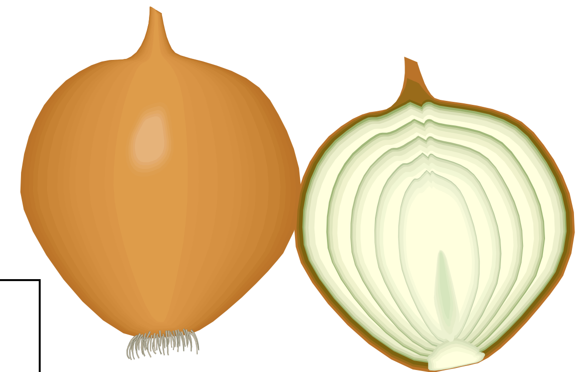
Variables	ρ, φ, z
Bases	$\hat{\rho}, \hat{\varphi}, \hat{z}$
Line element	$d\boldsymbol{l} = \overrightarrow{PP'} = \hat{\rho} d\rho + \hat{\varphi} \rho d\varphi + \hat{z} dz$
Surface element	$d\boldsymbol{S}_\rho = \hat{\rho} \rho d\varphi dz, \ d\boldsymbol{S}_\varphi = \hat{\varphi} d\rho dz, \ d\boldsymbol{S}_z = \hat{z} \rho d\rho d\varphi$
Volume element	$dV = \rho d\rho d\varphi dz$

Spherical Coordinates





Volume element:
Similar to onion



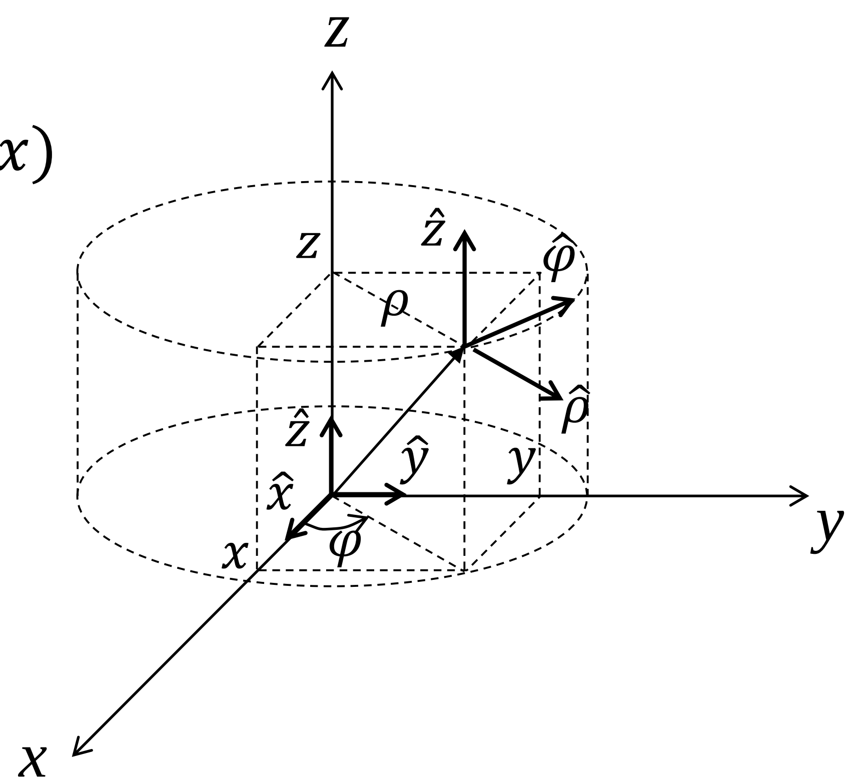
Variables	r, θ, φ
Bases	$\hat{r}, \hat{\theta}, \hat{\varphi}$
Line element	$d\boldsymbol{l} = \overrightarrow{PP'} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\varphi}r\sin\theta\, d\varphi$
Surface element	$d\boldsymbol{S}_r = \hat{r}r^2 \sin\theta\, d\theta d\varphi, d\boldsymbol{S}_\theta = \hat{\varphi}r\sin\theta\, drd\varphi, d\boldsymbol{S}_\varphi = \hat{z}rdrd\theta$
Volume element	$dV = r^2 \sin\theta\, drd\theta d\varphi$

Transform: $(x, y, z) \leftrightarrow (\rho, \varphi, z)$

Position

$$(\rho, \varphi, z) \rightarrow (x, y, z) \quad \begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases}$$

$$(x, y, z) \rightarrow (\rho, \varphi, z) \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \varphi = \tan^{-1}(y/x) \\ z = z \end{cases}$$



Transform: $(x, y, z) \leftrightarrow (\rho, \varphi, z)$

Vector

Calculate inner products of bases at first.

$$\begin{cases} \hat{x}\cdot\hat{\rho} = \cos\varphi \\ \hat{x}\cdot\hat{\varphi} = -\sin\varphi \\ \hat{x}\cdot\hat{z} = 0 \end{cases} \quad \begin{cases} \hat{y}\cdot\hat{\rho} = \sin\varphi \\ \hat{y}\cdot\hat{\varphi} = \cos\varphi \\ \hat{y}\cdot\hat{z} = 0 \end{cases} \quad \begin{cases} \hat{z}\cdot\hat{\rho} = 0 \\ \hat{z}\cdot\hat{\varphi} = 0 \\ \hat{z}\cdot\hat{z} = 1 \end{cases}$$

Let us transform from (x, y, z) expression $\boldsymbol{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$ to (ρ, φ, z) expression.

$$\boldsymbol{A} = \hat{\rho}(\hat{\rho}\cdot\boldsymbol{A}) + \hat{\varphi}(\hat{\varphi}\cdot\boldsymbol{A}) + \hat{z}(\hat{z}\cdot\boldsymbol{A})$$

After transformation

$$= \hat{\rho}\left(\hat{\rho}\cdot(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)\right)$$

Before transformation

$$+\hat{\varphi}\left(\hat{\varphi}\cdot(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)\right)$$
$$+\hat{z}\left(\hat{z}\cdot(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)\right)$$

Finish to substitute

$$= \hat{\rho}\left((\hat{\rho}\cdot\hat{x})A_x + (\hat{\rho}\cdot\hat{y})A_y + (\hat{\rho}\cdot\hat{z})A_z\right)$$
$$+\hat{\varphi}\left((\hat{\varphi}\cdot\hat{x})A_x + (\hat{\varphi}\cdot\hat{y})A_y + (\hat{\varphi}\cdot\hat{z})A_z\right)$$
$$+\hat{z}\left((\hat{z}\cdot\hat{x})A_x + (\hat{z}\cdot\hat{y})A_y + (\hat{z}\cdot\hat{z})A_z\right)$$

Inverse transformation is the same as well.

Transform: $(x, y, z) \leftrightarrow (r, \theta, \varphi)$

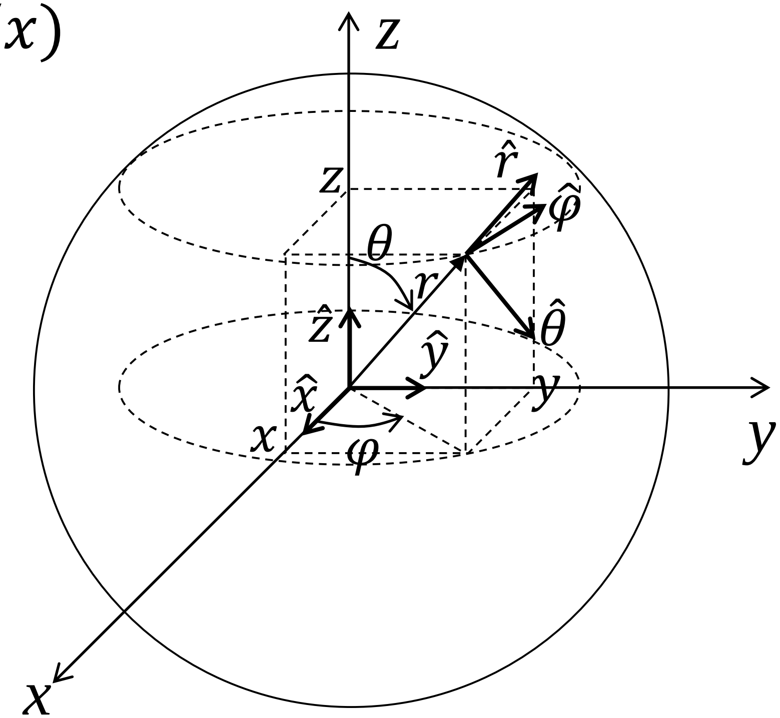
Position

$(r, \theta, \varphi) \to (x, y, z)$

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

$(x, y, z) \to (r, \theta, \varphi)$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2}) \\ \varphi = \tan^{-1}(y/x) \end{cases}$$



Transform: $(x, y, z) \leftrightarrow (r, \theta, \varphi)$

Vector

Calculate inner products of bases at first.

$$\begin{cases} \hat{x} \cdot \hat{r} = \sin\theta \cos\varphi \\ \hat{x} \cdot \hat{\theta} = \cos\theta \cos\varphi \\ \hat{x} \cdot \hat{\varphi} = -\sin\varphi \end{cases} \qquad \begin{cases} \hat{y} \cdot \hat{r} = \sin\theta \sin\varphi \\ \hat{y} \cdot \hat{\theta} = \cos\theta \sin\varphi \\ \hat{y} \cdot \hat{\varphi} = \cos\varphi \end{cases} \qquad \begin{cases} \hat{z} \cdot \hat{r} = \cos\theta \\ \hat{z} \cdot \hat{\theta} = -\sin\theta \\ \hat{z} \cdot \hat{\varphi} = 0 \end{cases}$$

The procedure is the same as that of the cylindrical coordinates.